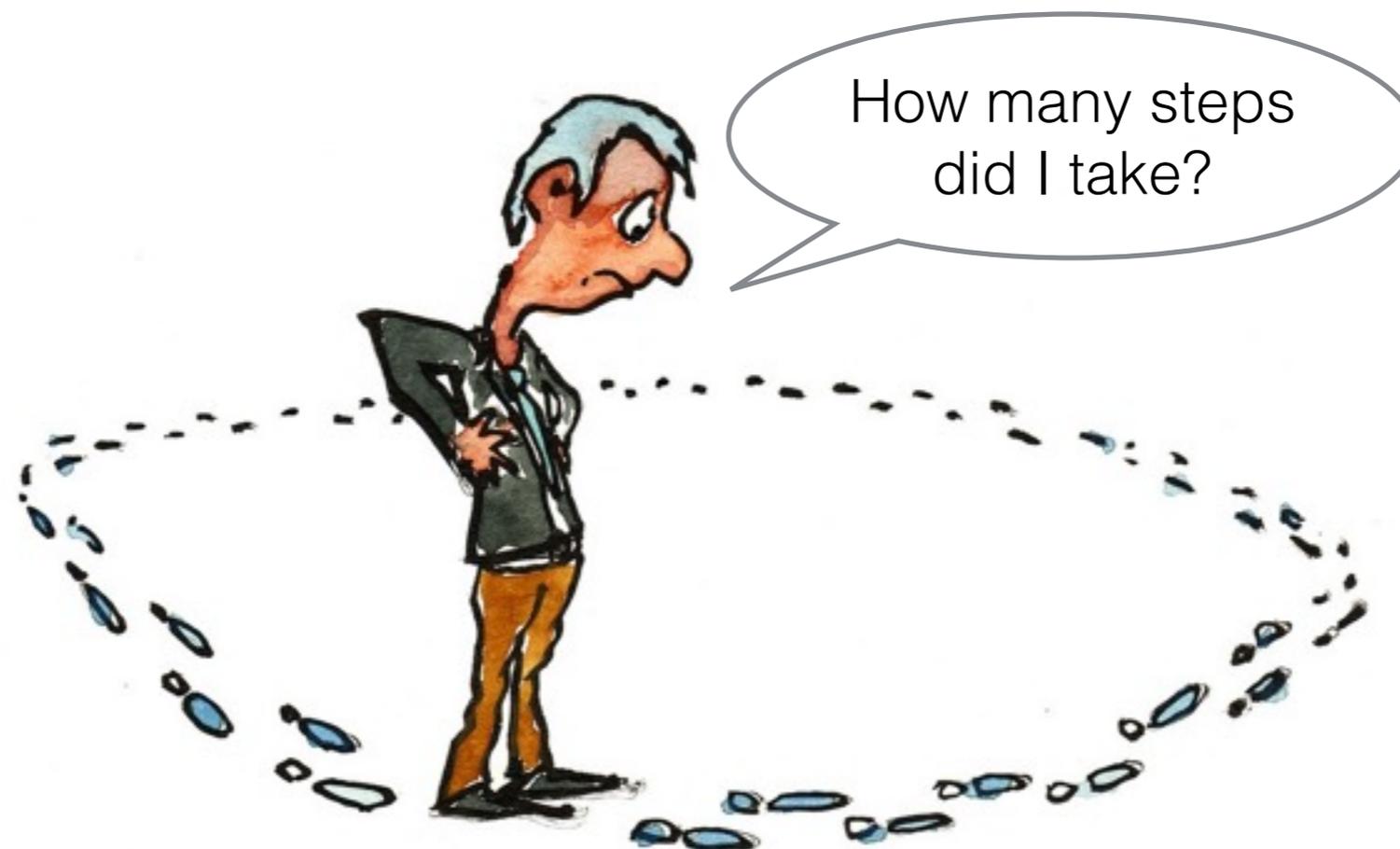


CONNECTING ANALYTIC AND ASYMPTOTIC BEHAVIOUR THROUGH MULTIVARIATE DIAGONALS

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Joint work with Mark Wilson, Marni Mishna,
George Labahn, and Bruno Salvy



PROLOGUE

*A (Brief) Introduction to Analytic
Combinatorics*

Generating Functions

A priori, a generating function is a *formal object* but it *may* also be the expansion of an analytic function at the origin.

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Example

$$P(z) = \sum_{n \geq 0} n! z^n$$

does not converge for any $z \neq 0$.

Example

$$B(z) = \sum_{n \geq 0} 2^n z^n = \frac{1}{1 - 2z}$$

converges for $|z| < 1/2$.

‘Analytic’ Combinatorics

If $S(z) = \sum_{k \geq 0} s_k z^k$ then

$$\begin{aligned} \frac{1}{2\pi i} \int_C S(z) \frac{dz}{z^{n+1}} &= \frac{1}{2\pi i} \int_C \sum_{k \geq 0} s_k z^{k-n-1} dz \\ &= \sum_{k \geq 0} \frac{1}{2\pi i} \int_C s_k z^{k-n-1} dz \\ &= s_n \end{aligned}$$

for a suitable closed curve C around the origin.

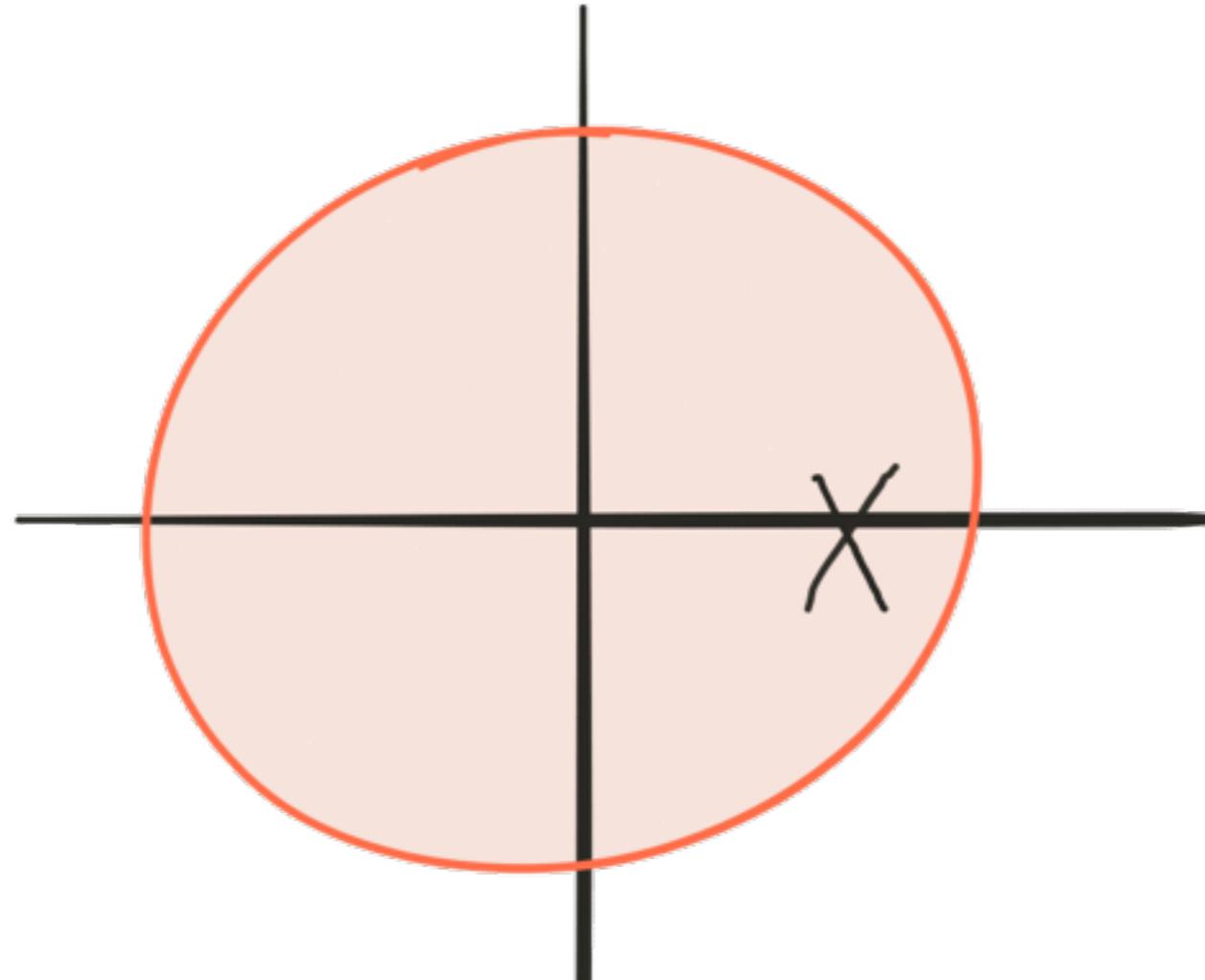
Theorem (Flajolet-Odlyzko / Darboux)

