

Weighted trees

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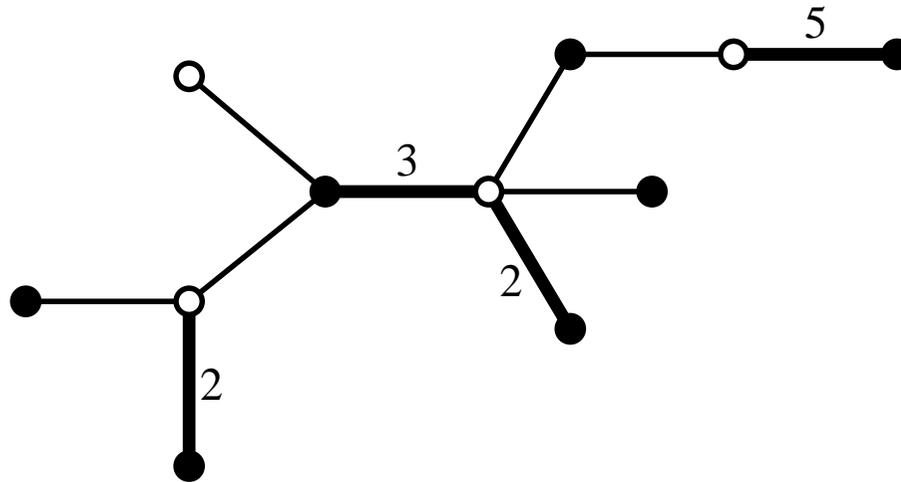
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A **weighted bicolored plane tree** is an object like that:



Vertices are **bicolored**.

Edges are endowed with **weights** (integers). Weights not indicated are equal to 1.

A **degree of a vertex** is the sum of the weights of incident edges.

Planar structure: the cyclic order of branches at each vertex is taken into account.

I. Polynomials

1965: B. J. Birch, S. Chowla, M. Hall Jr., A. Schinzel

Let A and B be two coprime polynomials, $A, B \in \mathbb{C}[x]$. What is the minimum possible degree of the difference $R = A^3 - B^2$?

Example (N. Elkies, 2000)

$$P = (x^{10} - 2x^9 + 33x^8 - 12x^7 + 378x^6 + 336x^5 + 2862x^4 + 2652x^3 + 14\,397x^2 + 9922x + 18\,553)^3,$$

$$Q = (x^{15} - 3x^{14} + 51x^{13} - 67x^{12} + 969x^{11} + 33x^{10} + 10\,963x^9 + 9729x^8 + 96\,507x^7 + 108\,631x^6 + 580\,785x^5 + 700\,503x^4 + 2\,102\,099x^3 + 1\,877\,667x^2 + 3\,904\,161x + 1\,164\,691)^2,$$

$$R = P - Q \quad \boxed{\deg(P - Q) = ?}$$

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$$\begin{aligned} R &= P - Q \\ &= 2^6 3^{15} (5x^6 - 6x^5 + 111x^4 + 64x^3 + 795x^2 + 1254x + 5477). \end{aligned}$$

Verification is trivial. What can be said about computation?
Let us try a straightforward approach...

Let us make a short Maple session:

```
zvonkin@sacha-laptop:~$ maple
  |\~/|      Maple 11 (IBM INTEL LINUX)
._|\|  |/\|_. Copyright (c) Maplesoft, a division of Waterloo Maple Inc. 2007
 \  MAPLE  / All rights reserved. Maple is a trademark of
 <----->  Waterloo Maple Inc.
   |         Type ? for help.
```

First, write a polynomial A of degree 10 with indeterminate coefficients:

```
> A:=sum(a[i]*x^i,i=0..10);
```

$$A := a[0] + a[1] x + a[2] x^2 + a[3] x^3 + a[4] x^4 + a[5] x^5 + a[6] x^6 + a[7] x^7 + a[8] x^8 + a[9] x^9 + a[10] x^{10}$$

Now, write a polynomial B of degree 15, also with indeterminate coefficients:

```
> B:=sum(b[j]*x^j,j=0..15);
```

$$\begin{aligned}
B := & b[0] + b[1] x + b[2] x^2 + b[3] x^3 + b[4] x^4 + b[5] x^5 + b[6] x^6 \\
& + b[7] x^7 + b[8] x^8 + b[9] x^9 + b[10] x^{10} + b[11] x^{11} + b[12] x^{12} \\
& + b[13] x^{13} + b[14] x^{14} + b[15] x^{15}
\end{aligned}$$

Take $R = A^3 - B^2$:

> R:=A^3-B^2;

$$\begin{aligned}
R := & (a[0] + a[1] x + a[2] x^2 + a[3] x^3 + a[4] x^4 + a[5] x^5 + a[6] x^6 \\
& + a[7] x^7 + a[8] x^8 + a[9] x^9 + a[10] x^{10})^3 - (b[0] + b[1] x + b[2] x^2 \\
& + b[3] x^3 + b[4] x^4 + b[5] x^5 + b[6] x^6 + b[7] x^7 + b[8] x^8 + b[9] x^9 \\
& + b[10] x^{10} + b[11] x^{11} + b[12] x^{12} + b[13] x^{13} + b[14] x^{14} + b[15] x^{15} \\
&)^2
\end{aligned}$$

We have $11 + 16 = 27$ unknowns but we will get only 24 equations. Therefore, using Elkies's result as a prompting, we set

```
> a[10]:=1;
                                a[10] := 1
> b[15]:=1;
                                b[15] := 1
> a[9]:=-2;
                                a[9] := -2
```

Let us now compute the coefficients of $R = A^3 - B^2$ in front of the degrees from 7 to 30. They will give us our 24 equations: we must make them all equal to zero.

```
> R:=collect(R,x):
> for k from 7 to 30 do c[k]:=expand(coeff(R,x,k)) end do;
```

```
c[7] := -2 b[3] b[4] - 2 b[5] b[2] - 2 b[6] b[1] - 2 b[0] b[7]
        + 6 a[6] a[0] a[1] + 6 a[5] a[0] a[2] + 3 a[5] a[1]2 + 3 a[7] a[0]2
```

$$+ 6 a[4] a[2] a[1] + 6 a[4] a[0] a[3] + 3 a[1] a[3]^2 + 3 a[3] a[2]^2$$

$$c[8] := 6 a[7] a[0] a[1] + 3 a[8] a[0]^2 - 2 b[0] b[8] - 2 b[1] b[7]$$

$$- 2 b[3] b[5] - b[4]^2 - 2 b[6] b[2] + 6 a[4] a[1] a[3]$$

$$+ 6 a[6] a[0] a[2] + 6 a[5] a[2] a[1] + 3 a[6] a[1]^2 + 3 a[0] a[4]^2$$

$$+ 3 a[4] a[2]^2 + 3 a[2] a[3]^2 + 6 a[5] a[0] a[3]$$

$$c[9] := -6 a[0]^2 + 6 a[8] a[0] a[1] - 2 b[2] b[7] - 2 b[3] b[6]$$

$$- 2 b[4] b[5] - 2 b[8] b[1] - 2 b[9] b[0] + 3 a[7] a[1]^2 + 3 a[5] a[2]^2$$

$$+ 3 a[4] a[1]^2 + 6 a[2] a[6] a[1] + 6 a[2] a[4] a[3] + 6 a[5] a[1] a[3]$$

$$+ 6 a[7] a[0] a[2] + 6 a[6] a[0] a[3] + 6 a[5] a[0] a[4] + a[3]^3$$

$$\begin{aligned}
c[10] := & 3 a[0]^2 - 12 a[0] a[1] - 2 b[2] b[8] - 2 b[3] b[7] - 2 b[4] b[6] \\
& - b[5]^2 - 2 b[9] b[1] - 2 b[10] b[0] + 6 a[7] a[2] a[1] \\
& + 6 a[7] a[0] a[3] + 6 a[6] a[0] a[4] + 6 a[6] a[1] a[3] \\
& + 6 a[5] a[4] a[1] + 6 a[5] a[2] a[3] + 6 a[8] a[0] a[2] + 3 a[6] a[2]^2 \\
& + 3 a[8] a[1]^2 + 3 a[0] a[5]^2 + 3 a[4] a[3]^2 + 3 a[2] a[4]^2
\end{aligned}$$

$$\begin{aligned}
c[11] := & 6 a[0] a[1] - 12 a[0] a[2] - 6 a[1]^2 - 2 b[0] b[11] - 2 b[1] b[10] \\
& - 2 b[2] b[9] - 2 b[3] b[8] - 2 b[4] b[7] - 2 b[5] b[6] \\
& + 6 a[4] a[0] a[7] + 3 a[4]^2 a[3] + 3 a[1] a[5]^2 + 3 a[5] a[3]^2 \\
& + 3 a[7] a[2]^2 + 6 a[8] a[2] a[1] + 6 a[4] a[5] a[2] + 6 a[6] a[0] a[5] \\
& + 6 a[6] a[4] a[1] + 6 a[6] a[2] a[3] + 6 a[1] a[7] a[3]
\end{aligned}$$

$$+ 6 a[8] a[0] a[3]$$

$$\begin{aligned}
c[12] := & 6 a[0] a[2] - 12 a[0] a[3] + 3 a[1]^2 - 12 a[2] a[1] - 2 b[1] b[11] \\
& - 2 b[2] b[10] - 2 b[5] b[7] - b[6]^2 - 2 b[8] b[4] - 2 b[9] b[3] \\
& - 2 b[12] b[0] + 6 a[2] a[6] a[4] + 6 a[4] a[7] a[1] + 6 a[4] a[5] a[3] \\
& + 6 a[6] a[5] a[1] + 6 a[2] a[7] a[3] + 3 a[2]^2 a[8] + 3 a[2] a[5]^2 \\
& + 3 a[0] a[6]^2 + 3 a[6] a[3]^2 + 6 a[4] a[0] a[8] + a[4]^3 \\
& + 6 a[8] a[1] a[3] + 6 a[7] a[0] a[5]
\end{aligned}$$

