

Extension des formules de caractères en types affines via la décomposition de Littlewood

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Journées Combinatoires de Bordeaux,
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- 1 Nekrasov-Okounkov type formulas
- 2 Generalizations through Littlewood decomposition

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Partitions

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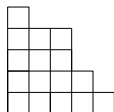


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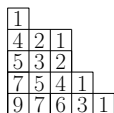


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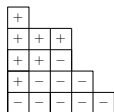


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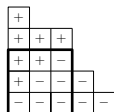


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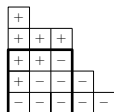


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$\mathcal{H}_t(\lambda)$ the multi-set of hook lengths which are multiple of t

Let $t \geq 2$ be an integer. A partition is a *t-core* if its hook lengths set **does not contain** t . It is equivalent to the fact that the hook lengths set does not contain any integral multiple of t , *i.e.* $\mathcal{H}_t(\lambda) = \emptyset$.

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[Han \(2009\)](#): expansion of η function in terms of hooks

Self-conjugate and doubled distinct partitions

Self-conjugate partitions

SC

1				
2				
4	1			
7	4	2	1	

$SC_{(t)}$: subset of
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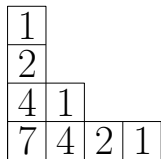
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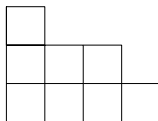
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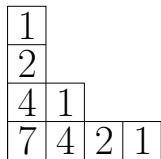
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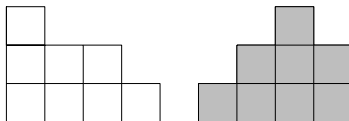


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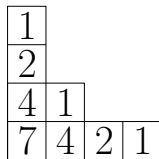
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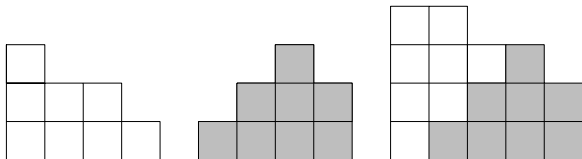


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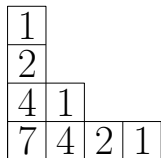
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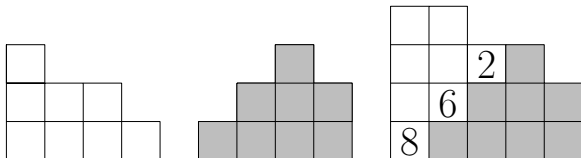


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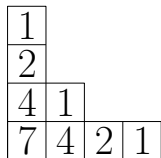
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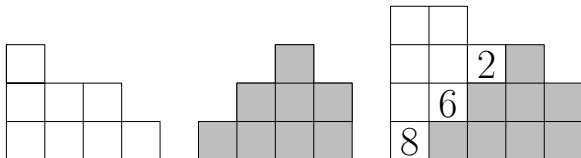


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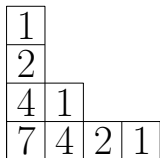
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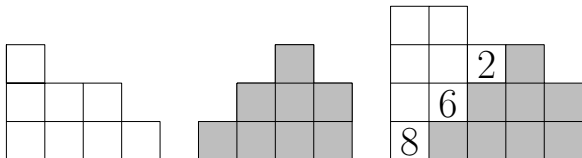


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$$\begin{pmatrix} b_1 + 1 & \dots & b_k + 1 \\ b_1 & \dots & b_k \end{pmatrix}$$

Dedekind η function

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Lehmer's conjecture (1947)

Coefficients of the expansion of η^{24} are nonzero.

Nekrasov-Okounkov type formulas

Theorem (Nekrasov-Okounkov, 2006; Westbury, 2006 ; Han, 2009 ; P., 2015)

For all complex number z , we have :

$$\prod_{k \geq 1} (1 - x^k)^{z-1} = \sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{z}{h^2}\right)$$

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$$\prod_{k \geq 1} (1 - x^k)^{2z^2+z} = \sum_{\lambda \in \mathcal{DD}} \delta_{\lambda} x^{|\lambda|/2} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{2z+2}{h \varepsilon_h}\right)$$

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Generalization of the type \tilde{C} Nekrasov-Okounkov formula

Theorem (P., 2015)