If $f(z)$ is analytic in $|z| \leq 1/R + \epsilon$, except at $z = 1/R$ where

$$f(z) = (1 - Rz)^\alpha + O((1 - Rz)^A) \quad \alpha \notin \{0, 1, \dots\}$$

as $z \rightarrow 1/R$, with $\alpha < A$, then

$$[z^n]f(z) = R^n \cdot \frac{n^{-\alpha-1}}{\Gamma(\alpha)} + O(R^n n^{-A-1})$$



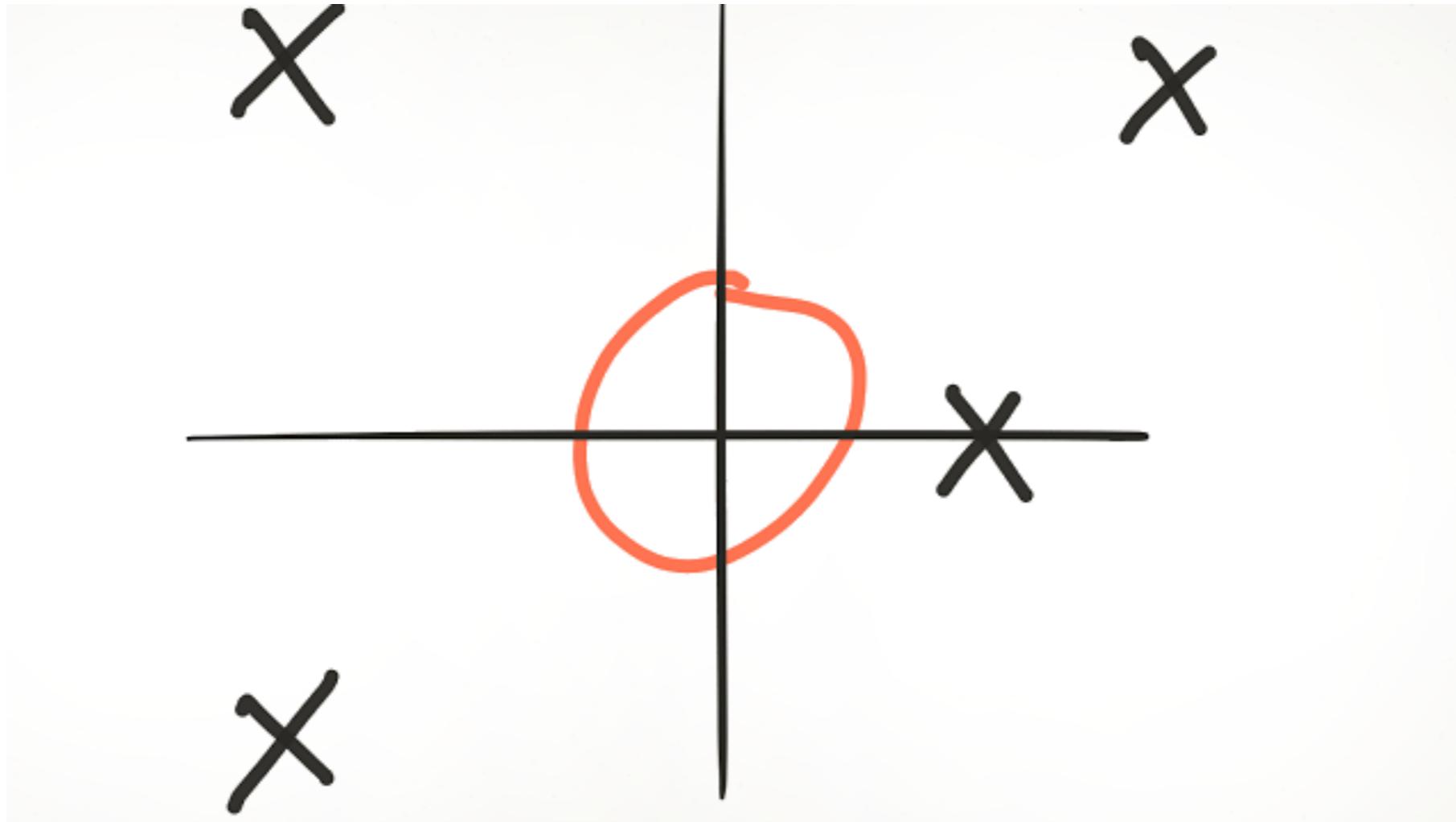
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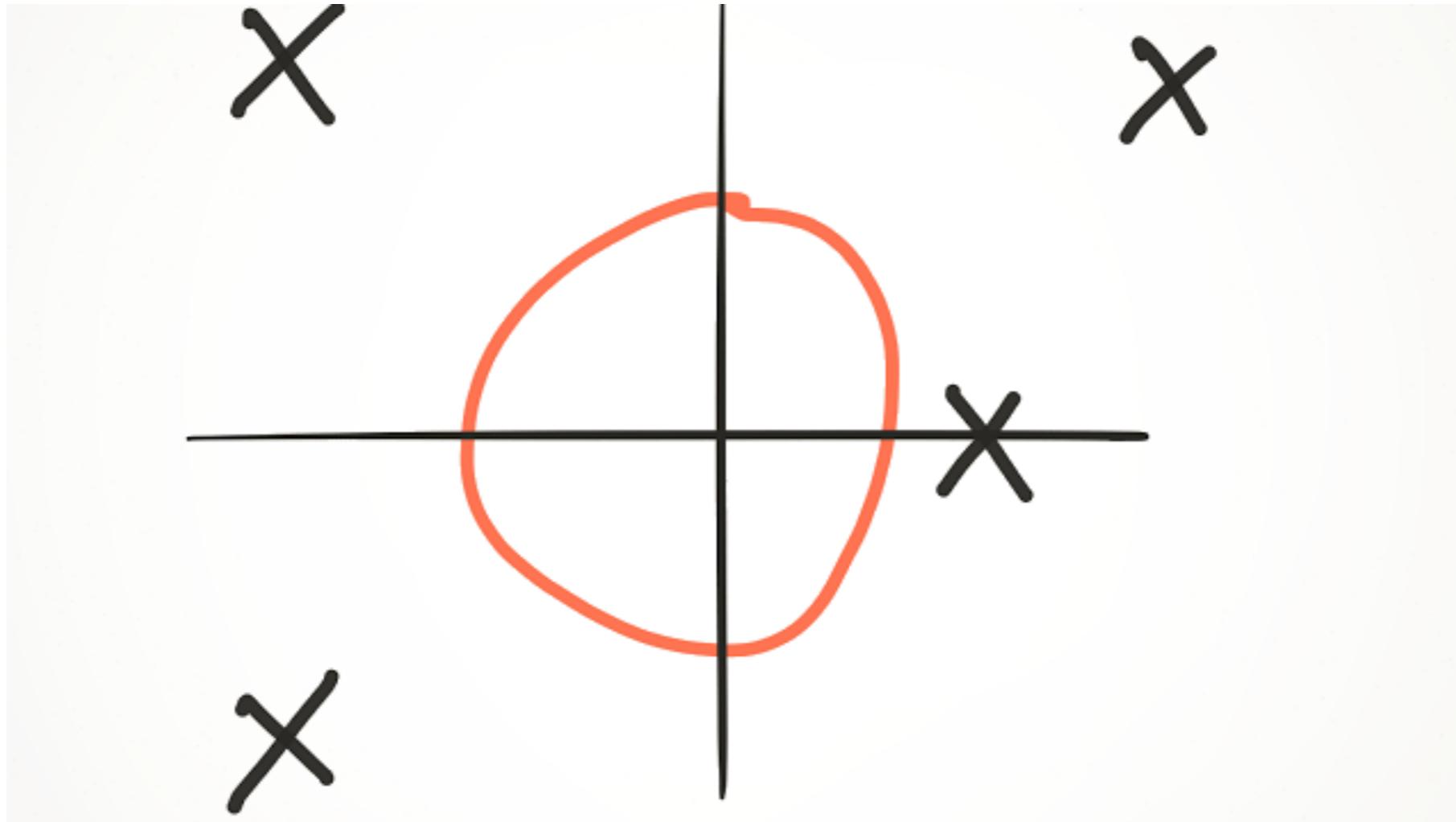
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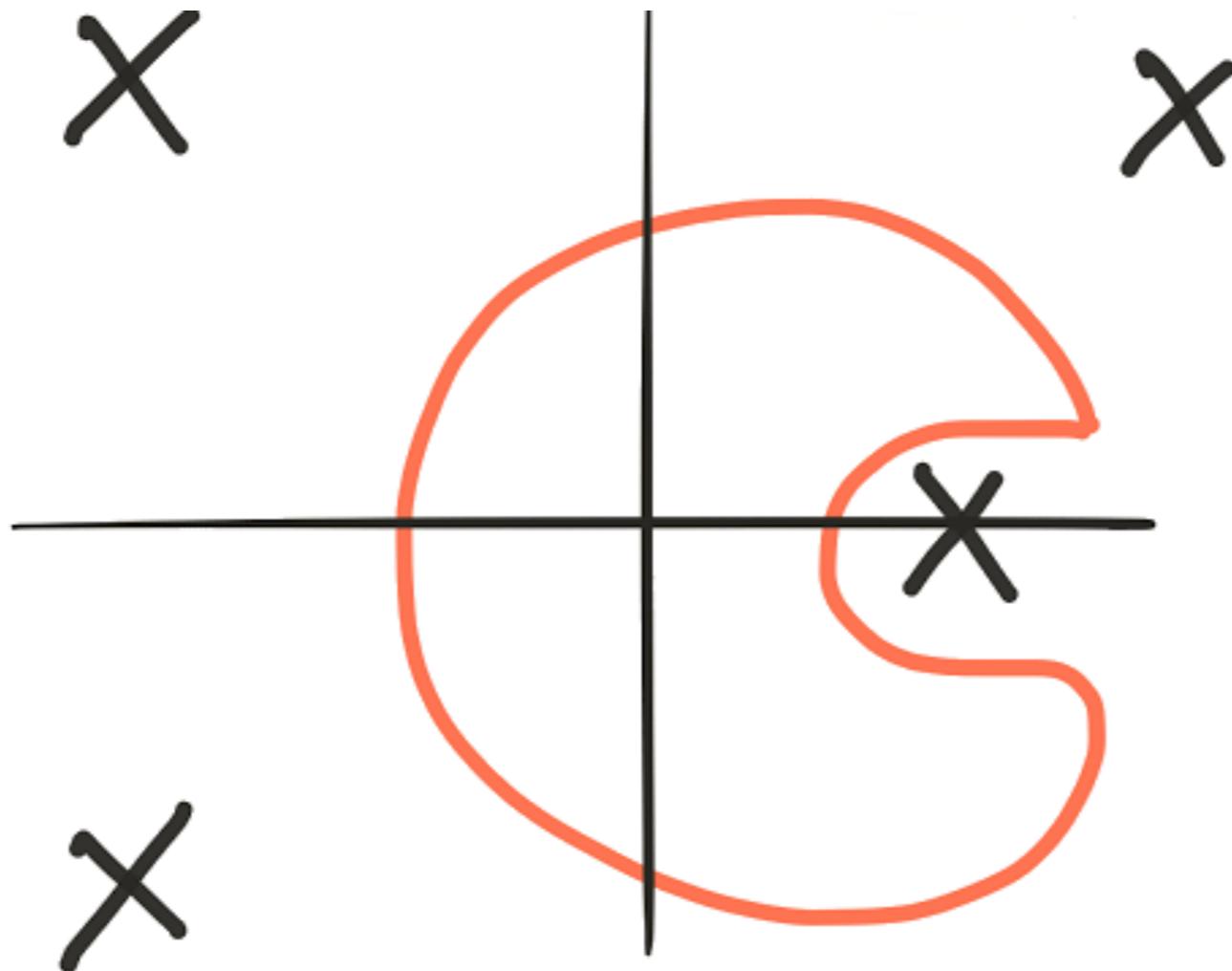
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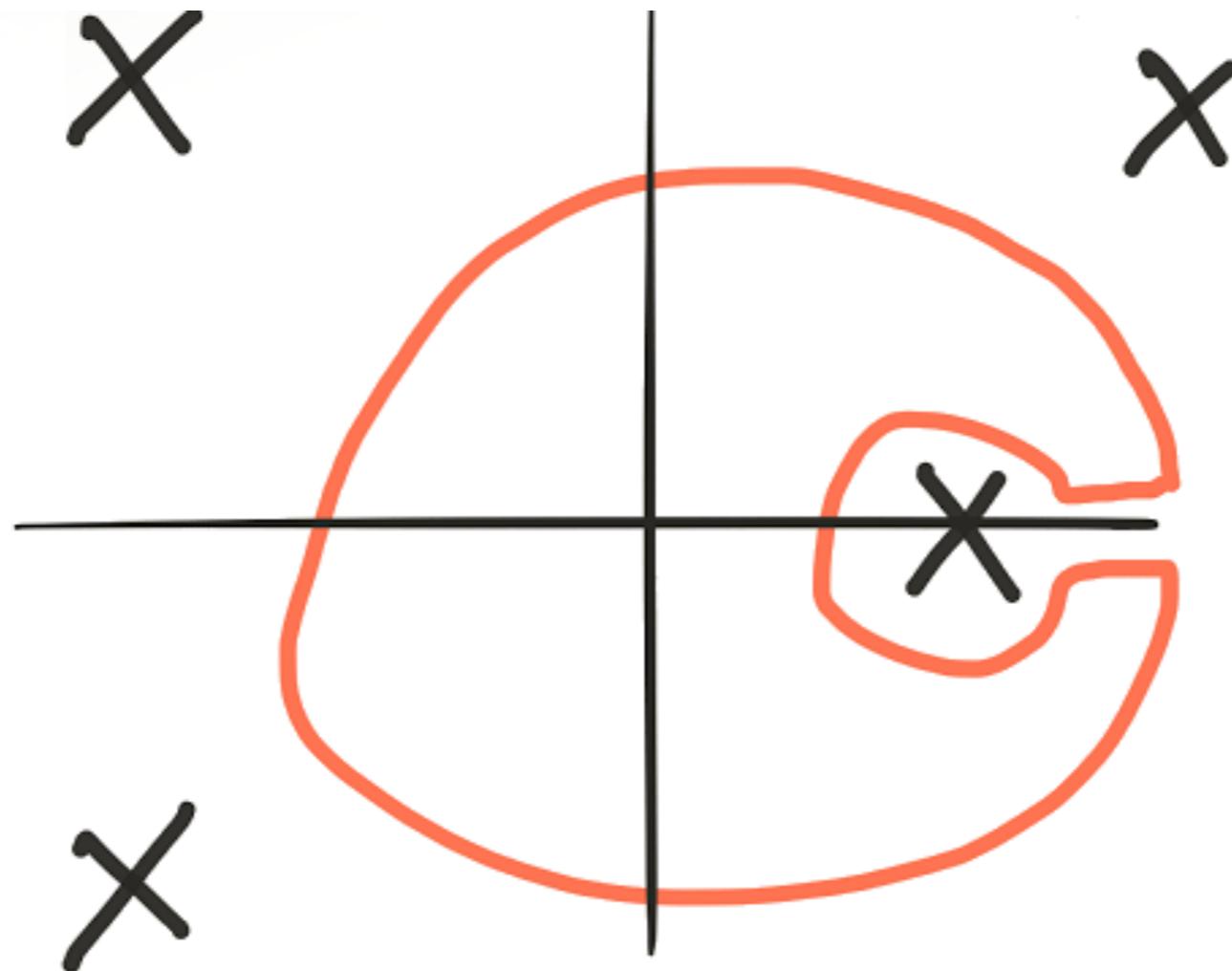
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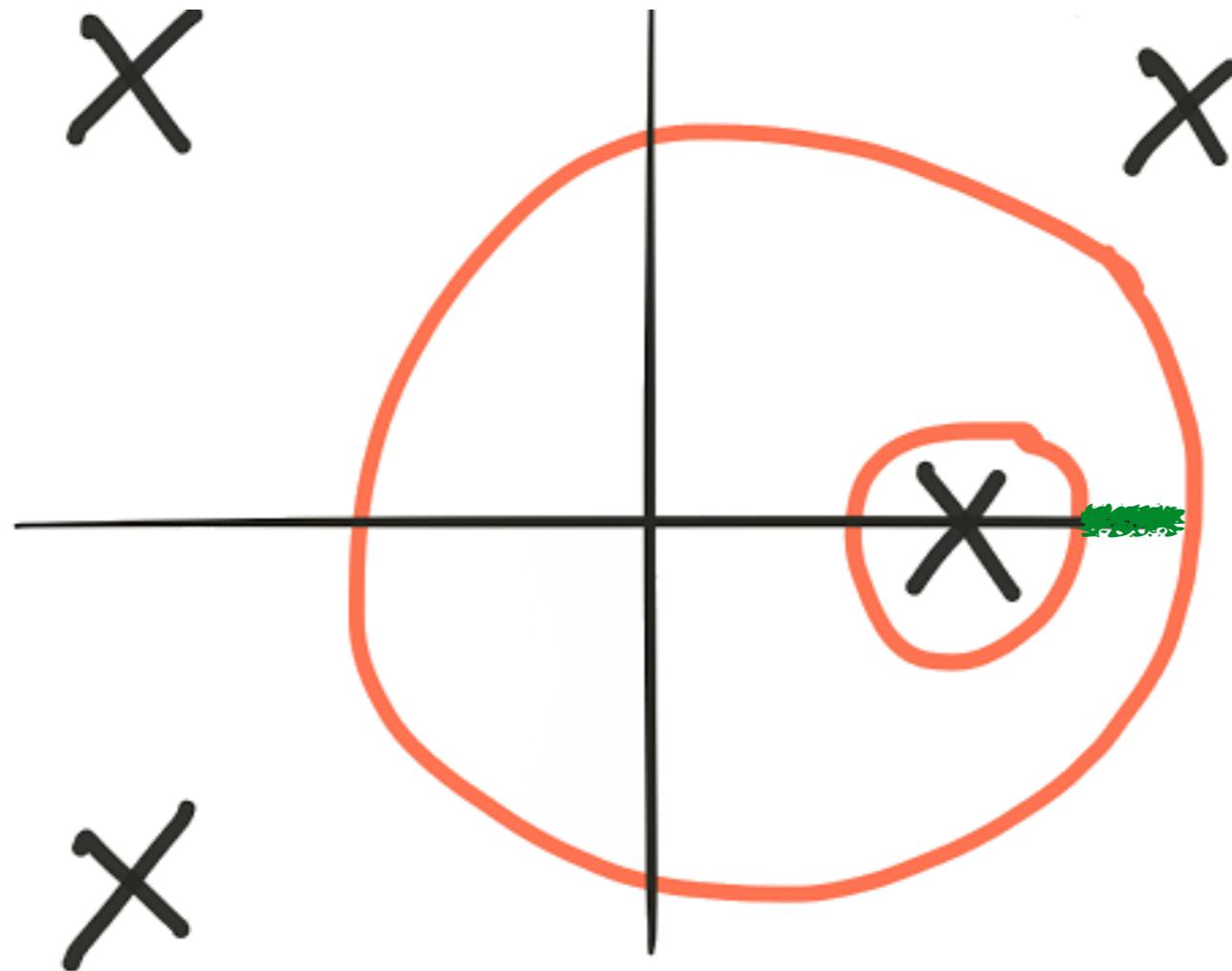
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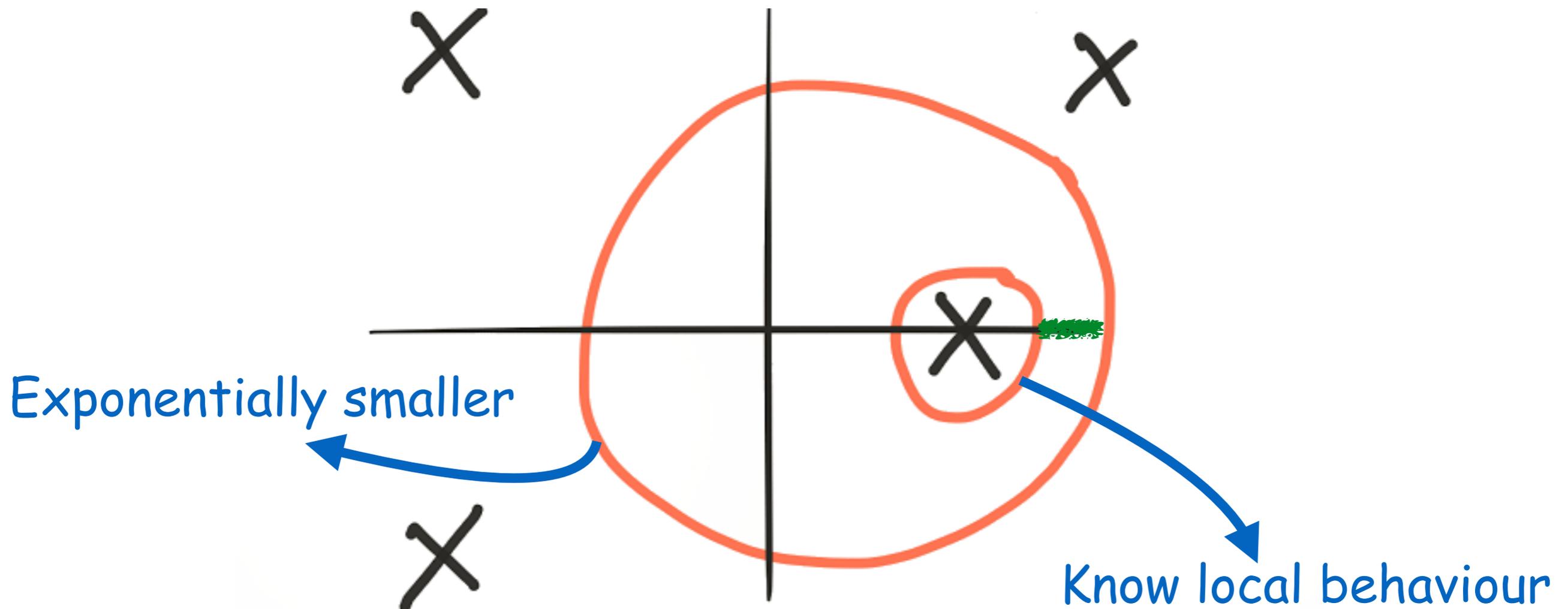
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Univariate AC

Theorem (Pringsheim)

If $F(z) = \sum_{n \geq 0} a_n z^n$ is analytic at the origin with radius of convergence $\rho > 0$ and $a_n \geq 0$ then $z = \rho$ is a singularity with minimal modulus.

Thus, the general method of AC (often) has one:

- (1) Find singularity closest to origin (using Pringsheim)
- (2) Calculate contribution of each singularity with this modulus

ACT 1

A Combinatorial Problem

Lattice Paths in a Quadrant

Given: A finite set of steps / directions $\mathcal{S} \subset \{\pm 1, 0\}^2 \setminus \{(0, 0)\}$

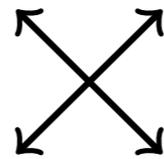
Count: $c_n = \#$ of walks staying in \mathbb{N}^2 , starting at the origin

Lattice Paths in a Quadrant

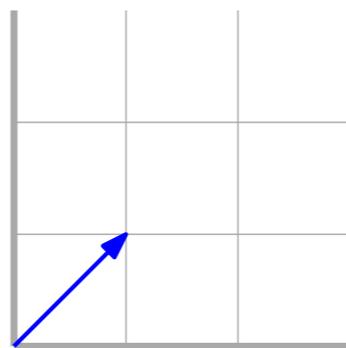
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For instance, given $\mathcal{S} = \{(1, -1), (1, 1), (-1, 1), (-1, -1)\} \subset \mathbb{Z}^2$



we have:



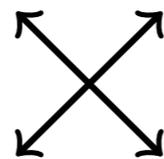
$$c_1 = 1$$

Lattice Paths in a Quadrant

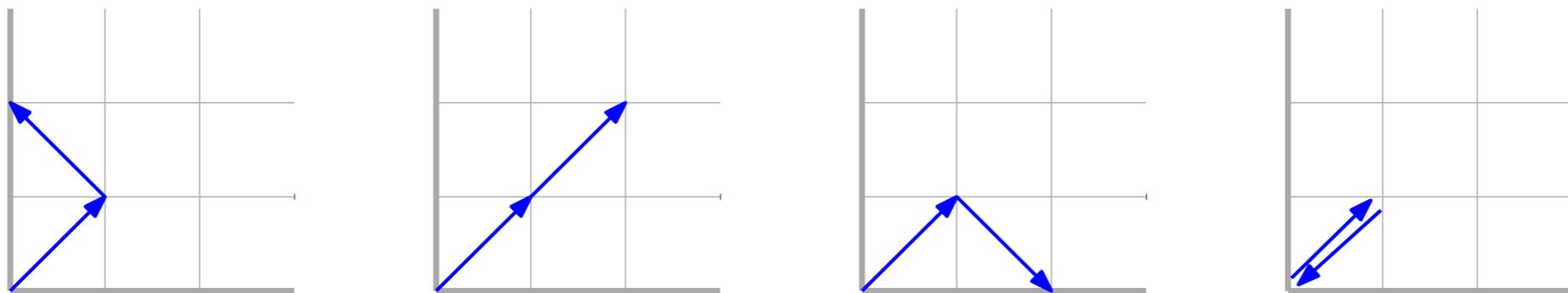
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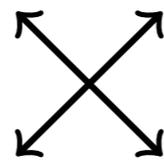
$$c_2 = 4$$

Lattice Paths in a Quadrant

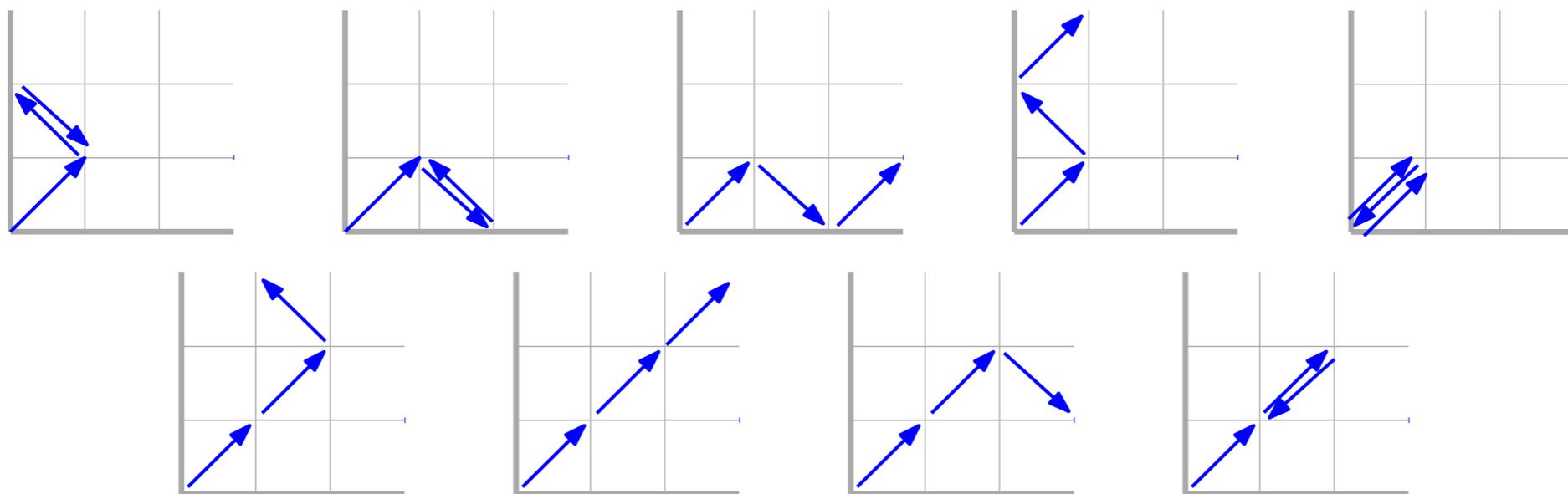
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Count: $c_n = \#$ of walks staying in \mathbb{N}^2 , starting at the origin

For instance, given $\mathcal{S} = \{(1, -1), (1, 1), (-1, 1), (-1, -1)\} \subset \mathbb{Z}^2$



we have:



$$c_3 = 9$$

Importance

Studied as far back as 18th and 19th centuries.