$$\begin{aligned}
c[13] := & 6 a[0] a[3] - 12 a[0] a[4] - 12 a[1] a[3] + 6 a[2] a[1] - 6 a[2]^2 \\
& - 2 b[0] b[13] - 2 b[1] b[12] - 2 b[2] b[11] + 6 a[7] a[0] a[6] \\
& - 2 b[6] b[7] - 2 b[8] b[5] - 2 b[9] b[4] - 2 b[10] b[3] + 3 a[7] a[3]^2
\end{aligned}$$

$$\begin{aligned}
& + 3 a[5] a[4]^2 + 3 a[6]^2 a[1] + 6 a[4] a[1] a[8] + 6 a[2] a[6] a[5] \\
& + 6 a[0] a[8] a[5] + 6 a[4] a[6] a[3] + 6 a[2] a[8] a[3] \\
& + 6 a[7] a[5] a[1] + 6 a[7] a[4] a[2] + 3 a[5]^2 a[3] \\
c[14] := & 6 a[0] a[4] - 12 a[0] a[5] + 6 a[1] a[3] + 3 a[2]^2 - 12 a[2] a[3] \\
& - 12 a[4] a[1] - 2 b[0] b[14] - 2 b[1] b[13] - 2 b[2] b[12] \\
& - 2 b[3] b[11] - b[7]^2 - 2 b[8] b[6] - 2 b[9] b[5] - 2 b[10] b[4] \\
& + 6 a[8] a[0] a[6] + 6 a[5] a[6] a[3] + 6 a[8] a[5] a[1] \\
& + 6 a[5] a[7] a[2] + 6 a[8] a[4] a[2] + 6 a[7] a[4] a[3] \\
& + 6 a[7] a[6] a[1] + 3 a[8] a[3]^2 + 3 a[5]^2 a[4] + 3 a[0] a[7]^2 \\
& + 3 a[2]^2 a[6] + 3 a[6]^2 a[4]
\end{aligned}$$

$$\begin{aligned}
c[15] := & 6 a[8] a[5] a[2] + 6 a[0] a[5] - 12 a[0] a[6] - 2 b[0] \\
& + 6 a[2] a[3] - 6 a[3]^2 + 6 a[4] a[1] - 12 a[4] a[2] - 12 a[5] a[1] \\
& - 2 b[1] b[14] - 2 b[2] b[13] - 2 b[3] b[12] - 2 b[8] b[7] \\
& - 2 b[9] b[6] - 2 b[10] b[5] - 2 b[11] b[4] + 6 a[8] a[6] a[1] + a[5]^3 \\
& + 3 a[6]^2 a[3] + 6 a[8] a[4] a[3] + 6 a[3] a[7] a[5] + 6 a[6] a[7] a[2] \\
& + 6 a[8] a[0] a[7] + 6 a[6] a[5] a[4] + 3 a[7] a[4]^2 + 3 a[7]^2 a[1]
\end{aligned}$$

$$\begin{aligned}
c[16] := & 6 a[3] a[8] a[5] + 6 a[0] a[6] - 12 a[0] a[7] - 2 b[1] + 3 a[3]^2 \\
& + 6 a[4] a[2] - 12 a[4] a[3] + 6 a[5] a[1] - 12 a[5] a[2] \\
& - 12 a[6] a[1] - 2 b[2] b[14] - 2 b[3] b[13] - 2 b[4] b[12] - b[8]^2 \\
& - 2 b[9] b[7] - 2 b[10] b[6] - 2 b[11] b[5] + 6 a[4] a[7] a[5]
\end{aligned}$$

$$\begin{aligned}
& + 6 a[3] a[7] a[6] + 6 a[2] a[8] a[6] + 6 a[1] a[7] a[8] + 3 a[2] a[7]^2 \\
& + 3 a[0] a[8]^2 + 3 a[4] a[8]^2 + 3 a[4] a[6]^2 + 3 a[6] a[5]^2
\end{aligned}$$

$$\begin{aligned}
c[17] := & 6 a[0] a[7] - 12 a[0] a[8] - 2 b[2] - 12 a[2] a[6] + 6 a[4] a[3] \\
& - 6 a[4]^2 + 6 a[5] a[2] - 12 a[5] a[3] + 6 a[6] a[1] - 12 a[7] a[1] \\
& - 2 b[3] b[14] - 2 b[4] b[13] - 2 b[8] b[9] - 2 b[10] b[7] \\
& - 2 b[11] b[6] - 2 b[12] b[5] + 6 a[3] a[8] a[6] + 6 a[4] a[8] a[5] \\
& + 3 a[1] a[8]^2 + 6 a[2] a[7] a[8] + 6 a[4] a[7] a[6] + 3 a[3] a[7]^2 \\
& + 3 a[7] a[5]^2 + 3 a[5] a[6]^2
\end{aligned}$$

$$\begin{aligned}
c[18] := & 6 a[5] a[7] a[6] + 12 a[0] + 6 a[0] a[8] - 12 a[1] a[8] - 2 b[3] \\
& + 6 a[2] a[6] + 3 a[4]^2 + 6 a[5] a[3] - 12 a[5] a[4] - 12 a[6] a[3]
\end{aligned}$$

$$\begin{aligned}
& + 6 a[7] a[1] - 12 a[7] a[2] - 2 b[4] b[14] - 2 b[5] b[13] \\
& - 2 b[8] b[10] - b[9]^2 - 2 b[11] b[7] - 2 b[12] b[6] + 6 a[4] a[8] a[6] \\
& + 6 a[3] a[7] a[8] + 3 a[2] a[8]^2 + 3 a[4] a[7]^2 + 3 a[8] a[5]^2 + a[6]^3
\end{aligned}$$

$$c[19] := -12 a[0] + 12 a[1] + 6 a[1] a[8] - 2 b[4] - 12 a[2] a[8]$$

$$\begin{aligned}
& + 6 a[5] a[4] - 6 a[5]^2 + 6 a[6] a[3] - 12 a[6] a[4] + 6 a[7] a[2] \\
& - 12 a[7] a[3] - 2 b[5] b[14] - 2 b[6] b[13] - 2 b[7] b[12] \\
& - 2 b[8] b[11] - 2 b[9] b[10] + 6 a[8] a[7] a[4] + 3 a[8]^2 a[3] \\
& + 6 a[8] a[6] a[5] + 3 a[5] a[7]^2 + 3 a[7] a[6]^2
\end{aligned}$$

$$c[20] := 3 a[0] - 12 a[1] + 12 a[2] - 2 b[5] + 6 a[2] a[8] + 3 a[5]^2$$

$$+ 6 a[6] a[4] - 12 a[6] a[5] + 6 a[7] a[3] - 12 a[7] a[4]$$

$$\begin{aligned}
& - 12 a[8] a[3] - 2 b[7] b[13] - 2 b[8] b[12] - 2 b[9] b[11] - b[10]^2 \\
& - 2 b[14] b[6] + 6 a[8] a[7] a[5] + 3 a[8] a[6]^2 + 3 a[6] a[7]^2 \\
& + 3 a[4] a[8]^2
\end{aligned}$$

$$\begin{aligned}
c[21] := & 3 a[1] - 12 a[2] + 12 a[3] - 2 b[6] - 12 a[4] a[8] + 6 a[6] a[5] \\
& - 6 a[6]^2 + 6 a[7] a[4] - 12 a[7] a[5] + 6 a[8] a[3] - 2 b[8] b[13] \\
& - 2 b[9] b[12] - 2 b[10] b[11] - 2 b[14] b[7] + 6 a[7] a[8] a[6] \\
& + 3 a[8]^2 a[5] + a[7]^3
\end{aligned}$$

$$\begin{aligned}
c[22] := & 3 a[2] - 12 a[3] + 12 a[4] - 2 b[7] + 6 a[4] a[8] + 3 a[6]^2 \\
& + 6 a[7] a[5] - 12 a[7] a[6] - 12 a[8] a[5] - 2 b[8] b[14] \\
& - 2 b[10] b[12] - b[11]^2 - 2 b[13] b[9] + 3 a[8]^2 a[6] + 3 a[8] a[7]^2
\end{aligned}$$

$$\begin{aligned}
c[23] := & -2 b[13] b[10] - 2 b[14] b[9] - 2 b[8] - 2 b[11] b[12] - 12 a[4] \\
& - 12 a[8] a[6] + 12 a[5] - 6 a[7]^2 + 3 a[7] a[8]^2 + 6 a[8] a[5] \\
& + 6 a[7] a[6] + 3 a[3]
\end{aligned}$$

$$\begin{aligned}
c[24] := & -2 b[14] b[10] - 2 b[9] - b[12]^2 - 2 b[13] b[11] + 3 a[4] \\
& + 6 a[8] a[6] - 12 a[5] + 3 a[7]^2 - 12 a[7] a[8] + 12 a[6] + a[8]^3
\end{aligned}$$

$$\begin{aligned}
c[25] := & -2 b[13] b[12] - 2 b[14] b[11] - 2 b[10] - 6 a[8]^2 - 12 a[6] \\
& + 12 a[7] + 6 a[7] a[8] + 3 a[5]
\end{aligned}$$

$$\begin{aligned}
c[26] := & \\
& -12 a[7] + 12 a[8] + 3 a[8]^2 + 3 a[6] - b[13]^2 - 2 b[14] b[12] - 2 b[11]
\end{aligned}$$

$$c[27] := -2 b[12] - 2 b[13] b[14] + 3 a[7] - 12 a[8] - 8$$

$$c[28] := -2 b[13] - b[14]^2 + 12 + 3 a[8]$$

$$c[29] := -2 b[14] - 6$$

$$c[30] := 0$$

Let us now compute the degrees of all these equations:

```
> for k from 7 to 29 do d[k]:=degree(c[k]) end do;
```

$$d[7] := 3$$

$$d[8] := 3$$

$$d[9] := 3$$

$$d[10] := 3$$

$$d[11] := 3$$

d[12] := 3

d[13] := 3

d[14] := 3

d[15] := 3

d[16] := 3

d[17] := 3

d[18] := 3

d[19] := 3

d[20] := 3

d[21] := 3

d[22] := 3

d[23] := 3

d[24] := 3

d[25] := 2

```
d[26] := 2
```

```
d[27] := 2
```

```
d[28] := 2
```

```
d[29] := 1
```

The degree of the whole system is the product of the degrees of the above equations:

```
> N:=mul(d[k],k=7..29);
```

```
N := 6198727824
```

```
> quit;
```

```
bytes used=762104, alloc=589716, time=0.05
```

```
zvonkin@sacha-laptop:~$
```

We thus obtain a system of algebraic equations of degree

$$N = 6\,198\,727\,824.$$

All this is rather discouraging, and even more so since in fact there are **exactly four non equivalent solutions**.

