

Combinatorial solutions to integrable hierarchies

Sergei Lando

National Research University Higher School of Economics,
Independent University of Moscow,
Moscow, Russia

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M. Kazarian, S. Lando,
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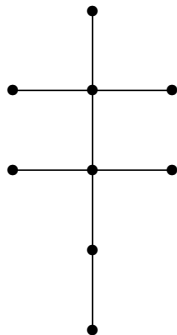
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- **(Bousquet-Mélou–Schaeffer problem, 2000)** What is the number of ways to represent a given permutation as a product of a given number of permutations?
- **(Enumeration of maps)** What is the number of maps on a given surface?

Example of Hurwitz numbers: Cayley's Theorem

Theorem *The number of labelled trees on N vertices is N^{N-2} .*

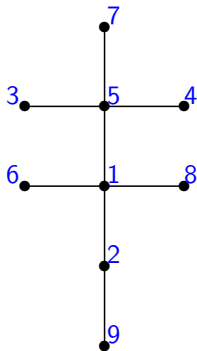
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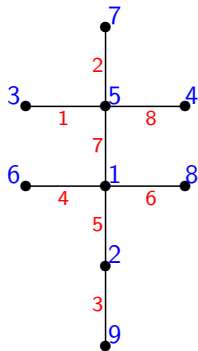
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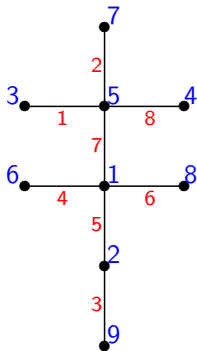
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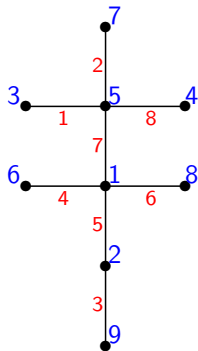
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Bijection

Trees with N labeled
vertices and $N - 1$
numbered edges

\longleftrightarrow

$(N - 1)$ -tuples of
transpositions in S_N
whose product
is a long cycle

Definition

For μ a partition of N , the *simple Hurwitz number* $h_{m;\mu}^\circ$ is defined by

$$h_{m;\mu}^\circ = \frac{1}{N!} |\{(\tau_1, \dots, \tau_m) \mid \text{the } \tau_i \text{ are transpositions,} \\ \tau_m \circ \dots \circ \tau_1 \text{ has cycle type } \mu\}|.$$

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Connected Hurwitz numbers enumerate certain connected graphs

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$$b_{m,k;\mu}^\circ = \frac{1}{N!} |\{(\tau_1, \dots, \tau_m) \mid \text{the } \tau_i \in S_N, \text{ their total degeneracy is } k, \text{ and } \tau_m \circ \dots \circ \tau_1 \text{ has cycle type } \mu\}|.$$

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The *degeneracy* of a partition $\nu \vdash N$, $\nu = (\nu_1, \dots, \nu_\ell)$, is $N - \ell$.

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For μ a partition of N , let $r_{m,n;\mu}^\circ$ denote the number of oriented rooted maps with m edges and n faces, and the partition defined by the valencies of the vertices being μ .

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Generating functions for Hurwitz numbers

Define exponential generating functions in a variable u (recording the number of transpositions) and infinitely many variables p_1, p_2, \dots (recording the parts of the partitions):

$$H^\circ(u; p_1, p_2, \dots) = \sum_{m=0}^{\infty} \sum_{\mu} h_{m;\mu}^\circ p_{\mu_1} p_{\mu_2} \cdots \frac{u^m}{m!};$$

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and

$$H(u; p_1, p_2, \dots) = \sum_{m=0}^{\infty} \sum_{\mu} h_{m; \mu} p_{\mu_1} p_{\mu_2} \cdots \frac{u^m}{m!}.$$

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Here $\mu = (\mu_1, \mu_2, \dots)$, $\mu_1 \geq \mu_2 \geq \dots$ runs over *all* partitions.