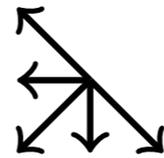
Applications to:

- Statistical mechanics (polymers in solution, Ising model, ...)
- Queueing theory / operations research
- Discrete Structures (trees, words, plane partitions, ...)
- Probability Theory (random walks, branching processes, ...)

Useful sandbox for developing methods for generating functions

A Reduction

A priori, there are $2^8 - 1 = 255$ different models. Some, like



are trivial. Some, like

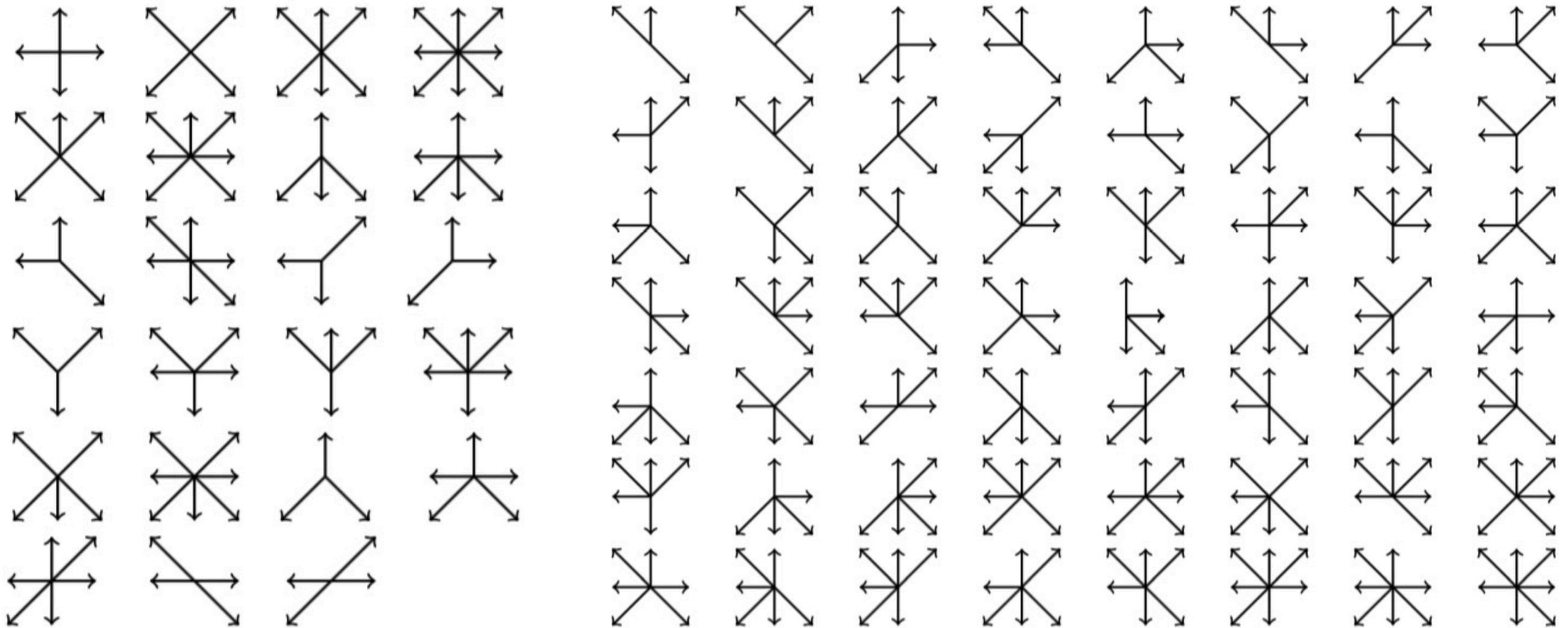


live in a half-space and can be solved by methods of Banderier and Flajolet (2001).

Finally, some are equivalent.



A Reduction



Bousquet-Mélou and Mishna (2010) showed there are **79** non-isomorphic two dimensional models.

The Kernel Method

Bousquet-Mélou and Mishna (2010) were able to show that many of these walks have a *D-finite* generating function.

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Let

$$C(x, y, t) = \sum_{i, j, n \geq 0} c_{i, j, n} x^i y^j t^n$$

where $c_{i, j, n}$ is the number of walks of length n staying in the quarter plane and ending at the point (i, j) .

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where $c_{i, j, n}$ is the number of walks of length n staying in the quarter plane and ending at the point (i, j) .

Then

$$C(1, 1, t) = \text{GF of all walks}$$

$$C(1, 0, t) = \text{GF of walks ending at } y = 0$$

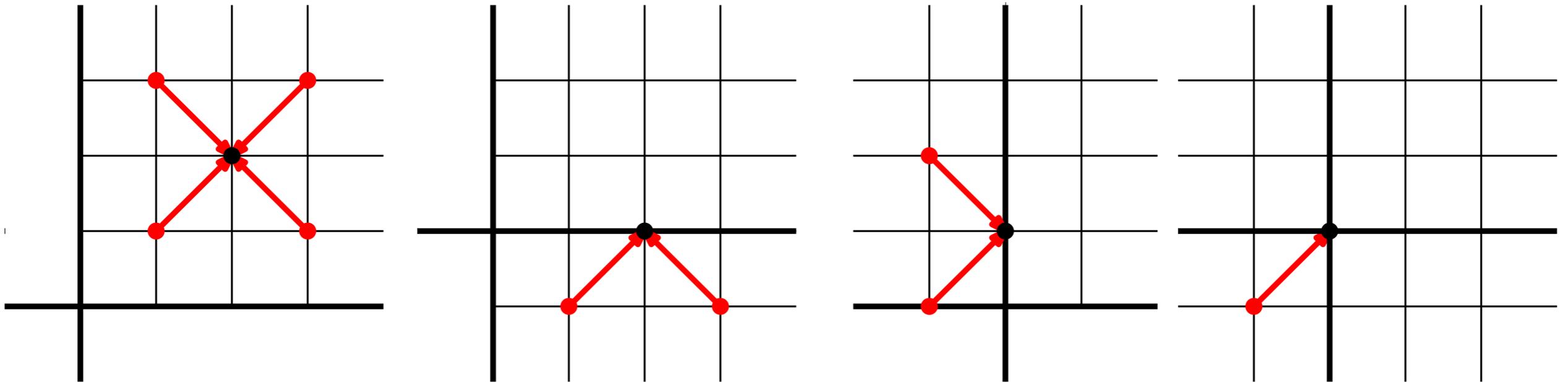
$$C(0, 1, t) = \text{GF of walks ending at } x = 0$$

$$C(0, 0, t) = \text{GF of walks ending at } (0, 0)$$

The Kernel Method

The recursive nature of a walk of length n ending at (i, j) implies that $C(x, y, t)$ satisfies a functional equation of the form

$$K(x, y, t) \cdot C(x, y, t) = 1 + A(x) \cdot C(x, 0, t) + B(y) \cdot C(0, y, t)$$



The Kernel Method

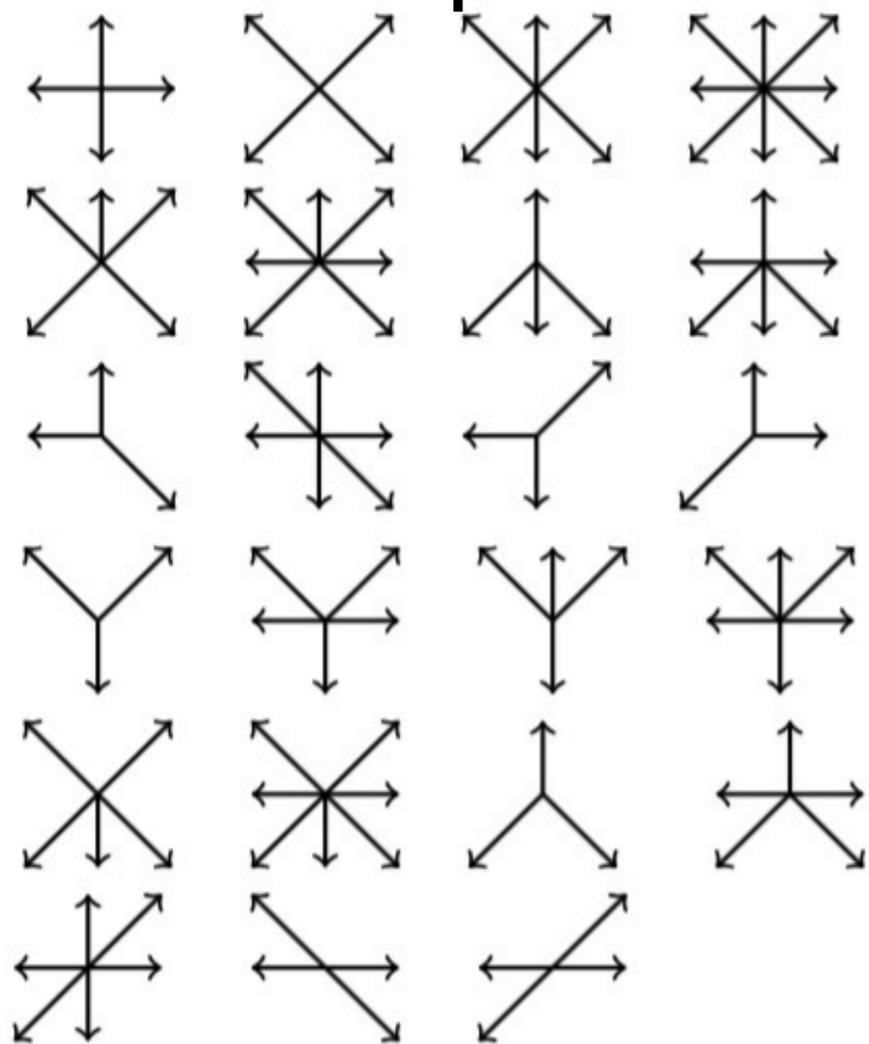
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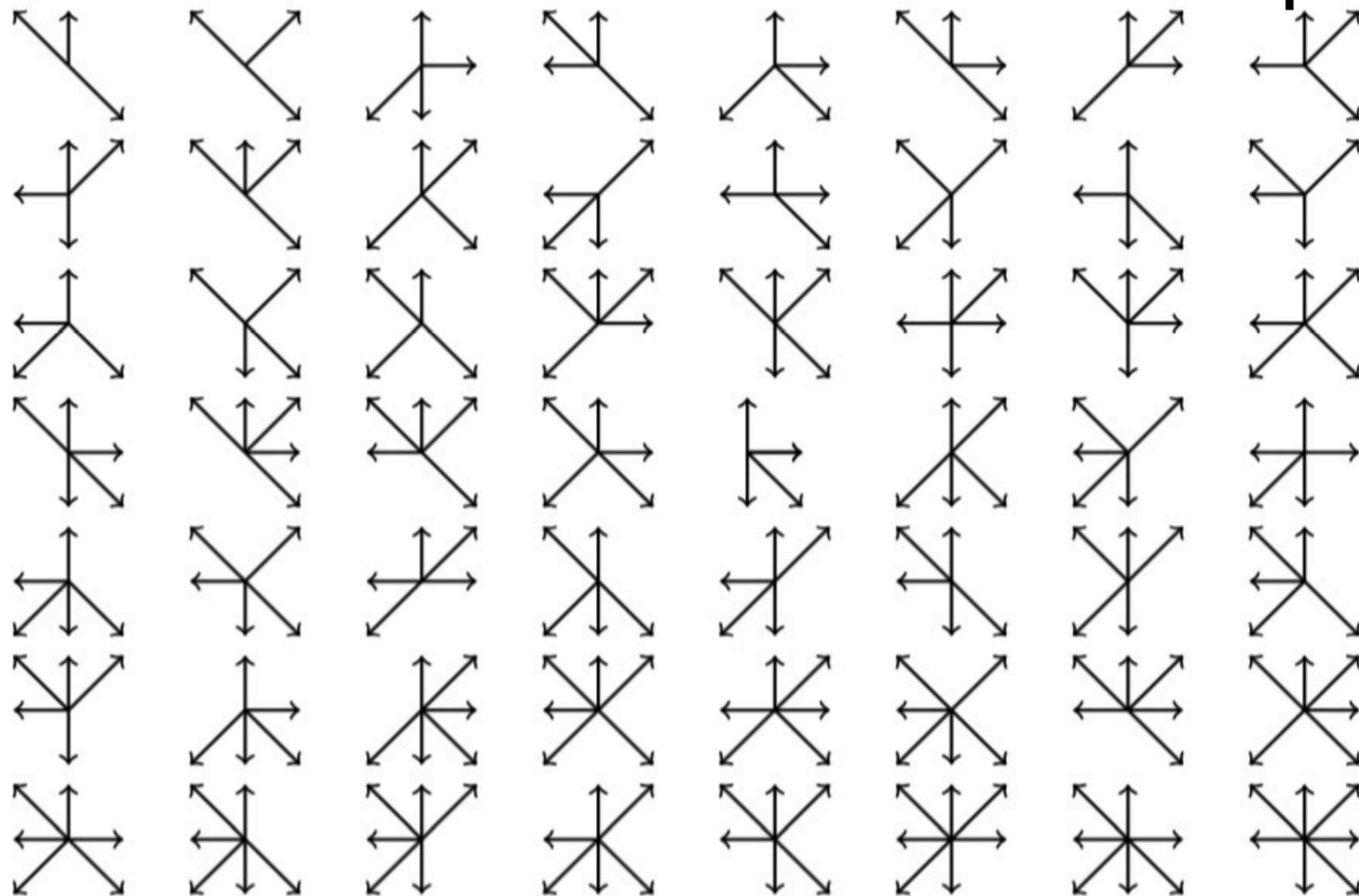
Bousquet-Mélou and Mishna use a group \mathcal{G} of bi-rational transformations of the plane associated to this model.

When the group is finite it can usually be combined with the functional equation to give a nice representation of $C(1, 1, t)$.