The problem is that our system does not ensure that the resulting polynomials are coprime. But if we *add* the condition of coprimality to the system, we will make it even more complicated.

If we try to further push down the degree of $A^3 - B^2$, we may well write the system but there will be no solutions at all.

It is also quite surprising that we have got a solution in \mathbb{Q} .

A “positive observation”: all solutions must be algebraic numbers.

Two conjectures (1965): Let

$$\deg A = 2k, \quad \deg B = 3k,$$

so that

$$\deg A^3 = \deg B^2 = 6k;$$

then

1. $\deg(A^3 - B^2) \geq k + 1$;
2. This bound is sharp.

In the previous example $k = 5$.

1965: The first conjecture was proved by H. Davenport.

1981: W. W. Stothers proved that the bound is attained for all k .

We see that the second conjecture turned out to be more difficult: it remained open for 16 years.

1995: The problem is generalized by U. Zannier:

Let two partitions of an integer n be given:

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_p), \quad \beta = (\beta_1, \beta_2, \dots, \beta_q),$$

$$\sum_{i=1}^p \alpha_i = \sum_{j=1}^q \beta_j = n,$$

and let P and Q be two coprime polynomials of degree n with complex coefficients, such that

$$P(x) = \prod_{i=1}^p (x - a_i)^{\alpha_i}, \quad Q(x) = \prod_{j=1}^q (x - b_j)^{\beta_j}.$$

Denote $R = P - Q$.

Question: What is the minimum possible degree of R ?

In the initial problem $\alpha = (3, 3, \dots, 3)$, $\beta = (2, 2, \dots, 2)$.

Two technical assumptions:

1. The greatest common divisor of $\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q$ is 1.
2. $p + q \leq n + 1$.

(We can do without them, but the statements would become more cumbersome.)

Theorem (U. Zannier, 1995)

1. $\deg R \geq (n + 1) - (p + q)$.
2. This bound is attained for any pair of partitions $\alpha, \beta \vdash n$ satisfying the above assumptions.

In the “cubes and squares” problem, $n = 6k$, $p = 2k$, $q = 3k$.

In fact, an equivalent result in terms of permutations was proved by G. Boccara in 1982. But Boccara did not know about the relations of his result to polynomials.

2010: F. Beukers, C. Stewart: a problem which is much more interesting from the number-theoretic point of view, namely:

Search for polynomials A and B such that

1. The degree of the difference $A^k - B^l$ attains its minimum;
2. A and B are defined over \mathbb{Q} .

We call a pair of polynomials (P, Q) of degree n such that

$$P(x) = \prod_{i=1}^p (x - a_i)^{\alpha_i}, \quad Q(x) = \prod_{j=1}^q (x - b_j)^{\beta_j},$$

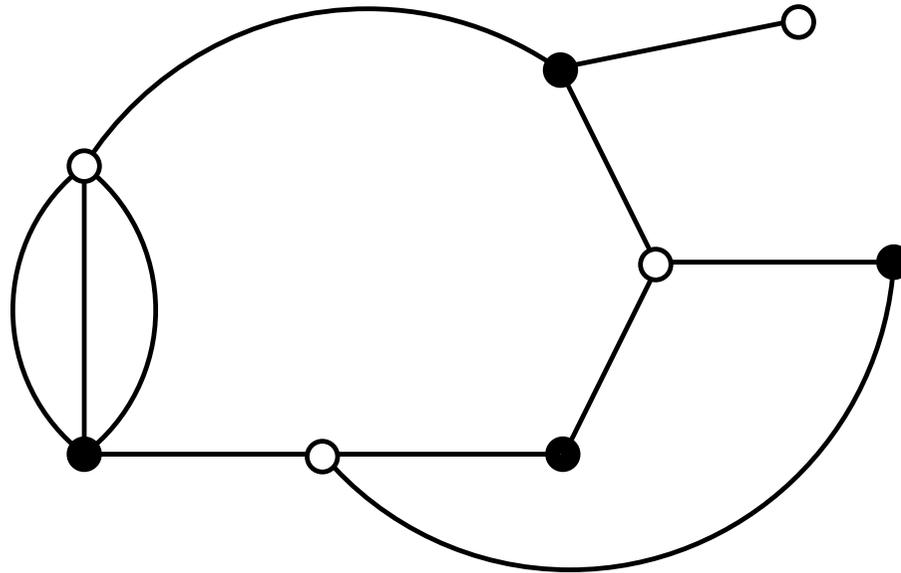
and

$$\deg(P - Q) = (n + 1) - (p + q)$$

a **Davenport-Zannier pair**, or a **DZ-pair**, and we call the pair of partitions (α, β) , $\alpha, \beta \vdash n$, its **passport**.

II. Maps and hypermaps

A **hypermap** is a map with an additional structure: **bicoloring** of the vertices.



A **face degree** is defined as a **half** of the number of surrounding edges.

Thus, the sum of degrees of the black vertices, of the white vertices, and of the faces, is equal to the number of edges.

All hypermaps in this talk will be planar.

The **passport of a hypermap** is the triple of partitions of n ,

$$\pi = (\alpha, \beta, \gamma), \quad \alpha, \beta, \gamma \vdash n$$

where n is the number of edges,

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_p), \quad \sum_{i=1}^p \alpha_i = n,$$

is the set of degrees of the black vertices,

$$\beta = (\beta_1, \beta_2, \dots, \beta_q), \quad \sum_{j=1}^q \beta_j = n,$$

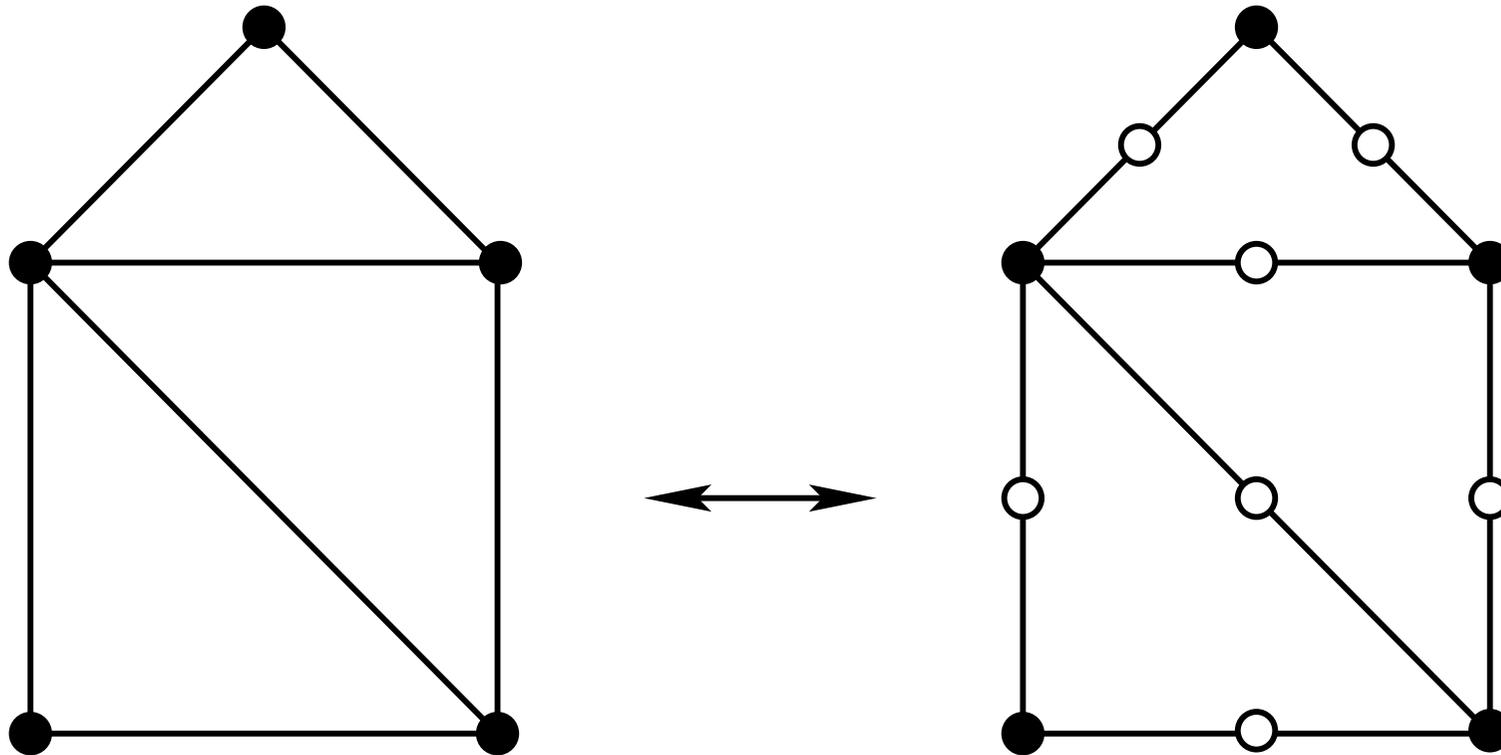
is the set of degrees of the white vertices, and

$$\gamma = (\gamma_1, \gamma_2, \dots, \gamma_{r+1}), \quad \sum_{k=1}^{r+1} \gamma_k = n,$$

is the set of degrees of the faces.