The two are related by

$$H^\circ = \exp(H).$$

Series expansion for Hurwitz numbers

It is clear from the definition that the coefficients of both H and H° are rational. The first terms of the power series expansions are

$$H^\circ(u; p_1, \dots) = 1 + p_1 + \frac{p_2 u}{2} + \frac{p_1^2}{2} + \frac{p_3 u^2}{2} + \frac{1}{2} p_2 p_1 u + \frac{p_1^3}{6} + \frac{p_2 u^3}{12} \\ + \frac{2p_4 u^3}{3} + \frac{1}{4} p_1^2 u^2 + \frac{1}{2} p_3 p_1 u^2 + \frac{1}{8} p_2^2 u^2 + \frac{1}{4} p_2 p_1^2 u + \frac{p_1^4}{24} + \dots$$

$$H(u; p_1, \dots) = p_1 + \frac{p_2 u}{2} + \left(\frac{p_1^2}{4} + \frac{p_3}{2} \right) u^2 + \left(\frac{2p_1 p_2}{3} + \frac{p_2}{12} + \frac{2p_4}{3} \right) u^3 \\ + \left(\frac{p_1^3}{6} + \frac{p_1^2}{48} + \frac{9p_3 p_1}{8} + \frac{p_2^2}{2} + \frac{3p_3}{8} + \frac{25p_5}{24} \right) u^4 + \dots$$

Generating functions for Bousquet-Mélou–Schaeffer numbers

For each $m = 0, 1, 2, \dots$, define exponential generating functions in a variable u (recording the total degeneracies) and infinitely many variables p_1, p_2, \dots (recording the parts of the partitions):

$$B_m^\circ(u; p_1, p_2, \dots) = \sum_{k, \mu} b_{m, k; \mu}^\circ p_{\mu_1} p_{\mu_2} \cdots \frac{u^k}{k!};$$

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$$B_m^\circ = \exp(B_m).$$

Generating functions for the numbers of rooted maps

Define exponential generating functions in a variable w (recording the number of edges), a variable z (recording the number of faces), and infinitely many variables p_1, p_2, \dots (recording the vertices' valencies):

$$R^\circ(w, z; p_1, p_2, \dots) = \sum_{m, n, \mu} \frac{r_{m, n; \mu}^\circ}{2n} p_{\mu_1} p_{\mu_2} \cdots \frac{w^m}{m!} \frac{z^n}{n!};$$

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$$R^\circ = \exp(R).$$

Theorem (A. Okounkov 2000, M. Kazarian, S. L. 2007)

The generating function H for the connected Hurwitz numbers is a one-parameter family of solutions to the Kadomtsev–Petviashvili hierarchy of partial differential equations. In particular,

$$\frac{\partial^2 H}{\partial p_2^2} = \frac{\partial^2 H}{\partial p_1 \partial p_3} - \frac{1}{2} \left(\frac{\partial^2 H}{\partial p_1^2} \right)^2 - \frac{1}{12} \frac{\partial^4 H}{\partial p_1^4}.$$

Theorem (I. P. Goulden, D. M. Jackson, 2008)

Each generating function B_m for the connected Bousquet-Mélou-Schaeffer numbers is a one-parameter family of solutions to the Kadomtsev-Petviashvili hierarchy of partial differential equations. In particular,

$$\frac{\partial^2 B_m}{\partial p_2^2} = \frac{\partial^2 B_m}{\partial p_1 \partial p_3} - \frac{1}{2} \left(\frac{\partial^2 B_m}{\partial p_1^2} \right)^2 - \frac{1}{12} \frac{\partial^4 B_m}{\partial p_1^4}$$

for $m = 1, 2, 3, \dots$

Theorem (I. P. Goulden, D. M. Jackson, 2008)

The generating function R for the numbers of connected maps is a two-parameter family of solutions to the Kadomtsev–Petviashvili hierarchy of partial differential equations. In particular,

$$\frac{\partial^2 R}{\partial p_2^2} = \frac{\partial^2 R}{\partial p_1 \partial p_3} - \frac{1}{2} \left(\frac{\partial^2 R}{\partial p_1^2} \right)^2 - \frac{1}{12} \frac{\partial^4 R}{\partial p_1^4}.$$

Genus expansion for connected maps

Connected maps have *genus*. The generating function R for connected maps admits the *genus expansion*

$$R^{\hbar}(w, z; p_1, p_2, \dots) = R_0(w, z; p_1, p_2, \dots) + \hbar^2 R_1(w, z; p_1, p_2, \dots) + \hbar^4 R_2(w, z; p_1, p_2, \dots) + \dots,$$

where R_g is the generating function for the numbers of maps on the genus g compact oriented surface, $g = 0, 1, 2, \dots$

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The genus expansion R^{\hbar} can be obtained from the generating function R by the substitution

$$R^{\hbar}(w, z; p_1, p_2, p_3, \dots) = \hbar^2 R\left(\hbar w, \hbar z; \frac{p_1}{\hbar^2}, \frac{p_2}{\hbar^3}, \frac{p_3}{\hbar^4} \dots\right)$$

(the Euler genus formula).