Finite Group

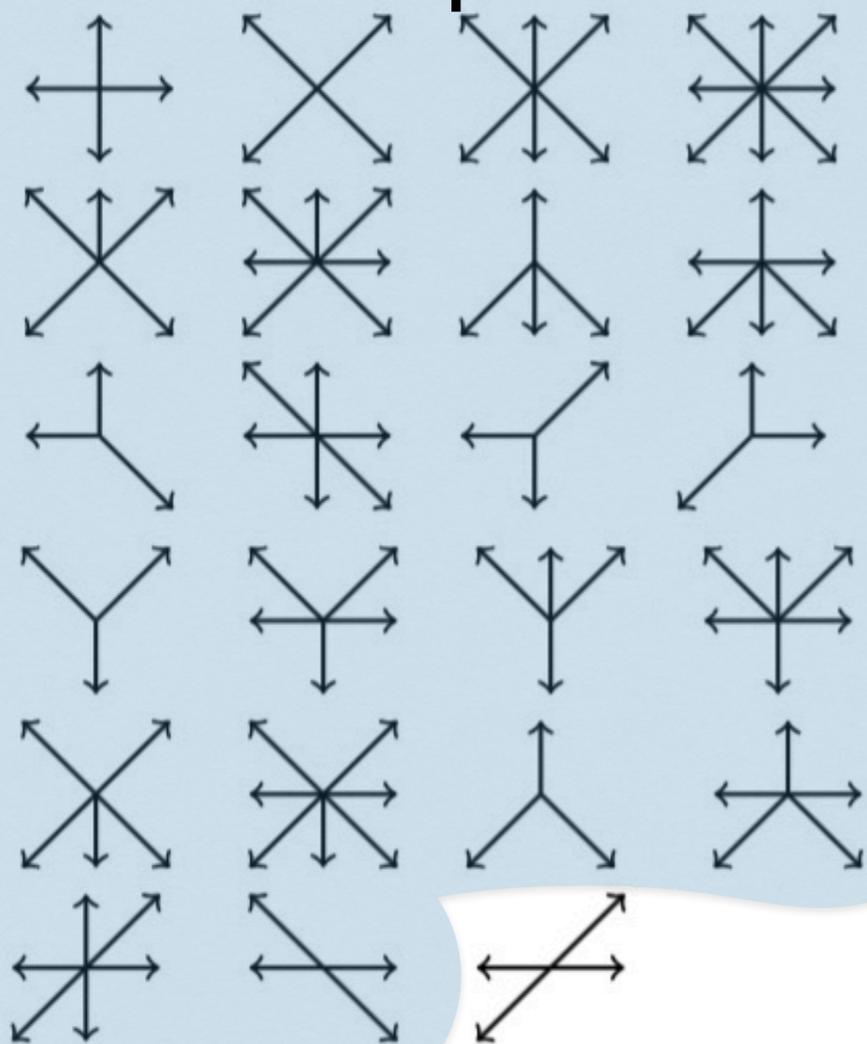


Infinite Group

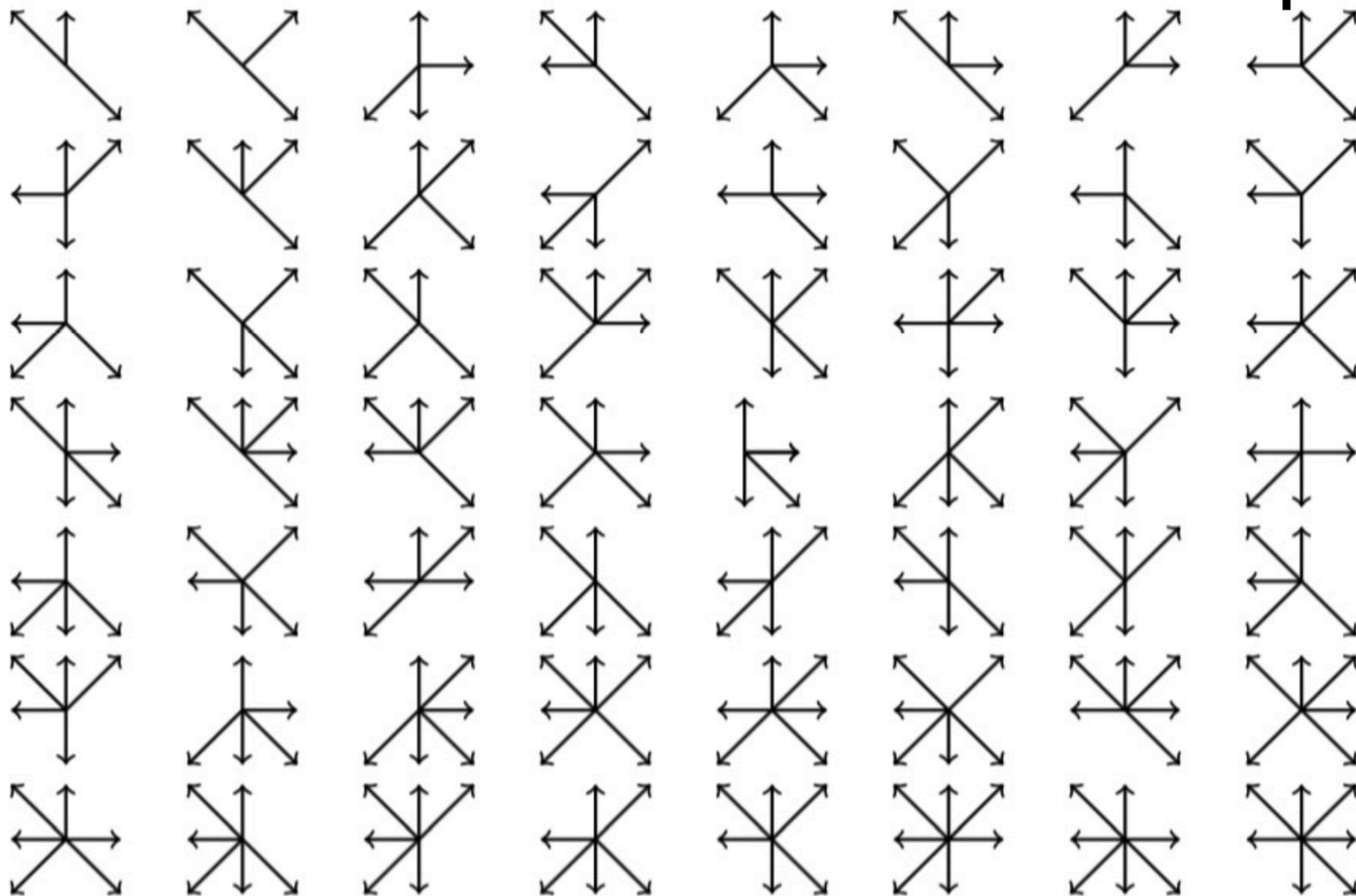


$C(x, y, t)$ is D-Finite (Bousquet-Mélou & Mishna 2010)

Finite Group

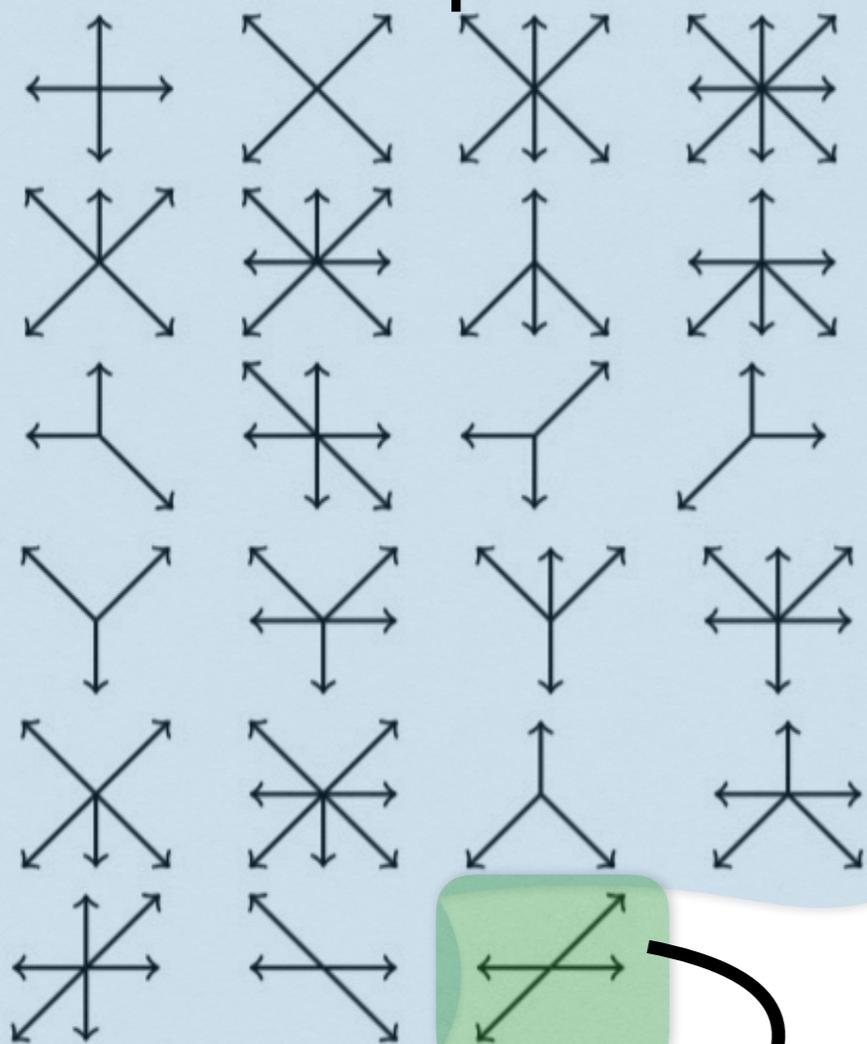


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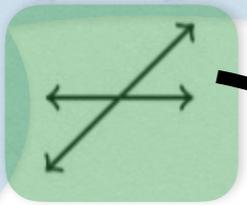
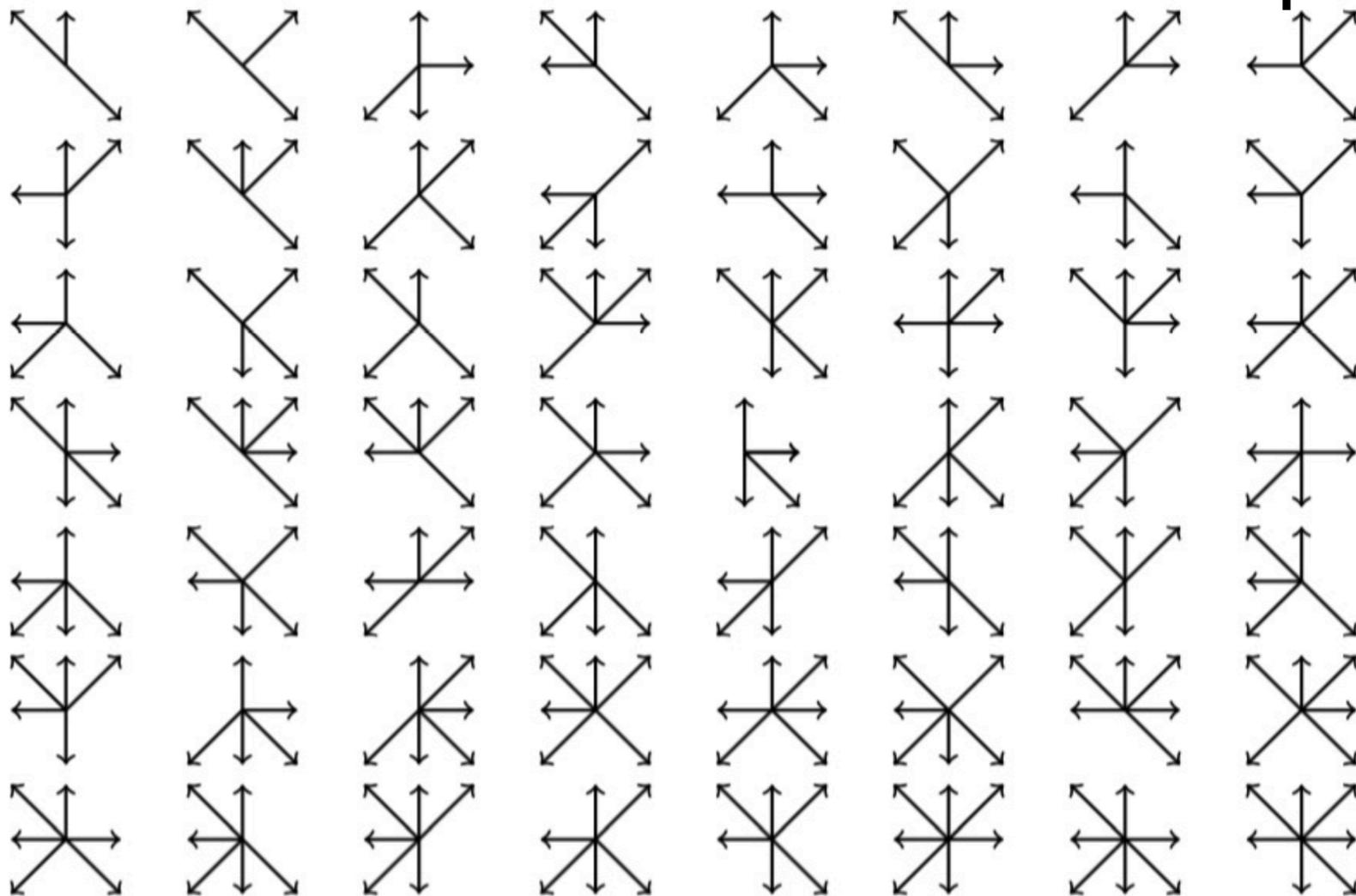


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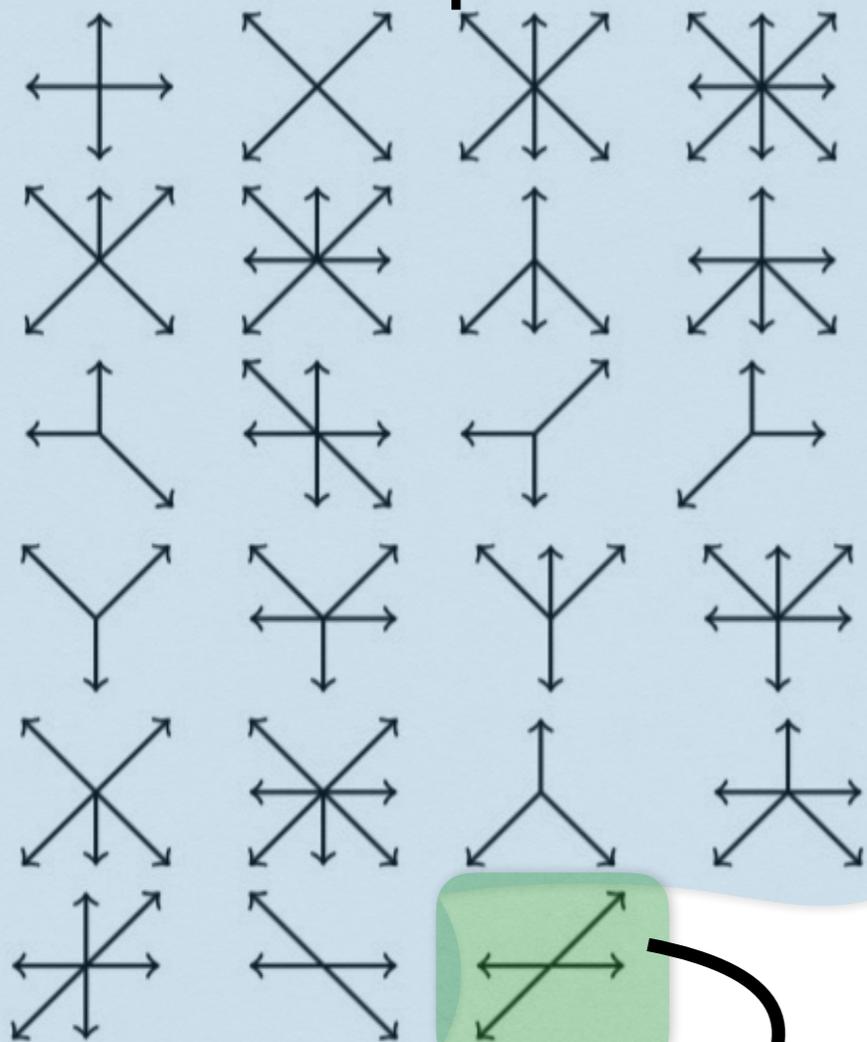
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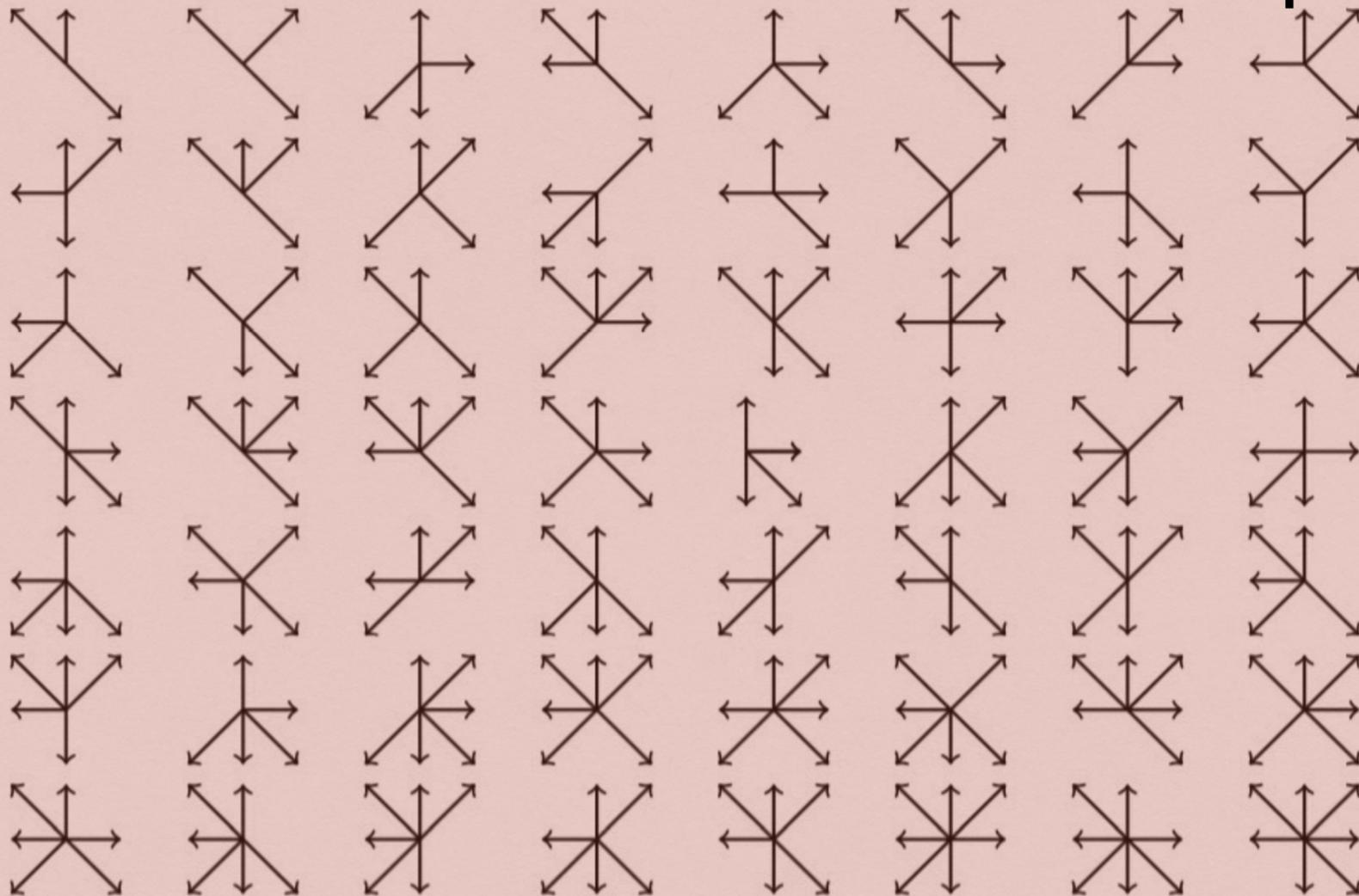
$C(x, y, t)$ is algebraic (Bostan & Kauers 2010)