We denote the number of faces by $r + 1$ and not by r since we would like to distinguish the outer face. Note that r is determined by p and q by virtue of Euler's formula.

An **ordinary map**, with only one sort of vertices, is a particular case of a hypermap: **all its white vertices are of degree 2**.



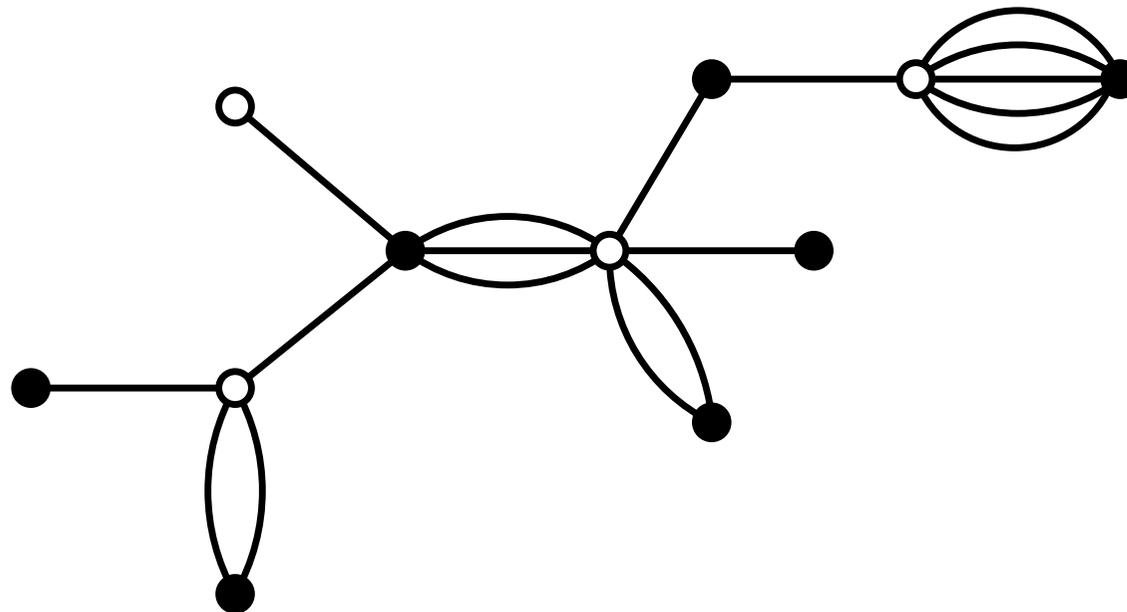
Remark. Two notions of the face degree are coherent.

In this talk we will be specially interested in the

hypermaps with all their faces

(except the outer one)

being of degree 1:

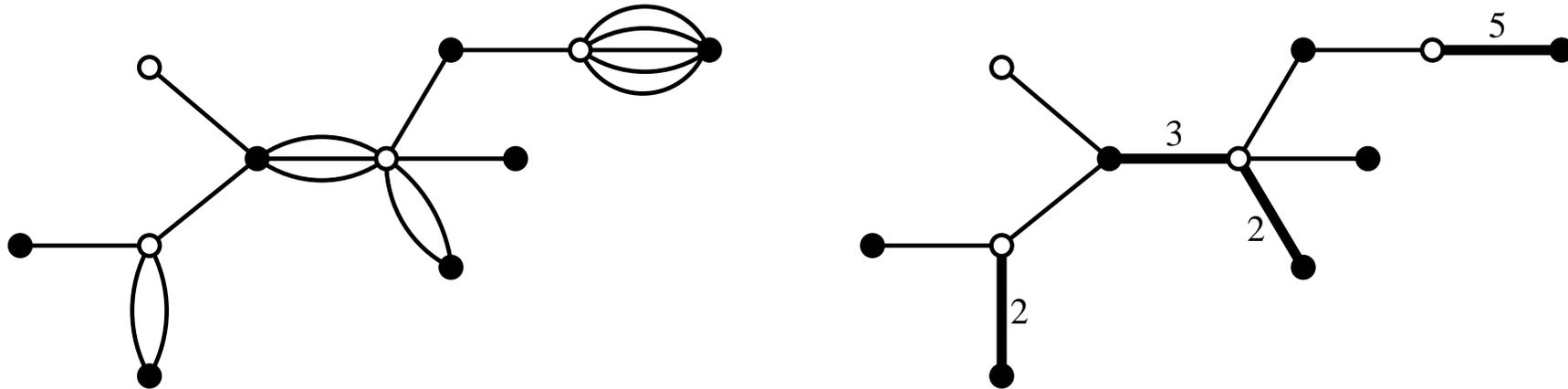


For the hypermaps of this type, the partition γ corresponding to the face degrees is as follows:

$$\gamma = (n - r, \underbrace{1, 1, \dots, 1}_r) = (n - r, 1^r).$$

In this case, the partition γ is completely determined by α and β (and even by the numbers p and q of their parts). Therefore, we will usually omit γ and write the passport of the hypermap as (α, β) .

It is very convenient to represent the hypermaps of the above type as **weighted trees**: the trees are much easier to handle.

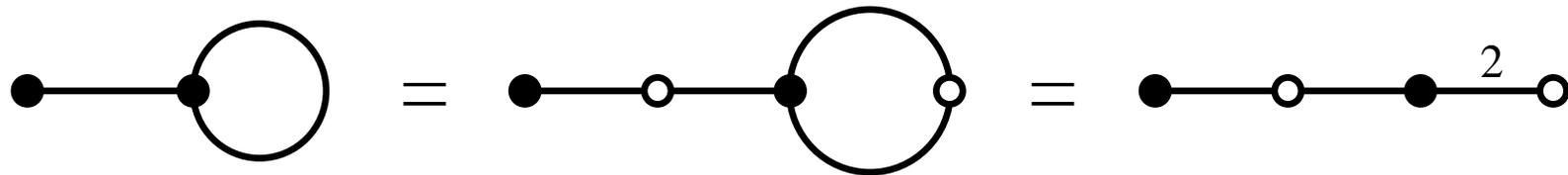


The **weight of an edge** is a positive integer.

The **degree of a vertex** is the sum of the weights of the edges incident to this vertex.

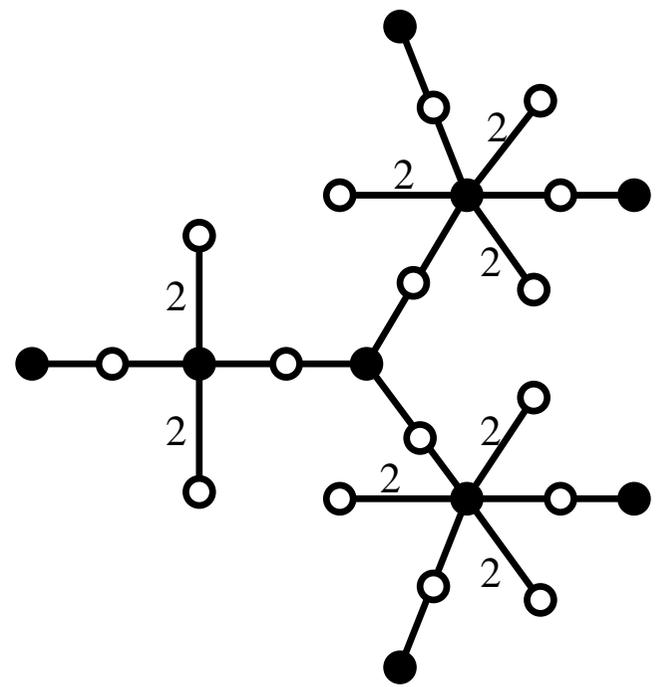
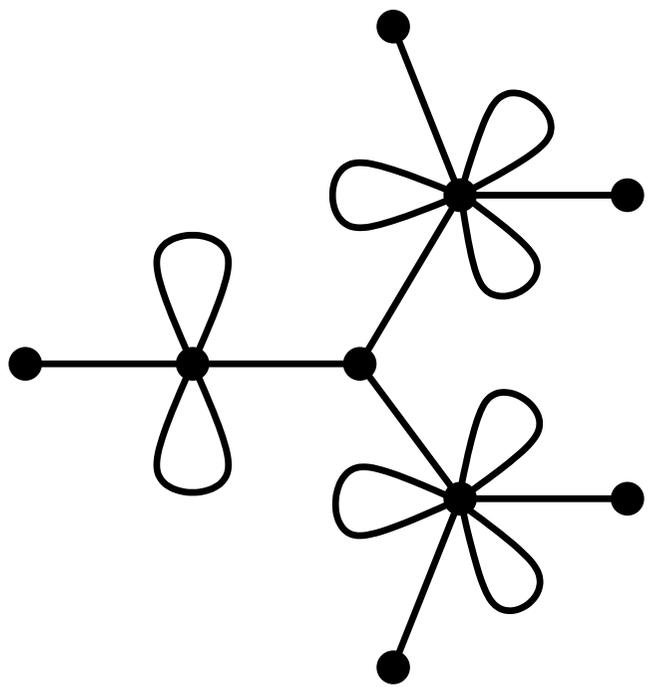
The **total weight** of a tree is equal to the number of edges of the underlying hypermap.