Theorem

The generating function R^{\hbar} for the numbers of connected maps is a two-parameter family of solutions to the perturbed Kadomtsev–Petviashvili hierarchy of partial differential equations. In particular,

$$\frac{\partial^2 R^{\hbar}}{\partial p_2^2} = \frac{\partial^2 R^{\hbar}}{\partial p_1 \partial p_3} - \frac{1}{2} \left(\frac{\partial^2 R^{\hbar}}{\partial p_1^2} \right)^2 - \frac{\hbar^2}{12} \frac{\partial^4 R^{\hbar}}{\partial p_1^4}$$

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due to the above substitution

$$R^{\hbar}(w, z; p_1, p_2, p_3, \dots) = \hbar^2 R \left(\hbar w, \hbar z; \frac{p_1}{\hbar^2}, \frac{p_2}{\hbar^3}, \frac{p_3}{\hbar^4}, \dots \right).$$

Genus expansion equations

For the genus 0 part, series expansion for the perturbed KP equation yields

$$\frac{\partial^2 R_0}{\partial p_2^2} = \frac{\partial^2 R_0}{\partial p_1 \partial p_3} - \frac{1}{2} \left(\frac{\partial^2 R_0}{\partial p_1^2} \right)^2$$

(the *dispersionless* first KP equation).

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For the genus 1 part, series expansion for the perturbed KP equation yields

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which is a **LINEAR** partial differential equation with respect to R_1 .

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Equations for P_2, P_3, \dots also are linear.

Genus expansion for Hurwitz and Bousquet-Mélou–Schaeffer numbers

$$\begin{aligned} H^{\hbar}(u; p_1, p_2, \dots) &= \hbar^2 H(\hbar u; p_1/\hbar^2, p_2/\hbar^3, \dots) \\ &= H_0(u; p_1, p_2, \dots) + \hbar^2 H_1(u; p_1, p_2, \dots) \\ &\quad + \hbar^4 H_2(u; p_1, p_2, \dots) + \dots, \end{aligned}$$

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$$\begin{aligned} B_m^{\hbar}(u; p_1, p_2, \dots) &= \hbar^2 B_m(\hbar u; p_1/\hbar^2, p_2/\hbar^3, \dots) \\ &= B_{m;0}(u; p_1, p_2, \dots) + \hbar^2 B_{m;1}(u; p_1, p_2, \dots) \\ &\quad + \hbar^4 B_{m;2}(u; p_1, p_2, \dots) + \dots, \end{aligned}$$

Theorem (Hurwitz's formula, 1891)

For a partition $\mu = (\mu_1, \dots, \mu_n)$ of N , and m smallest possible, we have

$$h_{m;\mu} = \frac{m!}{|Aut(\mu)|} \prod_{i=1}^n \frac{\mu_i^{\mu_i}}{\mu_i!} N^{n-3}.$$

Theorem (Bousquet-Mélou–Schaeffer formula, 2000)

For a partition $\mu = (\mu_1, \dots, \mu_n)$ of N , we have

$$b_{m,k;\mu} = \frac{m(m|\mu| - k + 1)^{\overline{\ell(\mu)-3}}}{|Aut(\mu)|} \prod_{i=1}^n \binom{m\mu_i - 1}{\mu_i}, \quad k = |\mu| + n - 2.$$

Theorem (A. Yu. Orlov, D. M. Shcherbin, 2001, I. P. Goulden, D. M. Jackson, 2008)

The generating function

$$\log \sum_{\mu} y_{\mu} s_{\mu}(p_1, p_2, \dots) s_{\mu}(q_1, q_2, \dots),$$

is a family of solutions to the KP hierarchy depending of infinitely many parameters $\dots, y_{-2}, y_{-1}, y_0, y_1, y_2, \dots$ and q_1, q_2, \dots

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- μ a partition, and the summation carries over all partitions;

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is a family of solutions to the KP hierarchy depending of infinitely many parameters $\dots, y_{-2}, y_{-1}, y_0, y_1, y_2, \dots$ and q_1, q_2, \dots .

- μ a partition, and the summation carries over all partitions;
- s_{μ} is the *Schur polynomial* corresponding to the partition μ ;

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The functions H, B_m, R belong to the Orlov–Shcherbin family.

Parameter values for solutions

The parameters are
for the Hurwitz numbers

$$(q_1, q_2, q_3, \dots) = (1, 0, 0, \dots)$$
$$y_c = e^{uc}, \quad c = \dots, -2, -1, 0, 1, 2, \dots$$

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for the Bousquet-Mélou-Schaeffer numbers

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$$y_c = (1 + uc)^m \quad c = \dots, -2, -1, 0, 1, 2, \dots$$

Theorem (M. Kazarian, S. L. 2015)

All the solutions to KP in the Orlov–Shcherbin family admit a genus expansion.

**Merci beaucoup
pour votre attention**