Let t be a positive integer. For any complex numbers y and z we have

$$\sum_{\lambda \in DD} \delta_{\lambda} x^{|\lambda|/2} \prod_{h \in \mathcal{H}_t(\lambda)} \left(y - \frac{tyz}{h \varepsilon_h} \right)$$
$$= \begin{cases} \prod_{k \geq 1} (1 - x^k)(1 - x^{kt})^{t'-1} (1 - x^{kt} y^{2k})^{(z-1)(zt+t-3)/2} & \text{if } t = 2t' + 1 \\ \prod_{k \geq 1} \frac{(1 - x^k)(1 - x^{kt})^{t'-1}}{1 - x^{kt'}} \left(\frac{(1 - y^{2k} x^{kt})^{zt'-1+t'}}{1 - y^k x^{kt'}} \right)^{z-1} & \text{if } t = 2t' \end{cases}$$

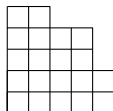
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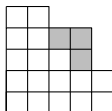
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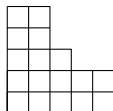
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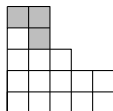
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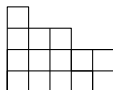
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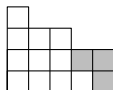
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The t -core of a doubled-distinct partition is doubled-distinct

Littlewood decomposition

Theorem (Littlewood, 1951, probably)

The *Littlewood decomposition* maps a partition λ to $(\tilde{\lambda}, \lambda^0, \lambda^1, \dots, \lambda^{t-1})$ such that:

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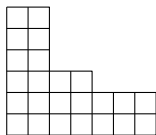
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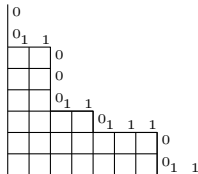
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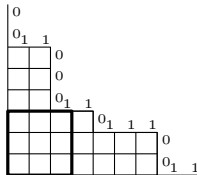
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$$w = \dots 00110001.101110011 \dots$$

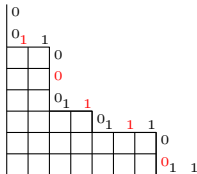
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The *Littlewood decomposition* maps a partition λ to $(\tilde{\lambda}, \lambda^0, \lambda^1, \dots, \lambda^{t-1})$ such that:

- (i) $\tilde{\lambda}$ is the t -core of λ and $\lambda^0, \lambda^1, \dots, \lambda^{t-1}$ are partitions;
- (ii) $|\lambda| = |\tilde{\lambda}| + t(|\lambda^0| + |\lambda^1| + \dots + |\lambda^{t-1}|)$
- (iii) $\{h/t, h \in \mathcal{H}_t(\lambda)\} = \mathcal{H}(\lambda^0) \cup \mathcal{H}(\lambda^1) \cup \dots \cup \mathcal{H}(\lambda^{t-1})$.

$t = 3$



$$w = \dots 00110001.101110011 \dots$$

$$w_0 = \dots 1 0 1 1 0 \dots$$

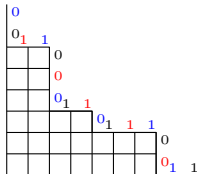
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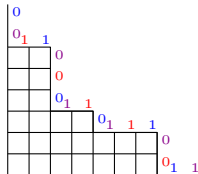
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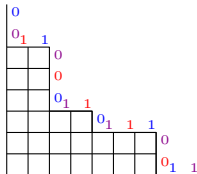
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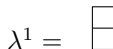
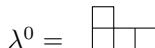


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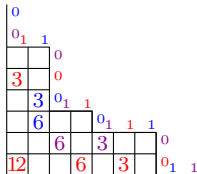
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$$\lambda^0 = \begin{array}{|c|c|c|} \hline 1 & & \\ \hline 4 & 2 & 1 \\ \hline \end{array}$$

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Properties of the Littlewood decomposition

When $\lambda \in DD$, its Littlewood decomposition $(\tilde{\lambda}, \lambda^0, \lambda^1, \dots, \lambda^{t-1})$ satisfies:

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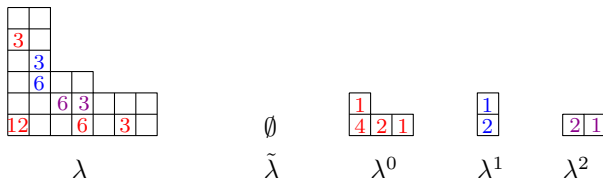
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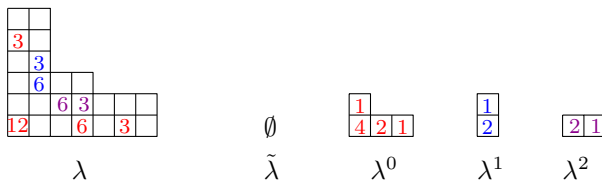
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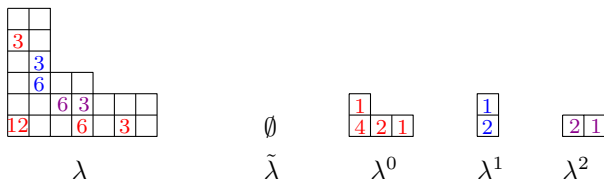
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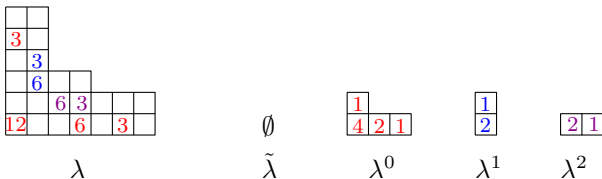
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- (iv) Let v be a box in λ^0 and V its canonically associated box in λ . v is strictly above the principal diagonal in λ^0 iff it is also the same for V in λ .



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When $\lambda \in DD$, its Littlewood decomposition $(\tilde{\lambda}, \lambda^0, \lambda^1, \dots, \lambda^{t-1})$ satisfies:

- (v) Let $v = (j, k)$ be a box in λ^i , with $1 \leq i \leq t'$ and $v^* = (k, j)$ a box in $\lambda^{2t'+1-i} = \lambda^{i^*}$. We denote by V and V^* the boxes of λ associated with them. If V is strictly above (resp. below) the principal diagonal of λ , then V^* is strictly above (resp. below) this diagonal.



Proof of our generalization

- Fix $t = 2t' + 1$ an integer, $\lambda \in DD$ and its Littlewood decomposition $(\tilde{\lambda}, \lambda^0, \lambda^1, \dots, \lambda^{t-1})$.

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- And sum over all doubled distinct partitions.

And for self-conjugate partitions ?

Same types of properties apply for self-conjugate partitions.

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Theorem (P., 2015)

Let t be a positive integer. For all complex numbers y and z , we have :

$$\sum_{\lambda \in SC} \delta_{\lambda} x^{|\lambda|} \prod_{h \in \mathcal{H}_t(\lambda)} \left(y - \frac{yzt}{h \varepsilon_h} \right)$$
$$= \begin{cases} \prod_{k \geq 1} \frac{1 - x^k}{1 - x^{2k}} (1 - x^{2kt})^{t'} (1 - y^{2k} x^{2kt})^{(z^2 - 1)t'} & \text{if } t = 2t' \\ \prod_{k \geq 1} \frac{1 - x^k}{1 - x^{2k}} \frac{(1 - x^{2kt})^{t'+1}}{1 - x^{kt}} \frac{(1 - y^{2k} x^{2kt})^{(tz^2 + z - t - 1)/2}}{(1 - y^k x^{kt})^{z-1}} & \text{if } t = 2t' + 1 \end{cases}$$

Corollary (P., 2015)

When $t = y = 1$, we recover the Nekrasov-Okounkov formula in types \tilde{C} and \tilde{C}^\vee .

Some consequences

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We have:

$$\sum_{\lambda \in DD} \delta_\lambda x^{|\lambda|/2} \prod_{h \in \mathcal{H}_t(\lambda)} \frac{bt}{h \varepsilon_h} = \exp(-tb^2 x^t / 2) \prod_{k \geq 1} (1 - x^k)(1 - x^{kt})^{t'-1}$$

Corollary

if t is odd,

$$\sum_{\substack{\lambda \in DD, |\lambda|=2tn \\ \#\mathcal{H}_t(\lambda)=2n}} \delta_\lambda \prod_{h \in \mathcal{H}_t(\lambda)} \frac{1}{h \varepsilon_h} = \frac{(-1)^n}{n! 2^n t^n}$$

if t is even,

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Thank you for your attention!