The same object may be drawn in various ways:



The same hypermap drawn in three different ways.

White vertices are always there but they may become “invisible” (or implicit).



III. Belyi functions

Let $\bar{\mathbb{C}}$ denote the complex Riemann sphere, $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, and let $f : \bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}}$ be a rational function with complex coefficients.

A **critical value** of f is a point $y \in \bar{\mathbb{C}}$ such that the equation

$$f(x) = y$$

has multiple roots.

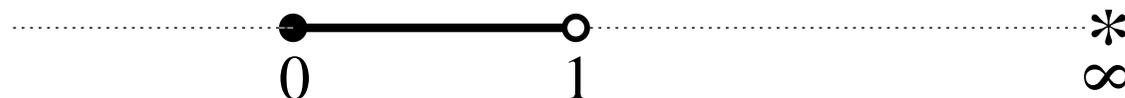
Belyi function is a rational function which has only **three critical values**, namely,

$$y = 0, \quad y = 1, \quad \text{and} \quad y = \infty.$$

The choice of $0, 1, \infty$ is a mere tradition. Indeed, any three points can be placed to any three given positions by making a linear fractional transformation.

Let us take a Belyi function f and consider the preimage

$$M = f^{-1}([0, 1]).$$



Then we get the following object:

- M is a planar hypermap with n edges, where $n = \deg f$.
- Black vertices are the points $x \in f^{-1}(0)$, and their degrees are equal to the multiplicities of the roots of $f(x) = 0$.
- White vertices are the points $x \in f^{-1}(1)$, and their degrees are equal to the multiplicities of the roots of $f(x) = 1$.
- Inside each face there is exactly one pole $x \in f^{-1}(\infty)$, and the degree of the face is equal to the multiplicity of this pole.

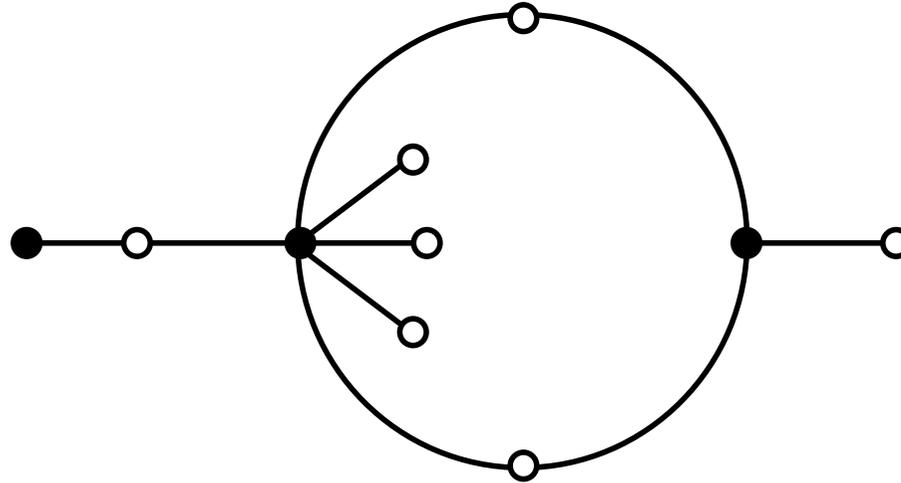
In the opposite direction:

- **Theorem.** For any planar hypermap M there exists a Belyi function f such that M is isomorphic to $f^{-1}([0, 1])$.

This function is unique up to a linear fractional transformation of the variable x .

- This statement is a particular case of **Riemann's Existence Theorem**.
- The coefficients of f can be made **algebraic numbers**.
- A striking consequence: the **Galois group** $\Gamma = \text{Aut}(\overline{\mathbb{Q}}|\mathbb{Q})$ acts on maps.
- The **theory of dessins d'enfants** (“children’s drawings” in French) studies combinatorial and other invariants of this action.

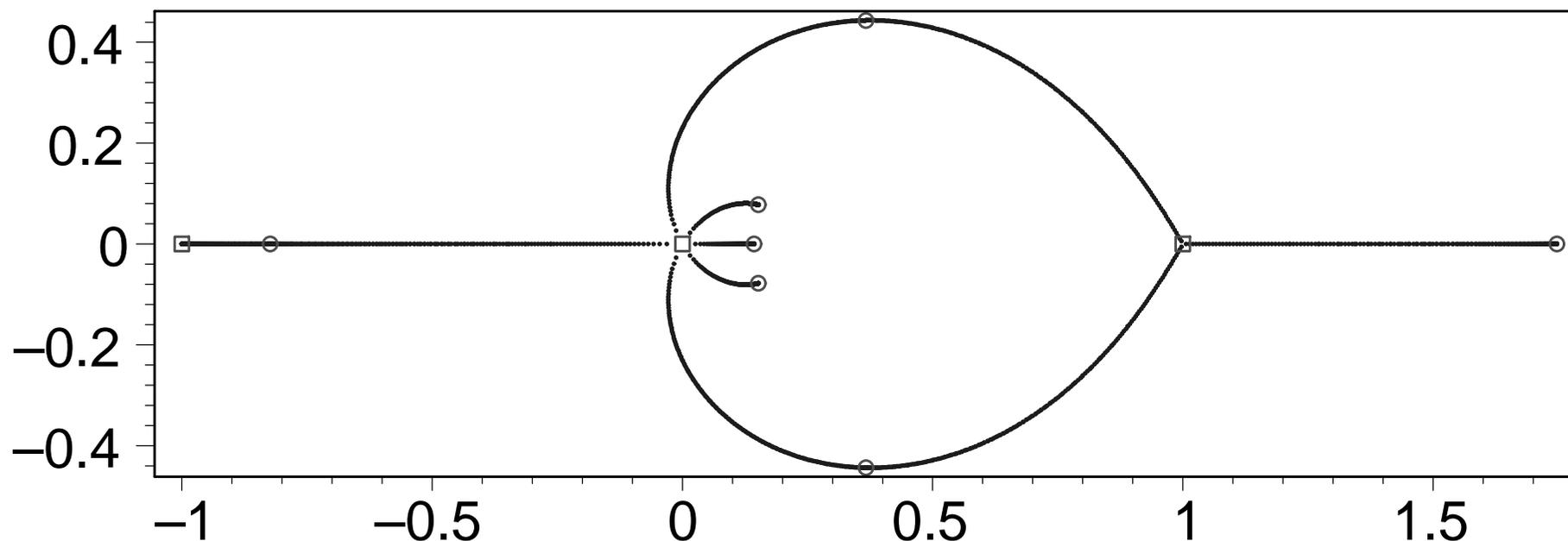
Example



$$f(x) = \frac{50000}{27} \cdot \frac{x^6 (x-1)^3 (x+1)}{(x^2 + 4x - 1)^5},$$

$$f(x) - 1 = \frac{1}{27} \cdot \frac{(11x^3 + x^2 - 3x + 3)^2 (413x^4 - 906x^3 + 352x^2 - 54x + 3)}{(x^2 + 4x - 1)^5}.$$

No other critical values except 0, 1 and ∞ .



The **dessin d'enfant** obtained as $f^{-1}([0, 1])$ for

$$f(x) = \frac{50\,000}{27} \cdot \frac{x^6 (x - 1)^3 (x + 1)}{(x^2 + 4x - 1)^5}.$$

Here the coefficients of f are rational, so f is **defined over** \mathbb{Q} . In general, the coefficients of Belyi functions are algebraic numbers.

IV. Back to polynomials

Recall the notation:

$$P(x) = \prod_{i=1}^p (x - a_i)^{\alpha_i}, \quad Q(x) = \prod_{j=1}^q (x - b_j)^{\beta_j}, \quad P - Q = R.$$

The main idea:

Instead of $P - Q = R$ write $P - R = Q$

Divide both parts of this equality by R

$$f = \frac{P}{R}, \quad f - 1 = \frac{Q}{R}.$$

The key observation is that f has already at least three critical values, namely, 0, 1 and ∞ .

So, what if f is a Belyi function?

First, what if it is not?

Denote:

- $n = \deg f$;
- $r =$ the number of distinct roots of R ;
- $n_y =$ the number of distinct preimages of $y \in \overline{\mathbb{C}}$.

Then the Riemann-Hurwitz formula implies that

$$r = (n + 1) - (p + q) + \sum_{y \notin \{0, 1, \infty\}} (n - n_y).$$

Notice that **we have already proved the bound**

$$\deg R \geq (n + 1) - (p + q).$$

$$r = (n + 1) - (p + q) + \sum_{y \notin \{0, 1, \infty\}} (n - n_y).$$

What do we need in order to attain the lower bound

$$\deg R = (n + 1) - (p + q) ?$$

The following two conditions are necessary and sufficient:

- No critical values except 0, 1 and ∞ , so that the sum

$$\sum_{y \notin \{0, 1, \infty\}} (n - n_y)$$

vanishes. **Hence, f is indeed a Belyi function.**

- **All roots of R are simple**, so that $r = \deg R$.