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Finite Group



Infinite Group



$C(x, y, t)$ is algebraic (Bostan & Kauers 2010)

$C(x, y, t)$ is not D-Finite (M. & Mishna 2014)
(Bostan, Raschel, Salvy 2014)

Bostan and Kauers (2009) Guessed Asymptotics

n	\mathcal{S}	Asymptotics	n	\mathcal{S}	Asymptotics	n	\mathcal{S}	Asymptotics
1		$\frac{4}{\pi} \cdot \frac{4^n}{n}$	9		$\frac{\sqrt{3}}{2\sqrt{\pi}} \cdot \frac{3^n}{\sqrt{n}}$	17		$\frac{4 \cdot A_n}{\pi} \cdot \frac{(2\sqrt{2})^n}{n^2}$
2		$\frac{2}{\pi} \cdot \frac{4^n}{n}$	10		$\frac{4}{3\sqrt{\pi}} \cdot \frac{4^n}{\sqrt{n}}$	18		$\frac{3\sqrt{3} \cdot B_n}{\pi} \cdot \frac{(2\sqrt{3})^n}{n^2}$
3		$\frac{\sqrt{6}}{\pi} \cdot \frac{6^n}{n}$	11		$\frac{\sqrt{5}}{2\sqrt{2}\pi} \cdot \frac{5^n}{\sqrt{n}}$	19		$\frac{\sqrt{8}(1+\sqrt{2})^{7/2}}{\pi} \cdot \frac{(2+2\sqrt{2})^n}{n^2}$
4		$\frac{8}{3\pi} \cdot \frac{8^n}{n}$	12		$\frac{\sqrt{5}}{3\sqrt{2}\pi} \cdot \frac{5^n}{\sqrt{n}}$	20		$\frac{6C_n}{\pi} \cdot \frac{(2\sqrt{6})^n}{n^2}$
5		$\frac{2\sqrt{2}}{\Gamma(1/4)} \cdot \frac{3^n}{n^{3/4}}$	13		$\frac{2\sqrt{3}}{3\sqrt{\pi}} \cdot \frac{6^n}{\sqrt{n}}$	21		$\frac{\sqrt{3}(1+\sqrt{3})^{7/2}}{2\pi} \cdot \frac{(2+2\sqrt{3})^n}{n^2}$
6		$\frac{3\sqrt{3}}{\sqrt{2}\Gamma(1/4)} \cdot \frac{3^n}{n^{3/4}}$	14		$\frac{\sqrt{7}}{3\sqrt{3}\pi} \cdot \frac{7^n}{\sqrt{n}}$	22		$\frac{\sqrt{6(379+156\sqrt{6})(1+\sqrt{6})^7}}{5\sqrt{95}\pi} \cdot \frac{(2+2\sqrt{6})^n}{n^2}$
7		$\frac{\sqrt{6\sqrt{3}}}{\Gamma(1/4)} \cdot \frac{6^n}{n^{3/4}}$	15		$\frac{3\sqrt{3}}{2\sqrt{\pi}} \cdot \frac{3^n}{n^{3/2}}$	23		$\frac{8}{\pi} \cdot \frac{4^n}{n^2}$
8		$\frac{4\sqrt{3}}{3\Gamma(1/3)} \cdot \frac{4^n}{n^{2/3}}$	16		$\frac{3\sqrt{3}}{2\sqrt{\pi}} \cdot \frac{6^n}{n^{3/2}}$			

Tab. 1: Asymptotics for the 23 D-finite models.

$$A_n = 4(1 - (-1)^n) + 3\sqrt{2}(1 + (-1)^n), \quad B_n = \sqrt{3}(1 - (-1)^n) + 2(1 + (-1)^n), \quad C_n = 12/\sqrt{5}(1 - (-1)^n) + \sqrt{30}(1 + (-1)^n)$$

Bostan and Kauers (2009) Guessed Asymptotics

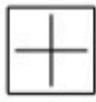
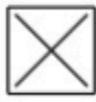
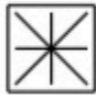
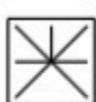
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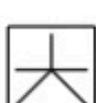
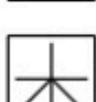
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$C(1, 1, t)$ is algebraic and has these asymptotics (Bousquet-Mélou Mishna & Bostan Kauers)

Bostan, Chyzak, van Hoeij, Kauers, Pech (2015) Guessed Asymptotics

\mathcal{S}	$C(0, 1, t)$	$C(1, 0, t)$	$C(0, 0, t)$	\mathcal{S}	$C(0, 1, t)$	$C(1, 0, t)$	$C(0, 0, t)$
	$\frac{8}{\pi} \cdot \frac{4^n}{n^2}$	$\frac{8}{\pi} \cdot \frac{4^n}{n^2}$	$\delta_n \frac{32}{\pi} \cdot \frac{4^n}{n^3}$		$\delta_n \frac{4}{\pi} \cdot \frac{4^n}{n^2}$	$\delta_n \frac{4}{\pi} \cdot \frac{4^n}{n^2}$	$\delta_n \frac{8}{\pi} \cdot \frac{4^n}{n^3}$
	$\frac{3\sqrt{6}}{2\pi} \cdot \frac{6^n}{n^2}$	$\delta_n \frac{2\sqrt{6}}{\pi} \cdot \frac{6^n}{n^2}$	$\delta_n \frac{3\sqrt{6}}{\pi} \cdot \frac{6^n}{n^3}$		$\frac{32}{9\pi} \cdot \frac{8^n}{n^2}$	$\frac{32}{9\pi} \cdot \frac{8^n}{n^2}$	$\frac{128}{27\pi} \cdot \frac{8^n}{n^3}$
	$\frac{3\sqrt{3}}{4\sqrt{\pi}} \frac{3^n}{n^{3/2}}$	$\delta_n \frac{4\sqrt{2}}{\pi} \frac{(2\sqrt{2})^n}{n^2}$	$\epsilon_n \frac{16\sqrt{2}}{\pi} \frac{(2\sqrt{2})^n}{n^3}$		$\frac{8}{3\sqrt{\pi}} \frac{4^n}{n^{3/2}}$	$\delta_n \frac{4\sqrt{3}}{\pi} \frac{(2\sqrt{3})^n}{n^2}$	$\delta_n \frac{12\sqrt{3}}{\pi} \frac{(2\sqrt{3})^n}{n^3}$
	$\frac{5\sqrt{10}}{16\sqrt{\pi}} \frac{5^n}{n^{3/2}}$	$\frac{\sqrt{2}A^{3/2}}{\pi} \frac{(2A)^n}{n^2}$	$\frac{2A^{3/2}}{\pi} \frac{(2A)^n}{n^3}$		$\frac{5\sqrt{10}}{24\sqrt{\pi}} \frac{5^n}{n^{3/2}}$	$\delta_n \frac{4\sqrt{30}}{5\pi} \frac{(2\sqrt{6})^n}{n^2}$	$\delta_n \frac{24\sqrt{30}}{25\pi} \frac{(2\sqrt{6})^n}{n^3}$
	$\frac{\sqrt{3}}{\sqrt{\pi}} \frac{6^n}{n^{3/2}}$	$\frac{2\sqrt{3}B^{3/2}}{3\pi} \frac{(2B)^n}{n^2}$	$\frac{2B^{3/2}}{\pi} \frac{(2B)^n}{n^3}$		$\frac{7\sqrt{21}}{54\sqrt{\pi}} \frac{7^n}{n^{3/2}}$	$\frac{D}{285\pi} \frac{(2C)^n}{n^2}$	$\frac{2E}{1805\pi} \frac{(2C)^n}{n^3}$
	$\frac{27\sqrt{3}}{8\sqrt{\pi}} \cdot \frac{3^n}{n^{5/2}}$	$\frac{27\sqrt{3}}{8\sqrt{\pi}} \cdot \frac{3^n}{n^{5/2}}$	$\sigma_n \frac{81\sqrt{3}}{\pi} \cdot \frac{3^n}{n^4}$		$\delta_n \frac{32}{\pi} \cdot \frac{4^n}{n^3}$	$\frac{32}{\pi} \cdot \frac{4^n}{n^3}$	$\delta_n \frac{768}{\pi} \cdot \frac{4^n}{n^5}$
	$\frac{27\sqrt{3}}{8\sqrt{\pi}} \cdot \frac{6^n}{n^{5/2}}$	$\frac{27\sqrt{3}}{8\sqrt{\pi}} \cdot \frac{6^n}{n^{5/2}}$	$\frac{27\sqrt{3}}{\pi} \cdot \frac{6^n}{n^4}$				

\mathcal{S}	$C(0, 1, t)$	\mathcal{S}	$C(0, 1, t)$
	$\left(\epsilon_n \frac{448\sqrt{2}}{9\pi} + \epsilon_{n-1} \frac{640}{9\pi} + \epsilon_{n-2} \frac{416\sqrt{2}}{9\pi} + \epsilon_{n-3} \frac{512}{9\pi} \right) \cdot \frac{(2\sqrt{2})^n}{n^3}$		$\left(\delta_n \frac{36\sqrt{3}}{\pi} + \delta_{n-1} \frac{54}{\pi} \cdot \frac{(2\sqrt{3})^n}{n^3} \right) \cdot \frac{(2\sqrt{3})^n}{n^3}$
	$\frac{4A^{7/2}}{\pi} \cdot \frac{(2A)^n}{n^3}$		$\left(\delta_n \frac{72\sqrt{30}}{5\pi} + \delta_{n-1} \frac{864\sqrt{5}}{25\pi} \right) \cdot \frac{(2\sqrt{6})^n}{n^3}$
	$\frac{3B^{7/2}}{2\pi} \cdot \frac{(2B)^n}{n^3}$		$\frac{6(4571+1856\sqrt{6})\sqrt{23-3\sqrt{6}}}{1805\pi} \cdot \frac{(2C)^n}{n^3}$

$$A = 1 + \sqrt{2}, \quad B = 1 + \sqrt{3}, \quad C = 1 + \sqrt{6}, \quad D = (156 + 41\sqrt{6})\sqrt{23 - 3\sqrt{6}}, \quad E = (583 + 138\sqrt{6})\sqrt{23 - 3\sqrt{6}}$$

$\delta_n = 1$ if $n \equiv 0 \pmod{2}$, $\sigma_n = 1$ if $n \equiv 0 \pmod{3}$, and $\epsilon_n = 1$ if $n \equiv 0 \pmod{4}$ – each is 0 otherwise

Diagonals of Rational Functions

For an element

$$A(x, y, t) = \sum_{n \geq 0} \left(\sum_{i, j} a_{i, j, n} x^i y^j \right) t^n \in \mathbb{Q}[x, \bar{x}, y, \bar{y}][[t]]$$

define the diagonal operator

$$\Delta A(x, y, t) := \sum_{n \geq 0} a_{n, n, n} t^n \in \mathbb{Q}[[t]]$$

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define the diagonal operator

$$\Delta A(x, y, t) := \sum_{n \geq 0} a_{n, n, n} t^n \in \mathbb{Q}[[t]]$$

For a step set \mathcal{S} , define the *characteristic* polynomial

$$S(x, y) = \sum_{(i, j) \in \mathcal{S}} x^i y^j$$

Example

If $\mathcal{S} = \{(\pm 1, 0), (0, \pm 1)\}$ then $S(x, y) = \bar{x} + x + \bar{y} + y$.

Diagonals of Rational Functions

Theorem

For each of the 23 D-finite models, except the algebraic models 5 — 8, there exists an explicit rational function $B_S(x, y)$ such that for $a, b \in \{0, 1\}$

$$C(a, b, t) = \Delta \left(\frac{B_S(x, y)}{(1-x)^a (1-y)^b (1-txyS(\bar{x}, \bar{y}))} \right)$$

Diagonals of Rational Functions

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For each of the 23 D-finite models, except the algebraic models 5 — 8, there exists an explicit rational function $B_{\mathcal{S}}(x, y)$ such that for $a, b \in \{0, 1\}$

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Example

When $\mathcal{S} = \{(\pm 1, 0), (0, \pm 1)\}$ then $B_{\mathcal{S}} = (1-x^2)(1-y^2)$ so

$$C(1, 1, t) = \Delta \left(\frac{(1+x)(1+y)}{1-txy(\bar{x}+x+\bar{y}+y)} \right)$$

$$C(0, 0, t) = \Delta \left(\frac{(1-x^2)(1-y^2)}{1-txy(\bar{x}+x+\bar{y}+y)} \right)$$

Asymptotic Analysis

Can we link the analytic properties of these rational functions to the asymptotics of our objects?

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Theorem (M. and Wilson 2015+)

All of the guessed asymptotics on previous slides are true.

Furthermore, we can “read off” certain asymptotic properties from these diagonal equations.

Asymp

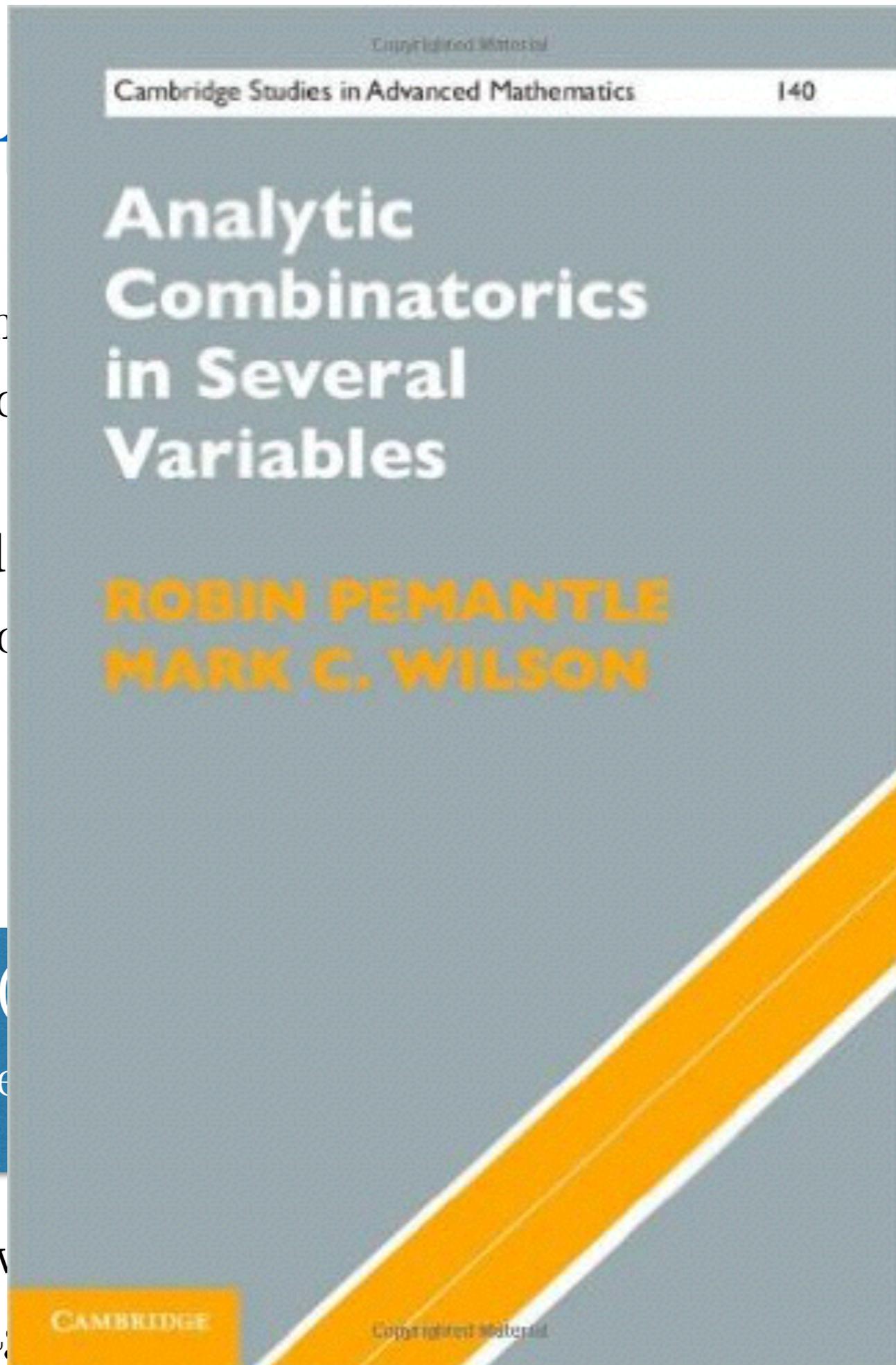
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Theorem

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properties

ACT 11

Analytic Combinatorics in Several Variables

Theorem (CIF In Several Variables)

Let $f(x, y, t)$ be holomorphic on an open neighbourhood of 0 .

Then there is a unique series $F(x, y, t) = \sum a_{i,j,n} x^i y^j z^n$ converging to f near the origin whose coefficients satisfy

$$c_n := a_{n,n,n} = \left(\frac{1}{2\pi i} \right)^3 \int_T \frac{f(x, y, t)}{(xyt)^n} \frac{dx dy dt}{xyt}$$

where T is a polydisk sufficiently close to the origin.

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Idea: Like the univariate case, find points where local behaviour determines asymptotics.

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Idea: Like the univariate case, find points where local behaviour determines asymptotics.

In general this is very very hard!

Minimal Points

Let \mathcal{D} be the open domain of convergence of the power series.
Points $\rho \in \partial\mathcal{D}$ are known as *minimal points*.

Minimal points give an ‘easy’ upper bound on asymptotics:

$$\begin{aligned} \sum_{i,j,n \geq 0} f_{i,j,n} x^i y^j t^n \text{ converges} &\implies |f_{i,j,n}| \cdot |x^i y^j t^n| \rightarrow 0 \\ &\implies |f_{n,n,n}| \leq |xyt|^{-n} \end{aligned}$$

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The Singular Variety

Assume that $f(x, y, t) = G(x, y, t)/H(x, y, t)$ is rational, and let \mathcal{V} be the set of zeroes of $H(x, y, t)$ where no coordinate is zero.

Assume that \mathcal{V} is a smooth manifold; i.e., that at no point

$$H_x = H_y = H_t = H = 0.$$

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Assume that \mathcal{V} is a smooth manifold; i.e., that at no point

$$H_x = H_y = H_t = H = 0.$$

At each $(x, y, t) \in \partial\mathcal{D}$ there is (at least) one point with the same coordinate-wise modulus in \mathcal{V} .

Which points of \mathcal{V} give the best bound? When is it tight?

Critical Points

A point $(x, y, t) \in \mathcal{V}$ is called **critical** if

$$xH_x = yH_y = tH_t$$

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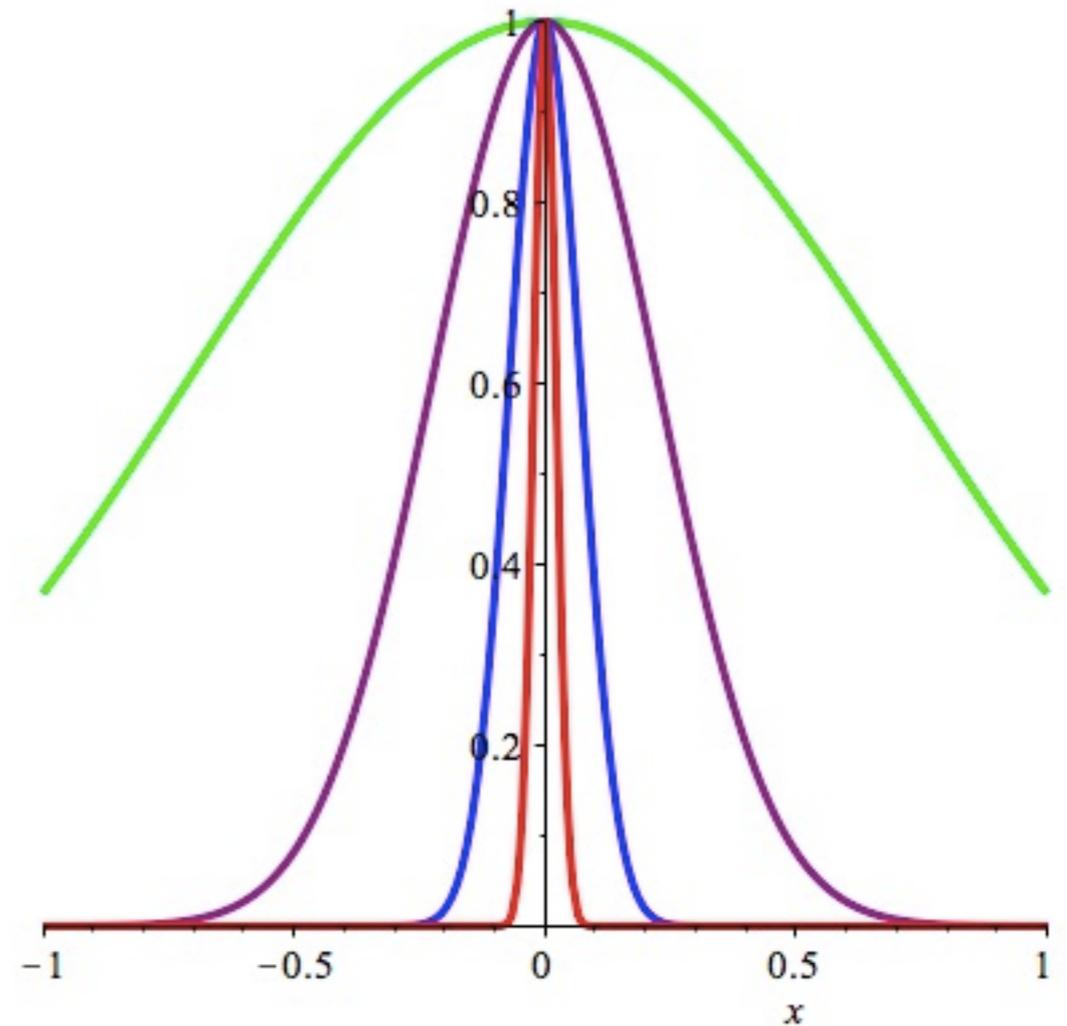
At such a point,

$$\int \frac{f(x, y, t) dx dy dt}{(xyt)^n} \frac{dx dy dt}{xyt}$$

$$\approx \int_{\mathcal{N}} A(u, v) e^{-n(a \cdot u^2 + b \cdot v^2) + O(|u+v|^3)} du dv$$

Idea: We understand asymptotics of Gaussian integrals very well

$$\int_{\mathcal{N}} A(\mathbf{z}) e^{-n(z_1^2 + \dots + z_d^2)} d\mathbf{z} = A(\mathbf{0}) \pi^{d/2} n^{-d/2} (1 + O(1/n))$$



Critical Minimal Points

Theorem

Let $W \subset \partial\mathcal{D}$ be the set of minimal points maximizing $|xyt|$.
Assume each point of W has the same coordinate-wise modulus.

If the subset $C \subset W$ of critical points in W is non-empty and finite then C is the set of contributing points, in the sense that

$$a_{n,n,n} \sim \sum_{(x,y,t) \in C} \text{formula}_n(x,y,t)$$

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There are extensions to more general cases with a finite number of critical minimal points.

Non-degenerate Smooth Formula

Near ζ parametrize $z_d = g(z_1, \dots, z_{d-1})$

$$\psi(z_1, \dots, z_{d-1}) = (z_1 \cdots z_{d-1}) \cdot g(z_1, \dots, z_{d-1})$$

The diagonal coefficients A_n behave like

$$A_n = (\zeta_1 \cdots \zeta_d)^{-n} n^{\frac{1-d}{2}} \left(\frac{(2\pi)^{(1-d)/2}}{\sqrt{\zeta^{3-d} |\mathcal{H}(\zeta)| / \zeta_d^2}} \frac{-G(\zeta)}{\zeta_d \frac{\partial H}{\partial z_d}(\zeta)} + O\left(\frac{1}{n}\right) \right)$$

where $|\mathcal{H}|$ is the determinant of the Hessian of ψ above.

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There are extensions to more general cases with a finite number of critical minimal points.

Problem: critical minimal points may not exist!

“Combinatorial” Case

Luckily, there are important cases where we are guaranteed critical minimal points.

Theorem

Suppose that $G(x, y, t)/H(x, y, t)$ has a power series expansion

$$\sum_{i,j,n \geq 0} a_{i,j,n} x^i y^j t^n$$

where each $a_{i,j,n} \geq 0$. Then there is at least one critical minimal point, and $(x, y, t) \in \partial \mathcal{D}$ only if $(|x|, |y|, |t|) \in \partial \mathcal{D} \cap \mathcal{V}$.

The Method of ACSV

- 1) Find contributing points. Can be done:
 - when dealing with minimal critical points* (if they exist)
 - when the denominator has only linear factors
 - in dimension 2 (DeVries, van der Hoeven, Pemantle)

- 2) Calculate the contribution of each, using pre-calculated formulae (can get very messy)

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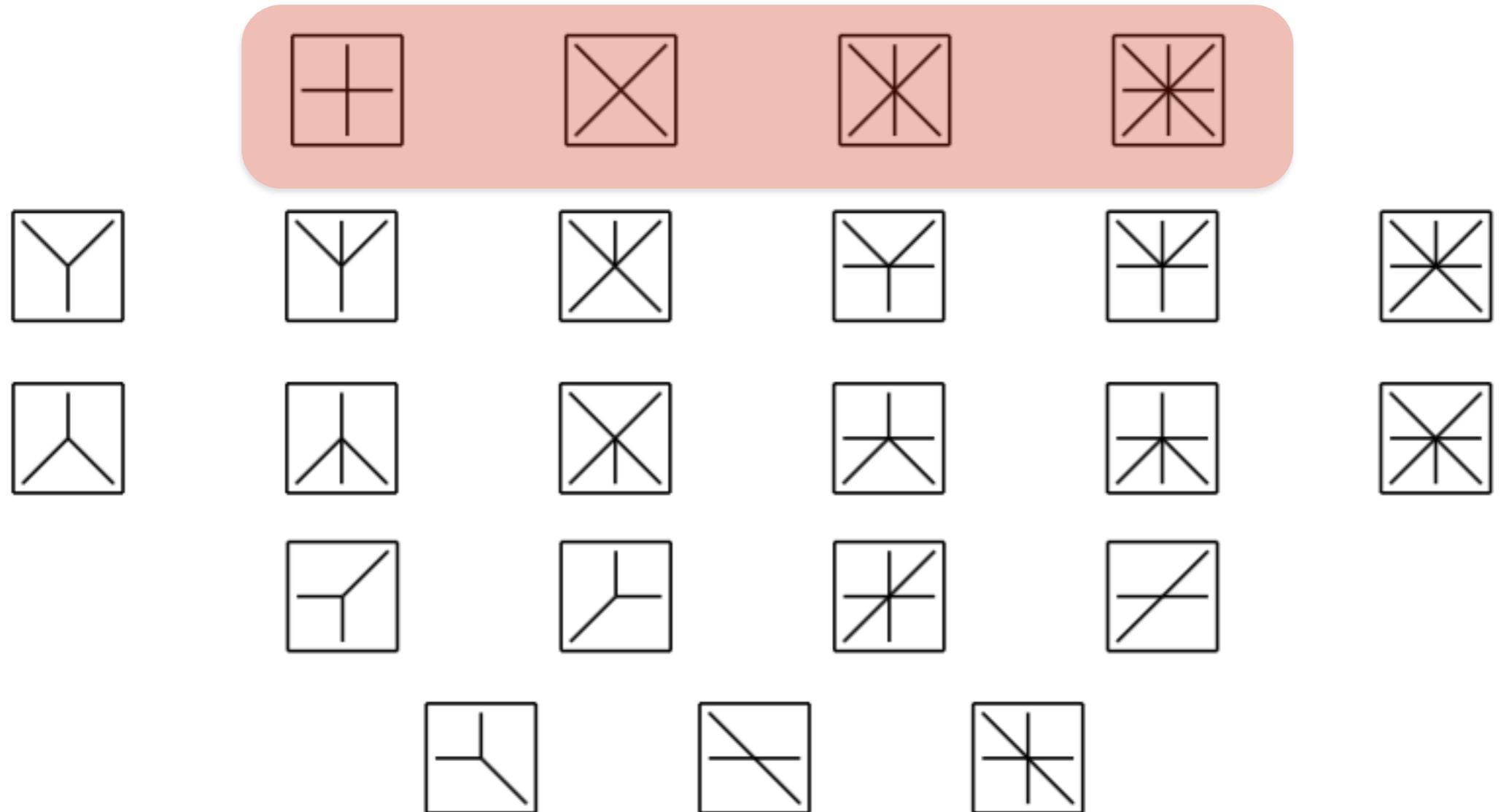
Work in Progress (with Bruno Salvy and George Labahn):

Make this effective and give complexity (see ISSAC 2016?)

ACT III

Back to Lattice Paths

The 23 D-Finite Models



Highly Symmetric — M. and Mishna 2014/15

$$c_n \sim C \cdot n^{-1} \cdot |S|^n$$

Highly Symmetric Walks

$$C(1, 1, t) = \Delta \left(\frac{(1+x)(1+y)}{1-txyS(x, y)} \right)$$

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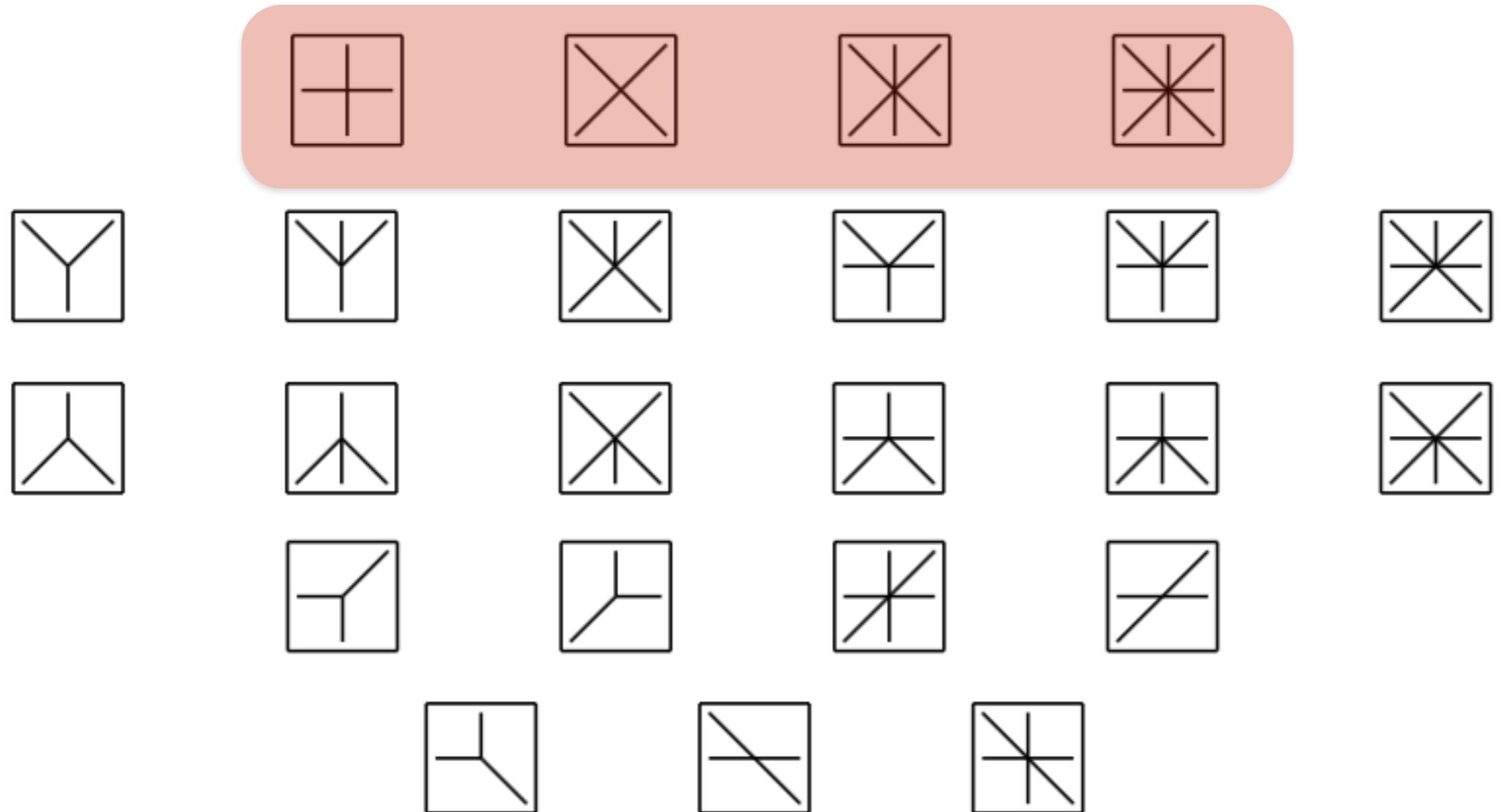
Theorem (M. and Mishna 2014)

Let $\mathcal{S} \subseteq \{-1, 0, 1\}^d \setminus \{0\}$ be symmetric with respect to each axis and take a positive step in each direction. Then

$$c_n \sim \left[\left(s^{(1)} \dots s^{(d)} \right)^{-1/2} \pi^{-d/2} |\mathcal{S}|^{d/2} \right] \cdot n^{-d/2} \cdot |\mathcal{S}|^n,$$

where $s^{(k)} = \#$ of steps which have k^{th} coordinate 1.