All roots of R are simple \iff all inner faces are of degree 1

- The corresponding hypermap has $n = \deg f$ edges and $p + q$ vertices:
 - p black vertices, of degrees $\alpha_1, \dots, \alpha_p$;
 - q white vertices, of degrees β_1, \dots, β_q ;

- Euler's formula:

$$(p + q) - n + \#(\text{faces}) = 2,$$

$$\#(\text{faces}) = (n + 2) - (p + q);$$

- the outer face corresponds to the pole at ∞ ;
the inner faces correspond to the roots of R ;
- Conclusion: $\deg R = (n + 1) - (p + q)$.

First result (A. Z.) A great simplification of Zannier's proof.

For a given (α, β) , the existence of a tree implies the attainability of the lower bound for $\deg R$.

It is much easier to draw trees than to work directly with polynomials.

Let us look how it works in the particular case

$$P = A^3, \quad Q = B^2, \quad R = A^3 - B^2.$$

We should draw a map with the following characteristics:

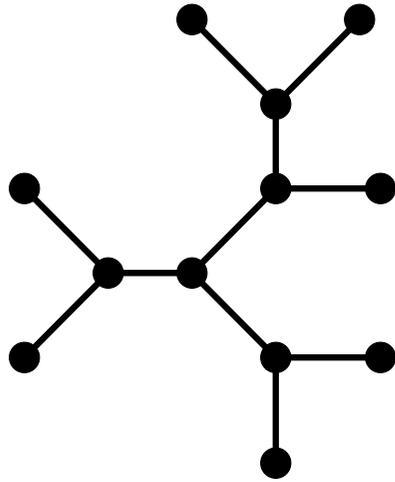
- all its black vertices are of degree 3;
- all its white vertices are of degree 2;

a very nice situation: we may forget about the white vertices and think of ordinary maps with only black vertices;

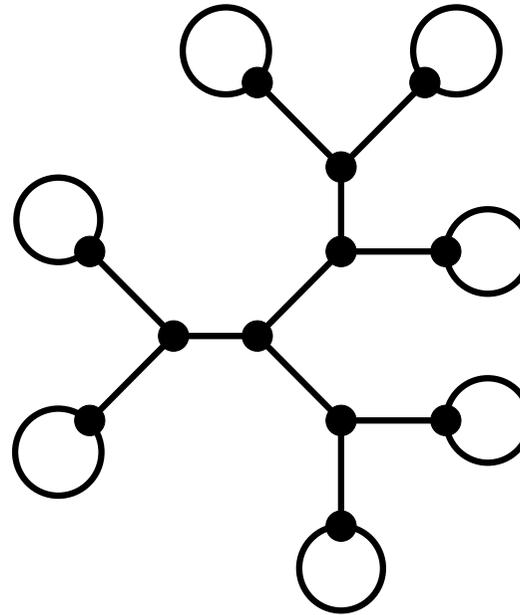
- all its faces except the outer one are of degree 1.

Recall that this is a problem which was studied by five famous mathematicians, and several less famous but still quite professional mathematicians, and nevertheless it remained open for 16 years (from 1965 to 1981).

Solution:



First stage



Second stage

(1) Draw a tree with inner nodes of degree 3. (2) Attach a loop to each leaf. Theorem is proved. \square

The general case, that of arbitrary partitions $\alpha, \beta \vdash n$, is slightly more elaborate but not much more difficult.

V. A number-theoretic application

Catalan's conjecture (1844): 2^3 and 3^2 are the only consecutive powers. (Proved in 2002.)

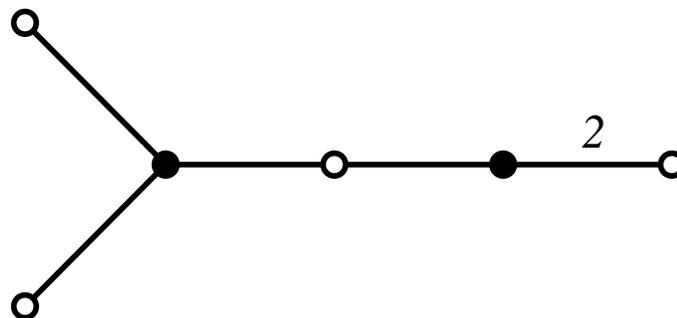
Hall's conjecture (1971): For any $\varepsilon > 0$ there exist infinitely many pairs of integers (a, b) such that

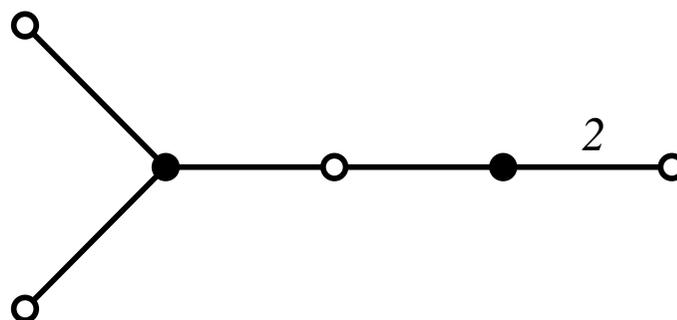
$$|a^3 - b^2| \leq C(\varepsilon) \cdot a^{1/2+\varepsilon}.$$

Theorem (Danilov, 1982): There exist infinitely many pairs of integers (a, b) such that

$$|a^3 - b^2| \leq C \cdot a^{1/2}.$$

The fundamental role in the proof is played by the following picture:





$$(2x^2 - 10)^3 - (2x^2 - 6x + 2)^2(2x^2 - 12x + 20) = 432x - 1080.$$

Substituting integer values of x we get “almost” a cube minus a square. We may also note that

$$432x - 1080 < C \cdot (2x^2 - 10)^{1/2}.$$

Let us try to make $2x^2 - 12x + 20$ a perfect square.

$$2x^2 - 12x + 20 = 2(x - 3)^2 + 2 = 2z^2 + 2.$$

$$2z^2 + 2 = y^2 \quad \Leftrightarrow \quad y^2 - 2z^2 = 2.$$

The equation $y^2 - 2z^2 = 2$ is a **Pell-like equation**. It has infinitely many solutions.

The smallest solution is $(y_0, z_0) = (2, 1)$. All the other solutions are obtained as

$$\begin{pmatrix} y_n \\ z_n \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}^n \cdot \begin{pmatrix} y_0 \\ z_0 \end{pmatrix}.$$

VI. Galois theory

In general, coefficients of Belyi functions are algebraic numbers.

Replace simultaneously all these numbers by their algebraic conjugates: we get another Belyi function, and hence another *dessin*.

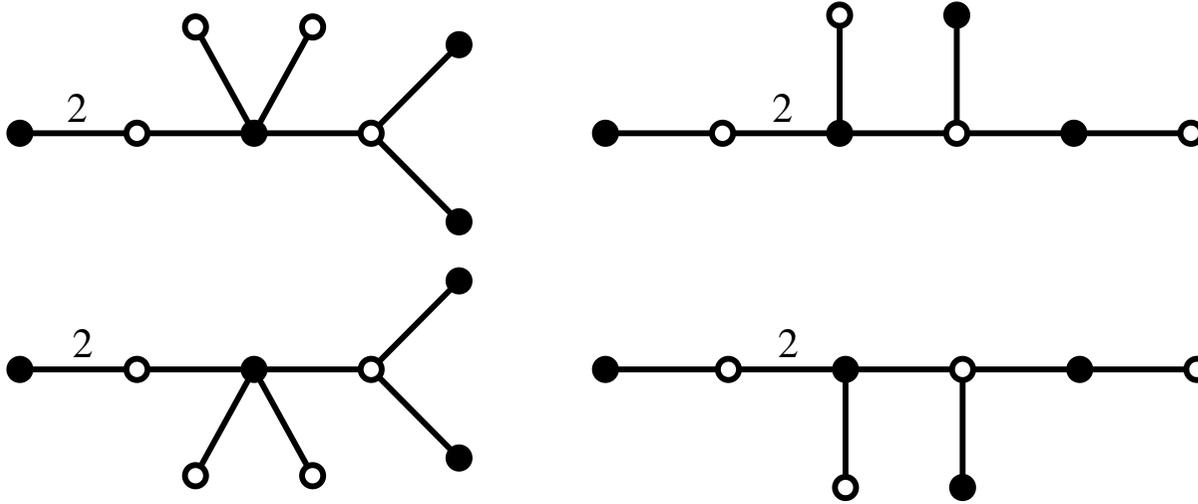
This is the **Galois action** on the dessins.

The number field to which the coefficients belong is not arbitrary: it is an *objective characteristic* of a dessin. It is called **field of moduli** of the dessin.

The degree of the field of moduli is equal to the size of the Galois orbit.

(In general, there are some technical difficulties concerning this field, but in our particular case of weighted trees these difficulties do not appear.)