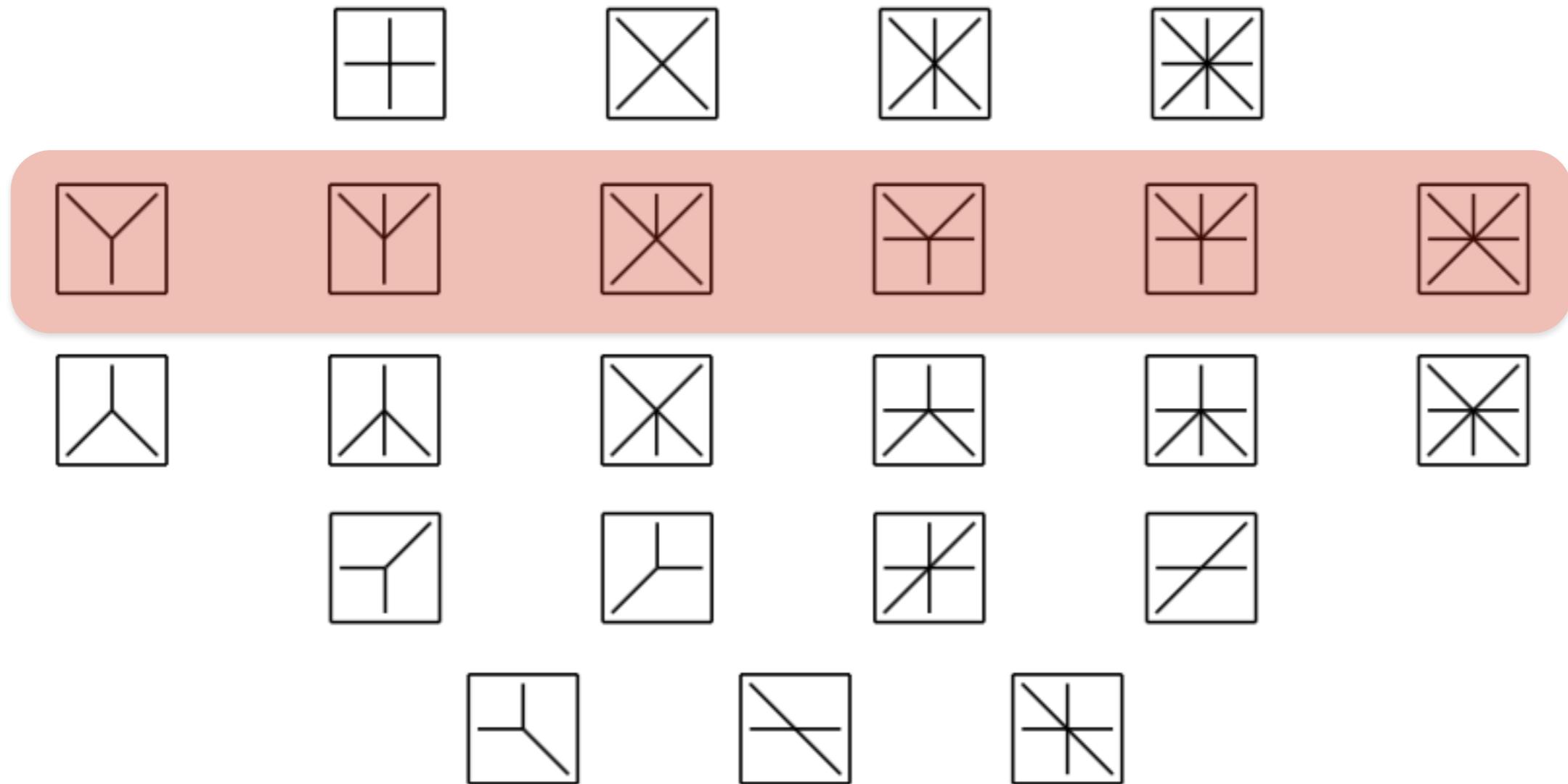
The 23 D-Finite Models



Highly Symmetric — M. and Mishna 2014/15

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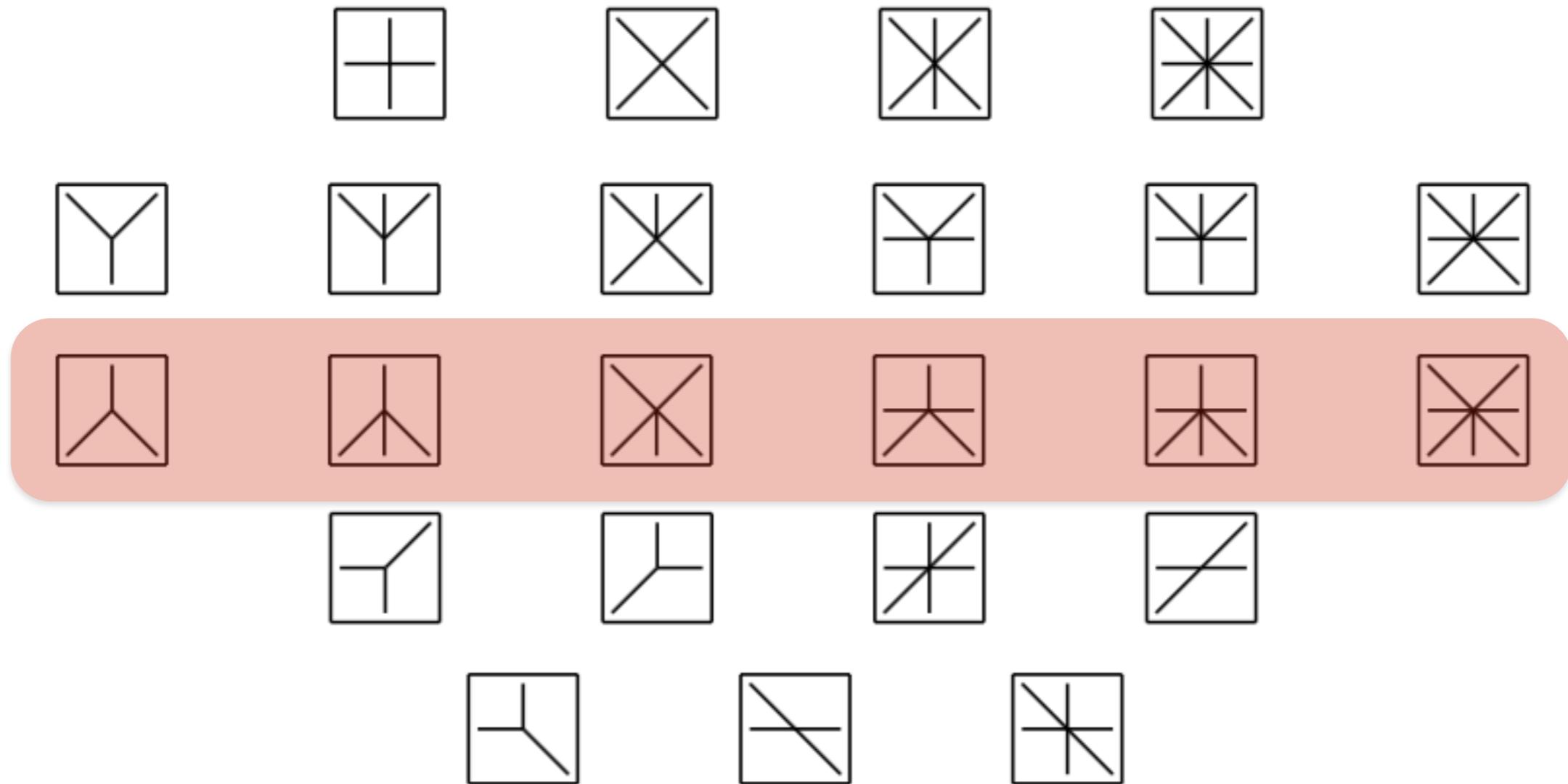
The 23 D-Finite Models



Positive Drift — M. and Wilson 2015

$$c_n \sim C \cdot n^{-3/2} \cdot |S|^n$$

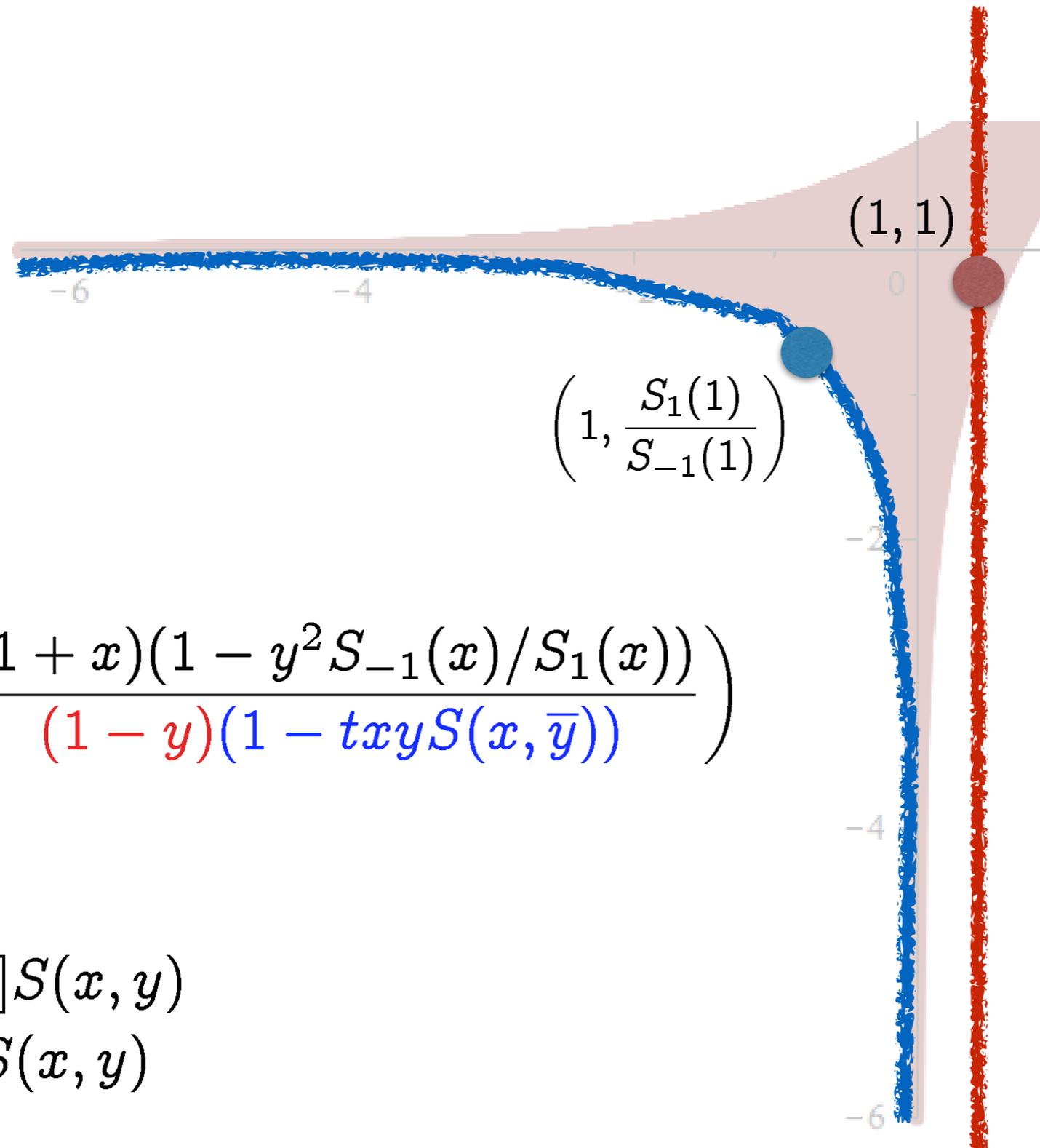
The 23 D-Finite Models



Negative Drift — M. and Wilson 2015

$$c_n \sim C_n \cdot n^{-2} \cdot S(1, \tau)^n$$

Negative Drift

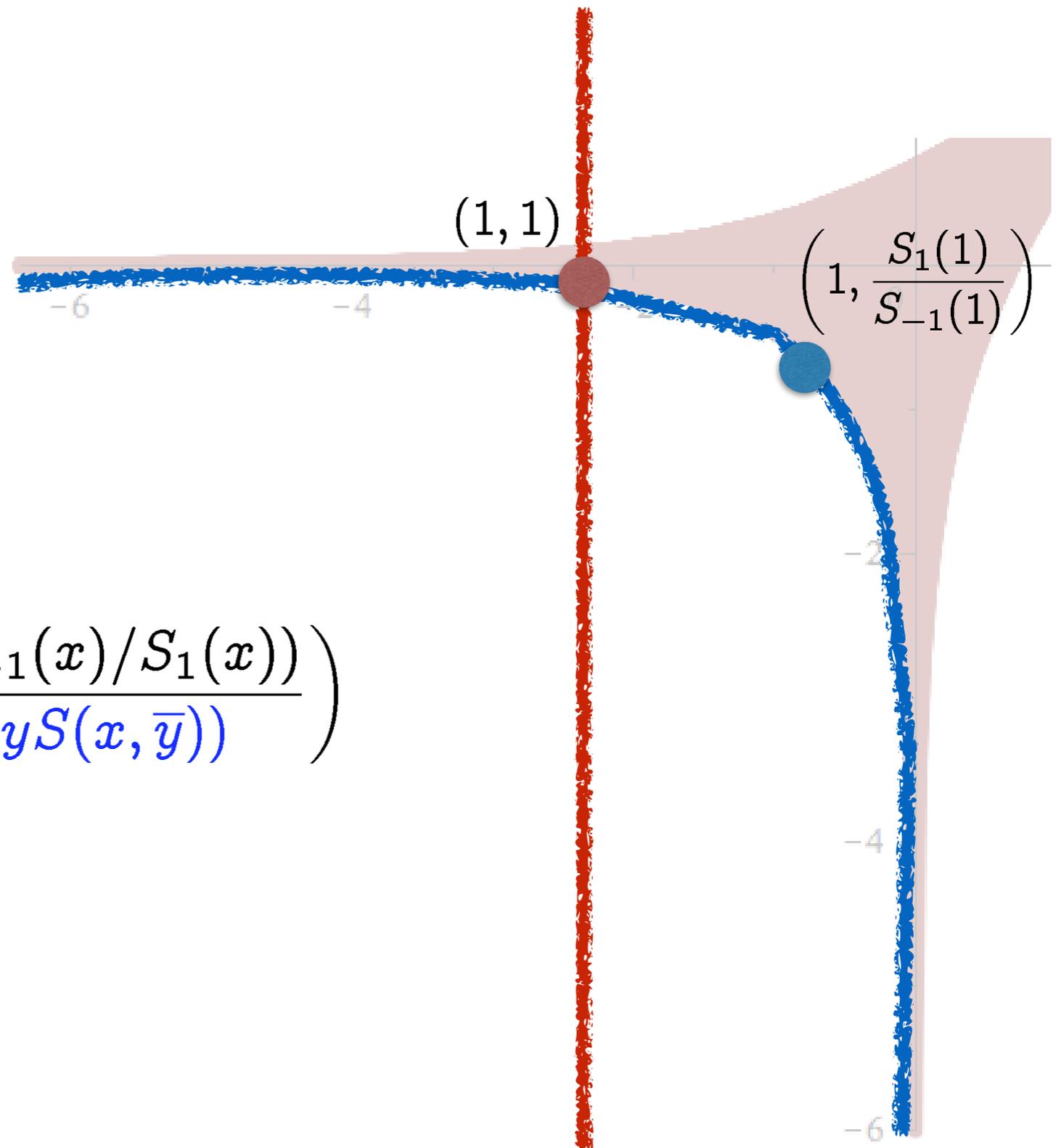


$$C(1, 1, t) = \Delta \left(\frac{(1+x)(1-y^2 S_{-1}(x)/S_1(x))}{(1-y)(1-txyS(x, \bar{y}))} \right)$$

$$S_{-1}(x) = [y^{-1}]S(x, y)$$

$$S_1(x) = [y^1]S(x, y)$$

Positive Drift

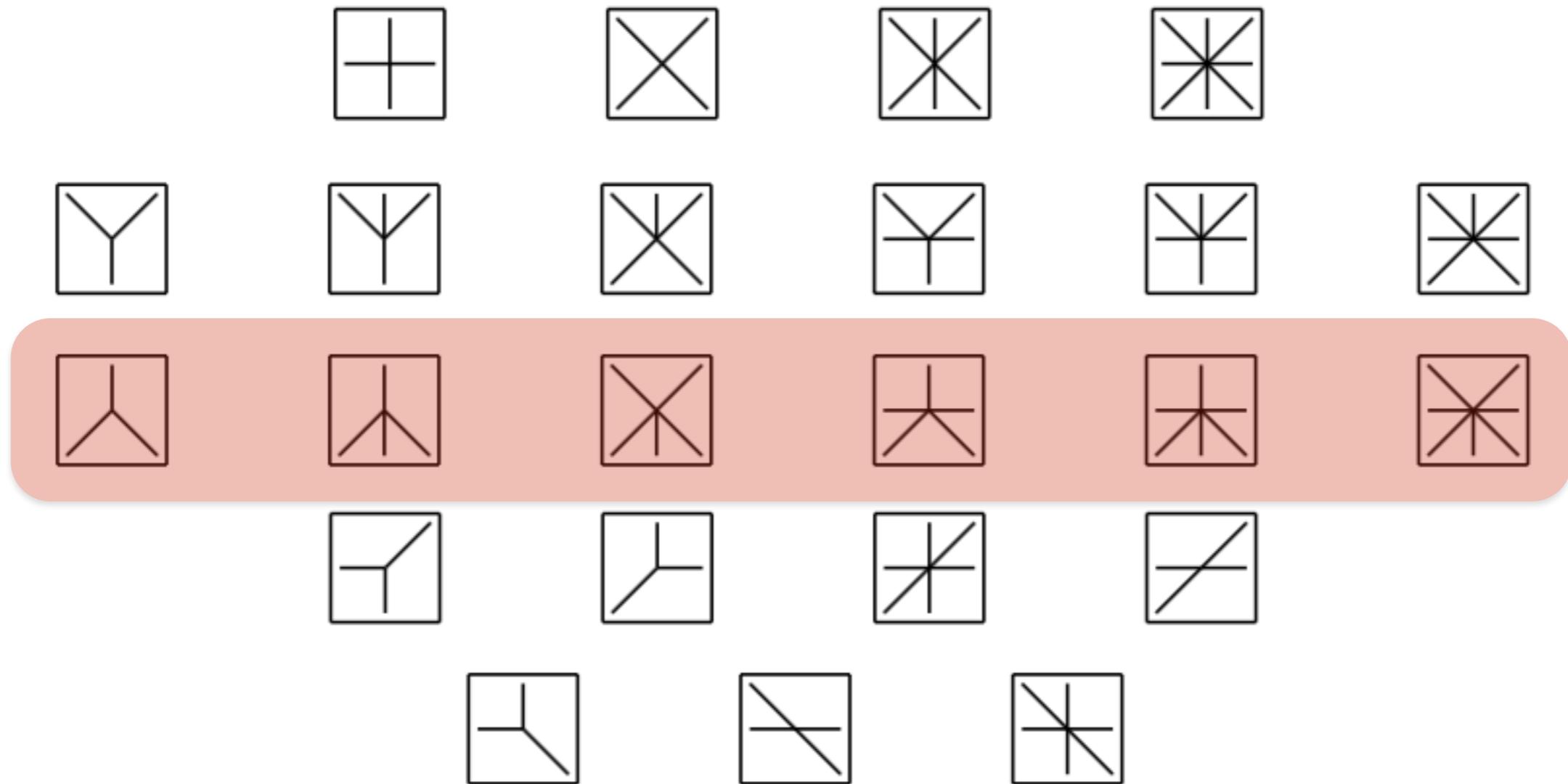


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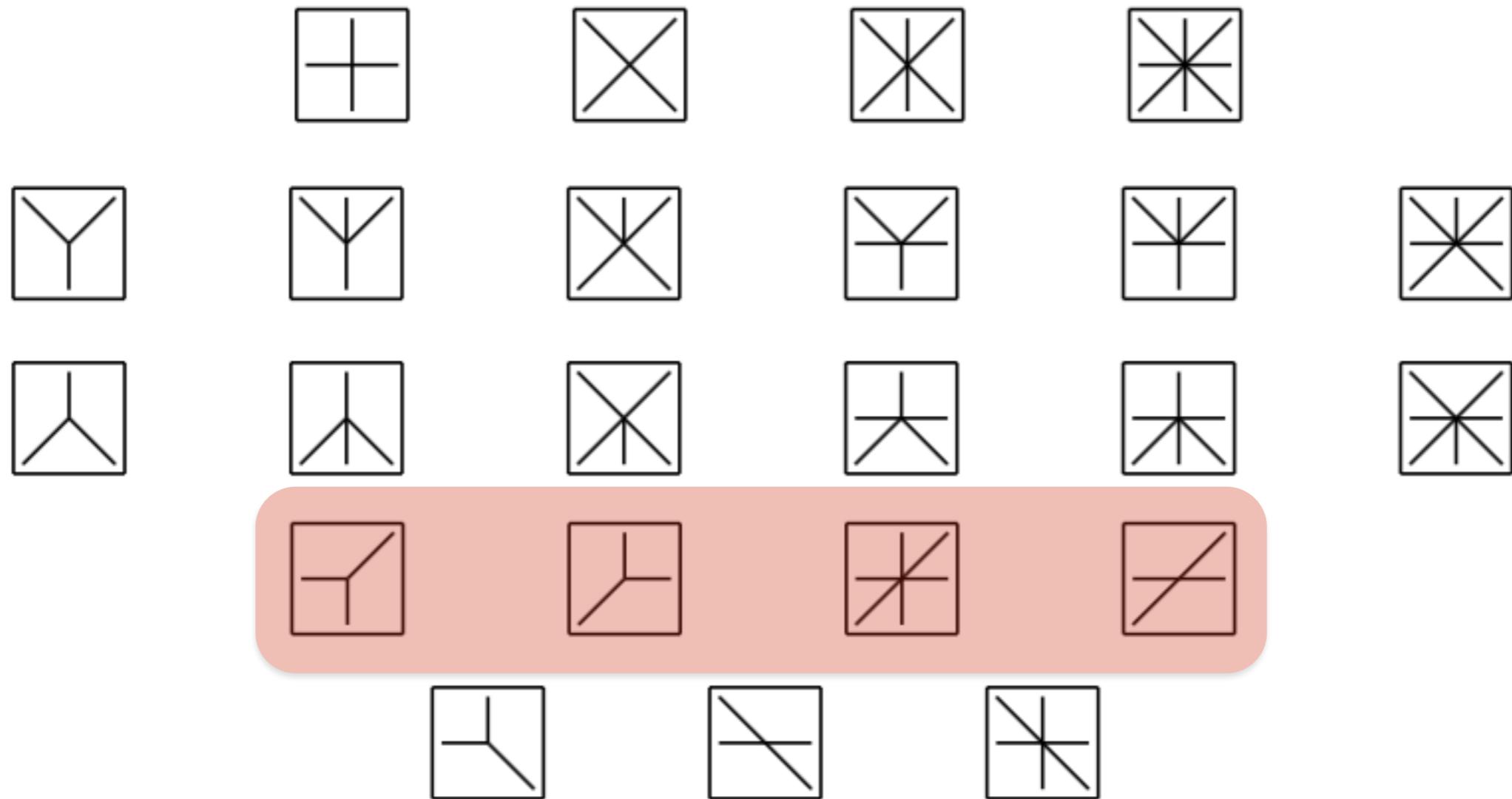
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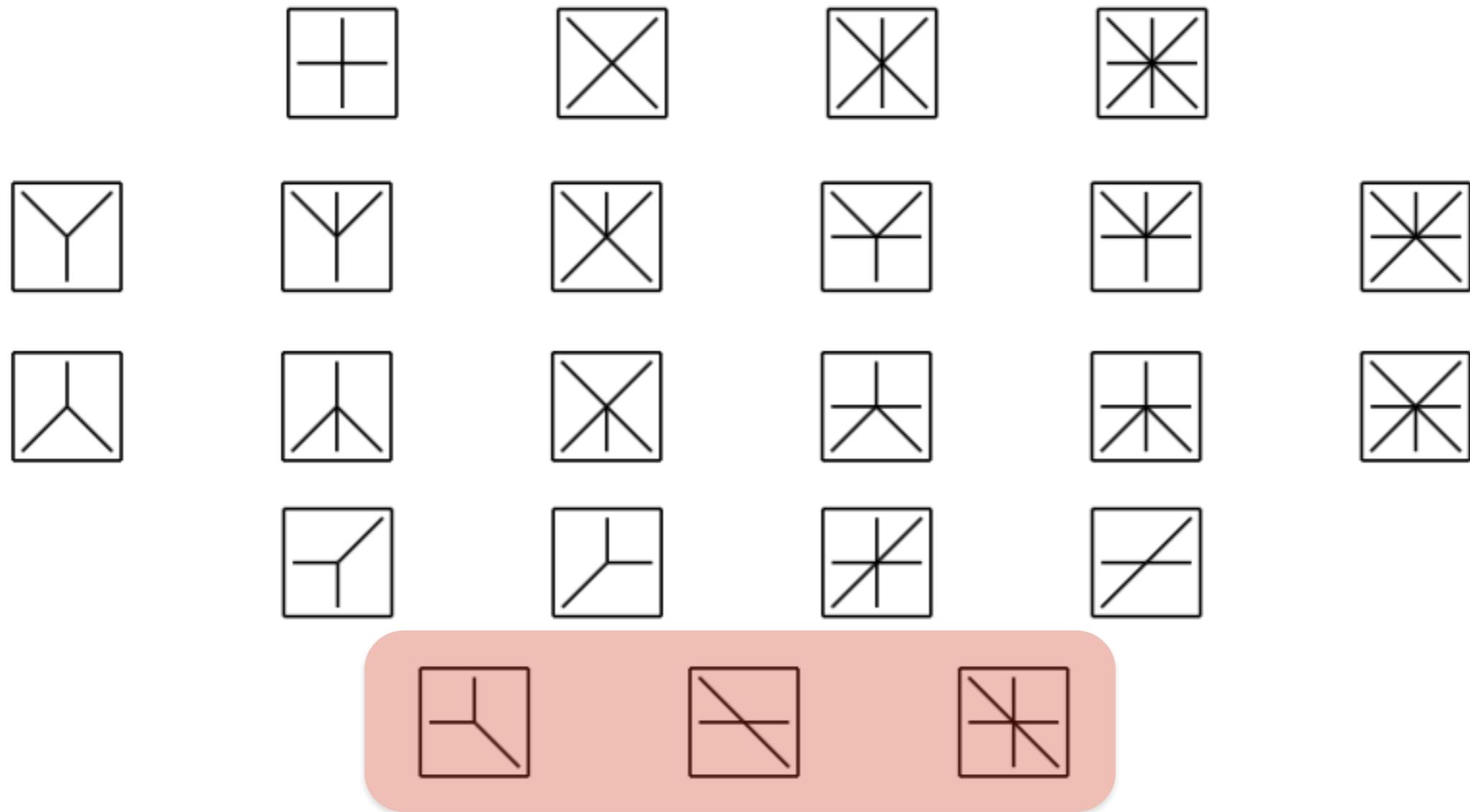
The 23 D-Finite Models



Algebraic Zero Orbit Sum Cases

Mishna / Bousquet-Mélou and Mishna / Bostan and Kauers

The 23 D-Finite Models



Three Sporadic Cases

M. and Wilson 2015 (and Bousquet-Mélou and Mishna)

Example: Negative Drift

Example

Let

$$\mathcal{S} = \{(0, 1), (-1, -1), (0, -1), (1, -1)\} = \begin{array}{c} \uparrow \\ \swarrow \downarrow \searrow \end{array}$$

Then

$$c_n = [t^n] \Delta \left(\frac{(1+x)(1-y^2(1+x+x^2))}{x(1-y)(1-t(x+y^2+xy^2+x^2y^2))} \right)$$

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Example: Negative Drift

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$$\mathcal{S} = \{(0, 1), (-1, -1), (0, -1), (1, -1)\} = \begin{array}{c} \uparrow \\ \swarrow \downarrow \searrow \end{array}$$

Then

$$\begin{aligned} c_n &= [t^n] \Delta \left(\frac{(1+x)(1-y^2(1+x+x^2))}{x(1-y)(1-t(x+y^2+xy^2+x^2y^2))} \right) \\ &= [t^{n+1}] \Delta \left(\frac{yt(1+x)(1-y^2(1+x+x^2))}{(1-y)(1-t(x+y^2+xy^2+x^2y^2))} \right) \end{aligned}$$

The contributing points turn out to be

$$\rho_1 = (1, 1/\sqrt{3}, 1/2) \text{ and } \rho_2 = (1, -1/\sqrt{3}, 1/2)$$

Example: Negative Drift

Example

The Sage package of Raichev calculates (after shifting)

$$\Psi_n^{(\rho_1)} = \frac{3\sqrt{3}(2 + \sqrt{3})}{\pi} \cdot \frac{(2\sqrt{3})^n}{n^2} + O\left(\frac{(2\sqrt{3})^n}{n^3}\right)$$

$$\Psi_n^{(\rho_2)} = \frac{3\sqrt{3}(2 - \sqrt{3})}{\pi} \cdot \frac{(-2\sqrt{3})^n}{n^2} + O\left(\frac{(2\sqrt{3})^n}{n^3}\right)$$

Example: Negative Drift

Example

The Sage package of Raichev calculates (after shifting)

$$\Psi_n^{(\rho_1)} = \frac{3\sqrt{3}(2 + \sqrt{3})}{\pi} \cdot \frac{(2\sqrt{3})^n}{n^2} + O\left(\frac{(2\sqrt{3})^n}{n^3}\right)$$

$$\Psi_n^{(\rho_2)} = \frac{3\sqrt{3}(2 - \sqrt{3})}{\pi} \cdot \frac{(-2\sqrt{3})^n}{n^2} + O\left(\frac{(2\sqrt{3})^n}{n^3}\right)$$

Thus,

$$\begin{aligned} c_n &= \frac{(2\sqrt{3})^n}{n^2} \cdot \frac{3\sqrt{3}}{\pi} \left(\sqrt{3}(1 - (-1)^n) + 2(1 + (-1)^n) + O(n^{-1}) \right) \\ &= \begin{cases} \frac{(2\sqrt{3})^n}{n^2} \cdot \left(\frac{12\sqrt{3}}{\pi} + O(n^{-1}) \right) & : n \text{ even} \\ \frac{(2\sqrt{3})^n}{n^2} \cdot \left(\frac{18}{\pi} + O(n^{-1}) \right) & : n \text{ odd} \end{cases} \end{aligned}$$

CONCLUSION

Conclusion

We are able to

- Prove (and re-discover) conjectured asymptotics of Bostan and Kauers from 2009
- Prove (and re-discover) conjectured asymptotics for returns to boundaries
- Derive compact representations for these combinatorial objects
- Explain combinatorial properties analytically

Extensions to:

- Step sets in d variables symmetric over every axis (M. & Mishna 2015)
- Step sets in d variables missing one symmetry* (M. & Wilson 2015+)

Conclusion

Other on-going work:

- Computational complexity of ACSV methods
(for combinatorial case and beyond) — use Computer Algebra
- Comparison of ACSV methods to creative telescoping algorithms
- Dealing with weights
- Dealing with longer steps
(joint work with Alin Bostan and Mireille Bosquet-Mélou)
- Walks in other regions (Weyl Chambers)
- **Applications to other areas of
Combinatorics / CS / Number Theory**

FIN

Example: A Sporadic Case

Example

Let

$$\mathcal{S} = \{(1, 0), (-1, 0), (-1, 1), (1, -1)\} = \begin{array}{c} \swarrow \\ \leftarrow \\ \searrow \end{array}$$

Then

$$c_n = [t^{n+2}] \Delta \left(\frac{yt^2(x+1)(y-x^2)(x-y)(x+y)}{1-t(x^2y+x^2+y^2+y)} \right)$$

Example: A Sporadic Case

Example

Let

$$\mathcal{S} = \{(1, 0), (-1, 0), (-1, 1), (1, -1)\} = \begin{array}{c} \swarrow \\ \leftarrow \\ \rightarrow \\ \searrow \end{array}$$

Then

$$c_n = [t^{n+2}] \Delta \left(\frac{yt^2(x+1)(y-x^2)(x-y)(x+y)}{1-t(x^2y+x^2+y^2+y)} \right)$$

This rational function has a smooth denominator and is combinatorial! We easily get contributing points:

$$\rho_1 = (1, 1, 1/4) \text{ and } \rho_2 = (-1, 1, 1/4)$$

Example: A Sporadic Case

Example

The Sage package of Raichev calculates (after shifting)

$$\Psi_n^{(\rho_1)} = \frac{8}{\pi} \cdot \frac{4^n}{n^2} + O\left(\frac{4^n}{n^3}\right)$$

$$\Psi_n^{(\rho_2)} = \frac{4}{\pi} \cdot \frac{4^n}{n^3} + O\left(\frac{4^n}{n^4}\right)$$

Thus,

$$c_n = \frac{4^n}{n^2} \cdot \frac{8}{\pi} + O\left(\frac{4^n}{n^3}\right)$$

Bousquet-Mélou and Mishna used Gosper-summability of a related sequence to prove this.

The MP Asymptotic Formula

$$(11) \quad \Psi_{n,N}^{(\mathbf{p})} := \mathbf{p}^{-n\mathbf{1}} \cdot n^{(k-d-1)/2} \cdot (2\pi)^{(k-d-1)/2} (\det \tilde{\Phi}''(\mathbf{0}))^{-1/2} \\ \times \left[\sum_{q=0}^{N-1} \left(n^{-q} \sum_{\substack{0 \leq j \leq \min(k-1, q) \\ \max(0, q-k) \leq i \leq q \\ i+j \leq q}} L_i(\tilde{A}_j, \tilde{\Phi}) \binom{k-1}{j} \begin{bmatrix} k-j \\ k+i-q \end{bmatrix} (-1)^{q-j-i} + O(n^{-N}) \right) \right],$$

where

$$L_i(\tilde{A}_j, \tilde{\Phi}) = \sum_{l=0}^{2i} \frac{\mathcal{H}^{i+l}(\tilde{A}_j \tilde{\Phi}^l)(\mathbf{0})}{(-1)^i 2^{i+l} l! (i+l)!}$$

with \mathcal{H} the differential operator

$$\mathcal{H} = - \sum_{1 \leq a, b \leq d+k-1} (\tilde{\Phi}''(\mathbf{0})^{-1})_{a,b} \partial_a \partial_b,$$

The MP Asymptotic Formula

$$H_j(\mathbf{z}, t) = U_j(\mathbf{z}, t) \left(t - \frac{1}{h_j(\mathbf{z})} \right),$$

$$\check{G}(\mathbf{z}, t) = \frac{G(\mathbf{z}, t)}{U_1(\mathbf{z}, t) \cdots U_k(\mathbf{z}, t)} \prod_{j=1}^k \frac{-h_j(\mathbf{z})}{t}$$

$$h(\mathbf{z}, \boldsymbol{\sigma}) = \sigma_1 h_1 + \cdots + \sigma_{k-1} h_{k-1} + \left(1 - \sum_{j=1}^{k-1} \sigma_j \right) h_k$$

$$A_j(\mathbf{z}, t) = (-1)^{k-1} t^{-k+j} \left(\frac{\partial}{\partial t} \right)^j \check{G}(\mathbf{z}, t^{-1})$$

$$e(\boldsymbol{\theta}) = (p_1 \exp(i\theta_1), \dots, p_d \exp(i\theta_d))$$

$$\tilde{h}(\boldsymbol{\theta}, \boldsymbol{\sigma}) = h(e(\boldsymbol{\theta}), \boldsymbol{\sigma})$$

$$\tilde{A}_j(\boldsymbol{\theta}, \boldsymbol{\sigma}) = A_j(e(\boldsymbol{\theta}), \tilde{h}(\boldsymbol{\theta}, \boldsymbol{\sigma}))$$

$$\tilde{\Phi}(\boldsymbol{\theta}, \boldsymbol{\sigma}) = -\log(p_t \hat{h}(\boldsymbol{\theta}, \boldsymbol{\sigma})) + i \sum_{m=1}^d \theta_m$$

$$\underline{\tilde{\Phi}}(\boldsymbol{\theta}, \boldsymbol{\sigma}) = \tilde{\Phi}(\boldsymbol{\theta}, \boldsymbol{\sigma}) - \frac{1}{2} (\boldsymbol{\theta}, \boldsymbol{\sigma}) \tilde{\Phi}''(\mathbf{0}) (\boldsymbol{\theta}, \boldsymbol{\sigma})^T,$$