Example: a Galois orbit of de degree 4



$$P = x^4(x + 1)^2(x^2 + ax + b)$$

$$Q = (x^2 + cx + d)^3(x^2 + ex + f)$$

$$R = gx + h$$

$$a, b, c, d, e, f, g, h \in \mathbb{Q} \left(\sqrt{-455 + 952\sqrt{-14}} \right).$$

More exactly, this means that

$$\begin{aligned}a &= a_0 + a_1t + a_2t^2 + a_3t^3, \\b &= b_0 + b_1t + b_2t^2 + b_3t^3, \\c &= c_0 + c_1t + c_2t^2 + c_3t^3, \\&\dots\dots \\h &= h_0 + h_1t + h_2t^2 + h_3t^3,\end{aligned}$$

where all the coefficients a_0, a_1, \dots, h_3 in the right-hand side are rational, and **substituting in these expressions four values**

$$t = \pm\sqrt{-455 \pm 952\sqrt{-14}}$$

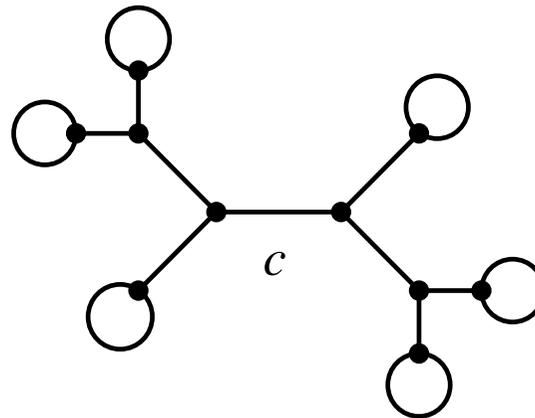
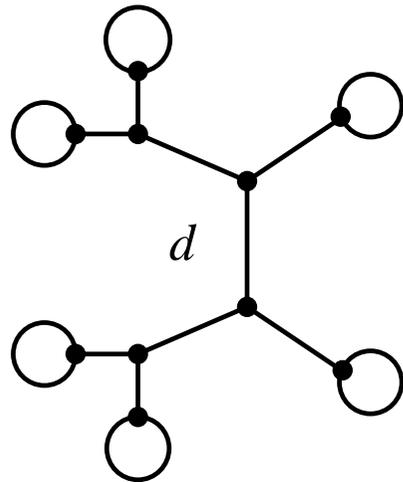
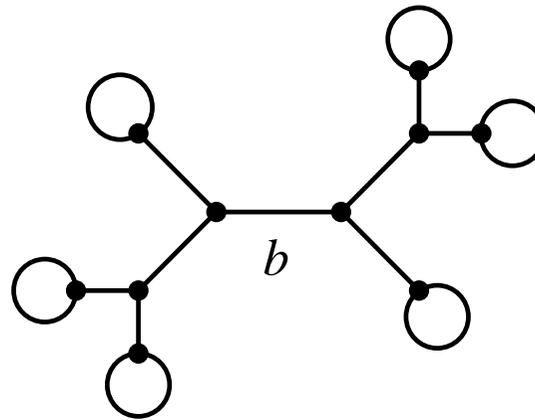
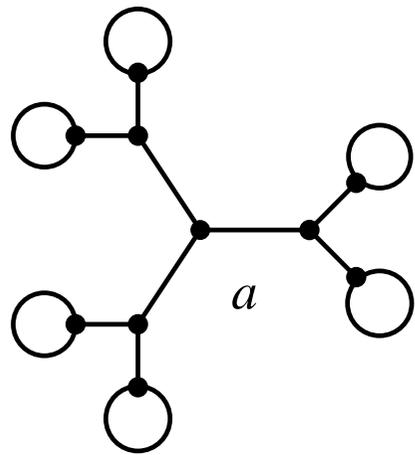
we get four Belyi functions corresponding to the above maps.

There are many **combinatorial Galois invariants**:

- **Passport** = triple (α, β, γ) of partitions of n which represent the degrees of black vertices, of white vertices, and of faces.

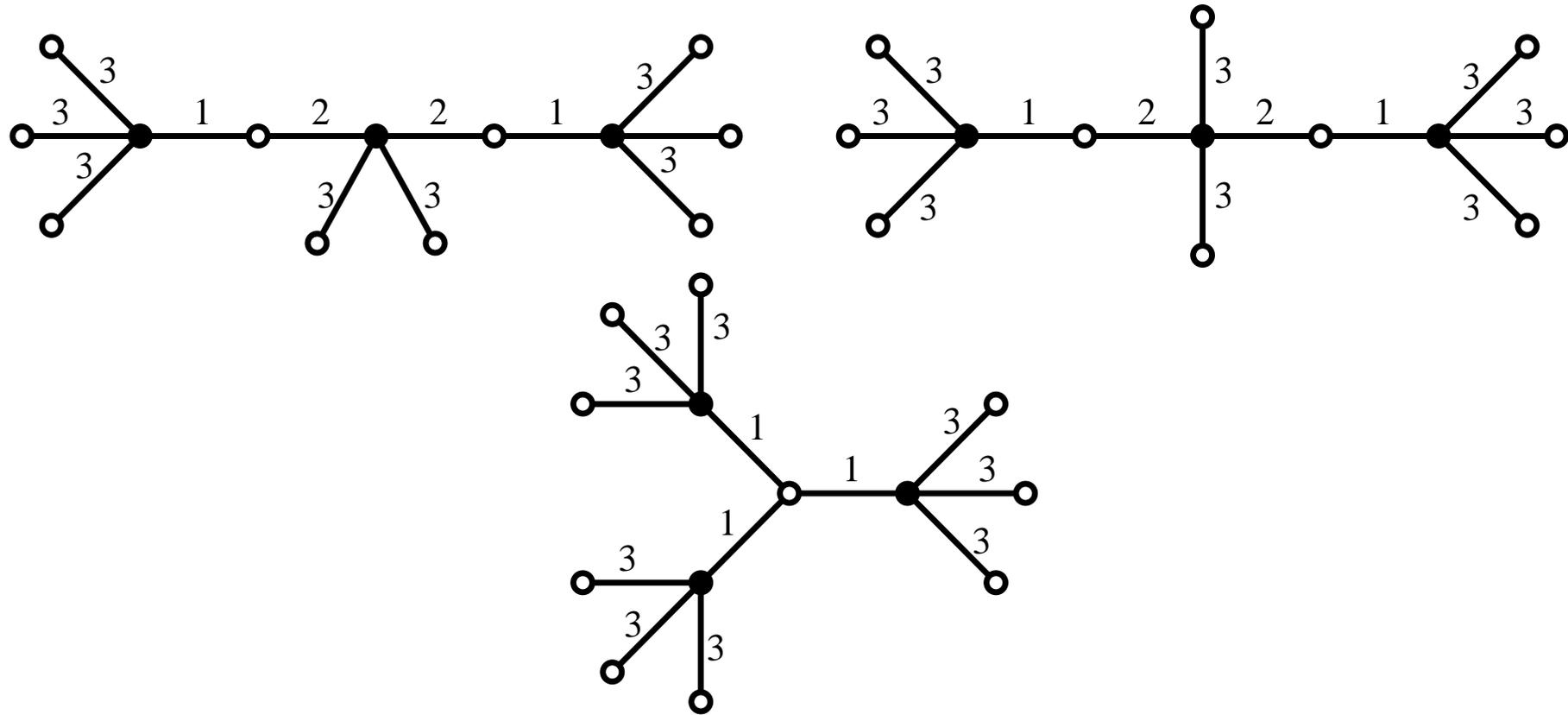
In our case γ is determined by α and β : $\gamma = (n - r, 1^r)$ where $r = (n + 1) - (p + q)$.

- Symmetry.
- Composition of coverings.
- Monodromy group.
- Self-duality.
- Etc.



Here $\deg A = 10$, $\deg B = 15$, $\deg(A^3 - B^2) = 6$. Computations:

(a)	B. Birch	1965	defined over \mathbb{Q}
(d)	N. Elkies	2000	defined over \mathbb{Q}
(b, c)	T. Shioda	2005	defined over $\mathbb{Q}(\sqrt{-3})$



Here $\deg A = 3$, $\deg B = 10$, $\deg(A^{10} - B^3) = 18$.

All three trees are defined over \mathbb{Q} .

Computation: Beukers & Stewart, 2010.

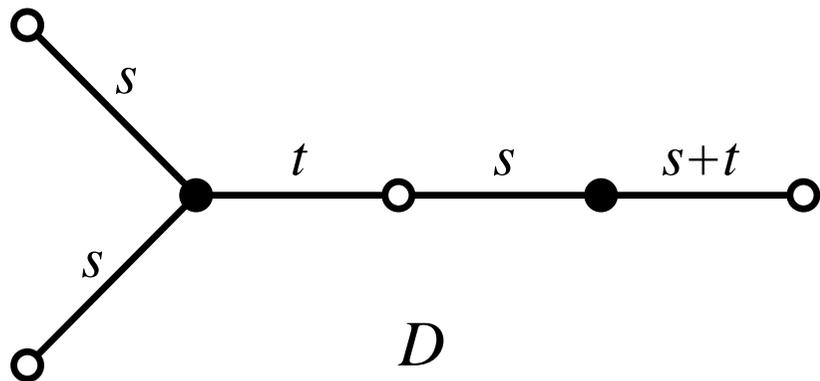
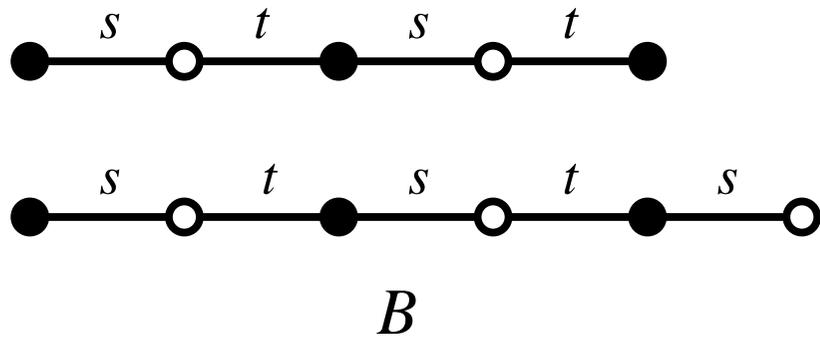
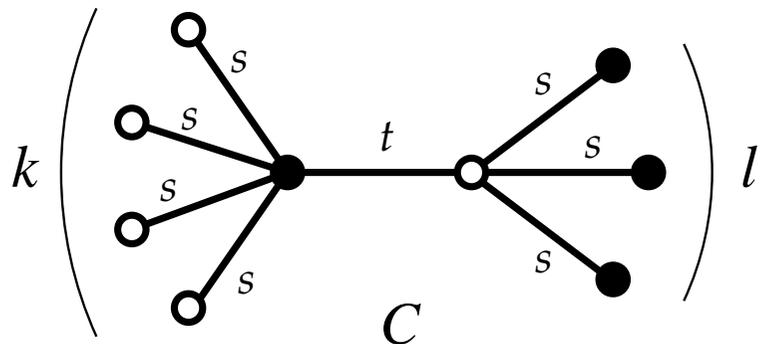
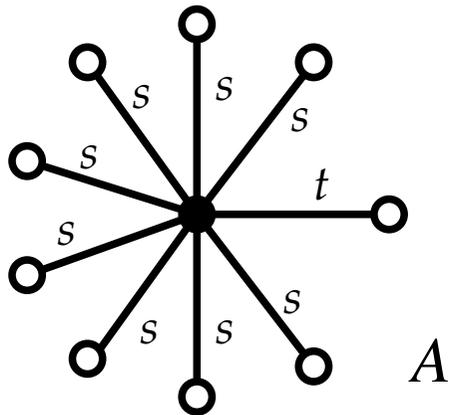
Proposition. If, for a given passport (α, β) , the corresponding tree is unique, then it is defined over \mathbb{Q} .

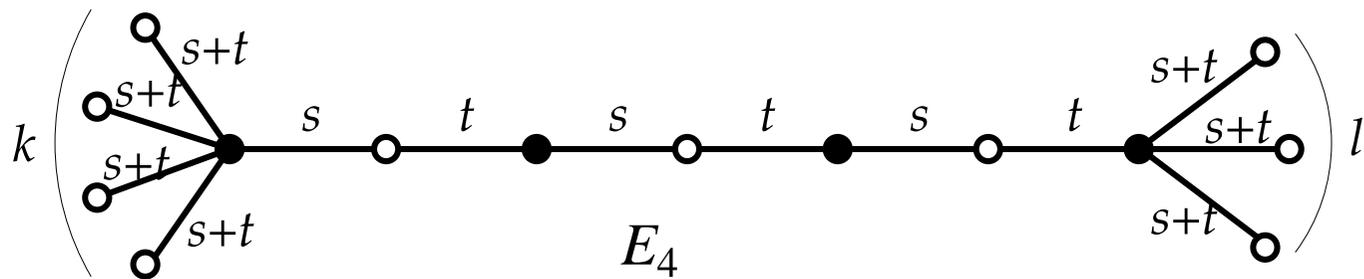
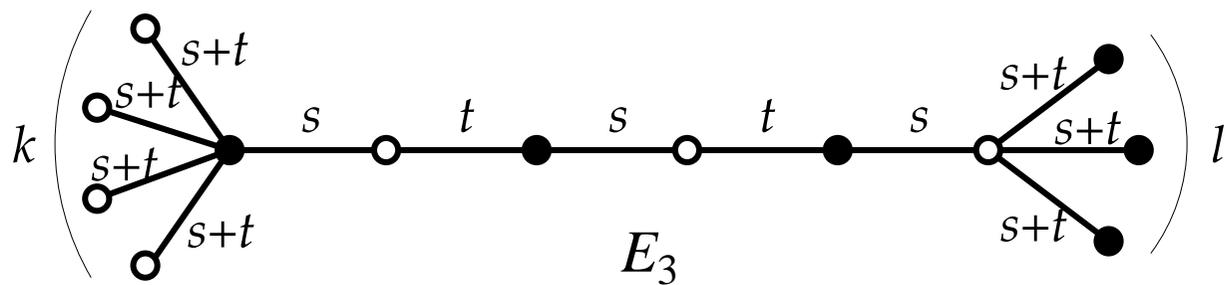
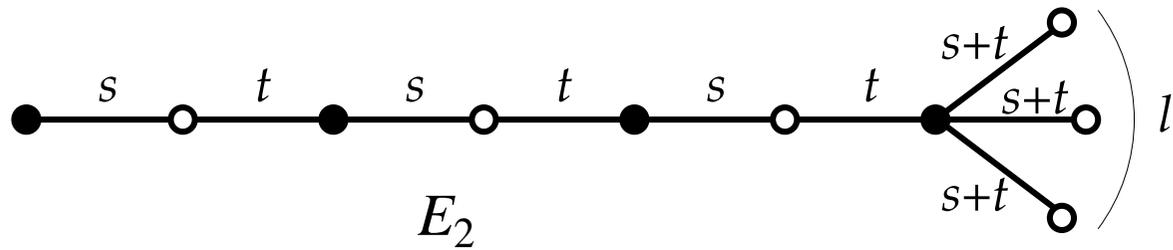
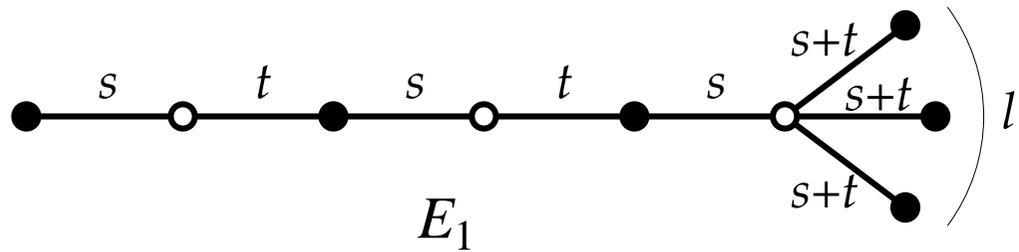
We call such trees **unitrees**.

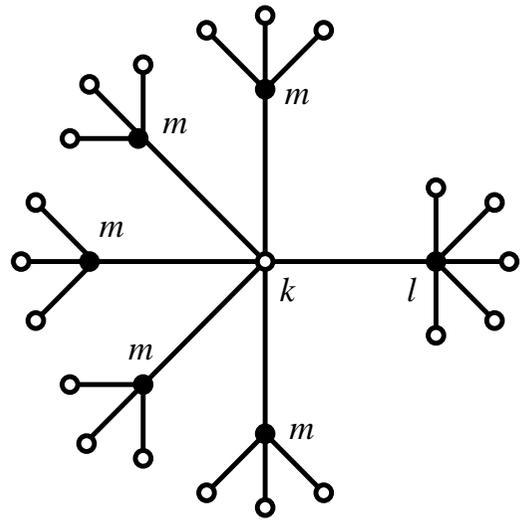
Second result (F. Pakovich, A. Z.): A complete classification of unitrees. There are:

- 10 infinite series, and
- 10 sporadic trees.

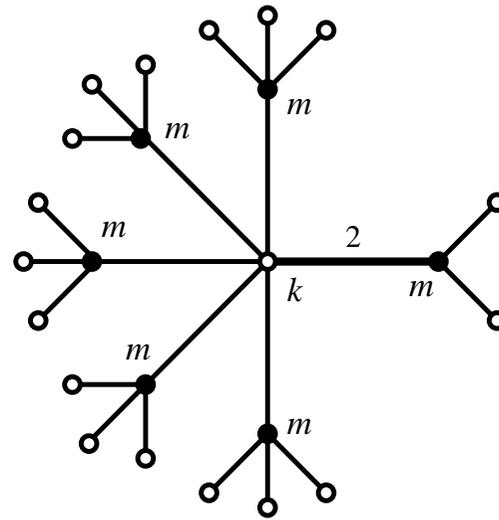
A very long and cumbersome proof. **Pictures follow...**



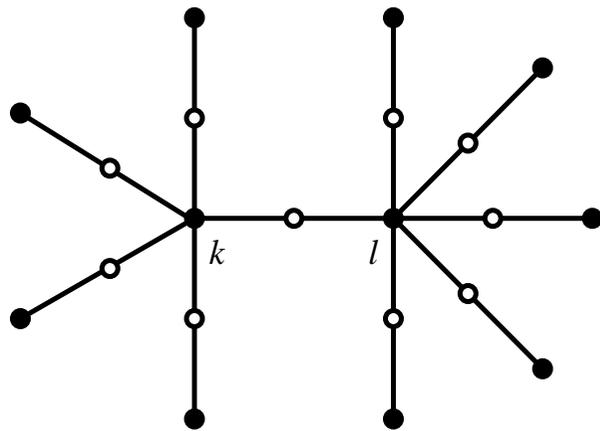




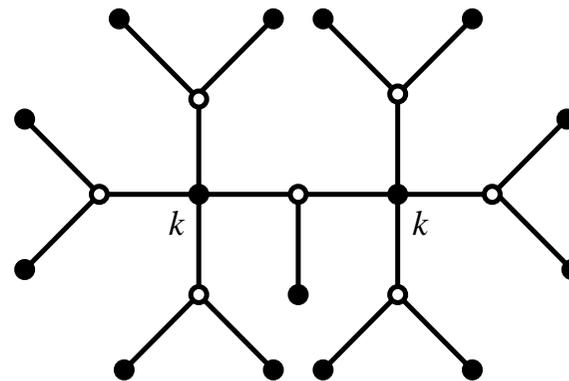
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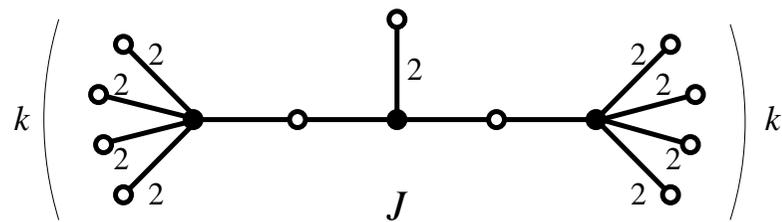
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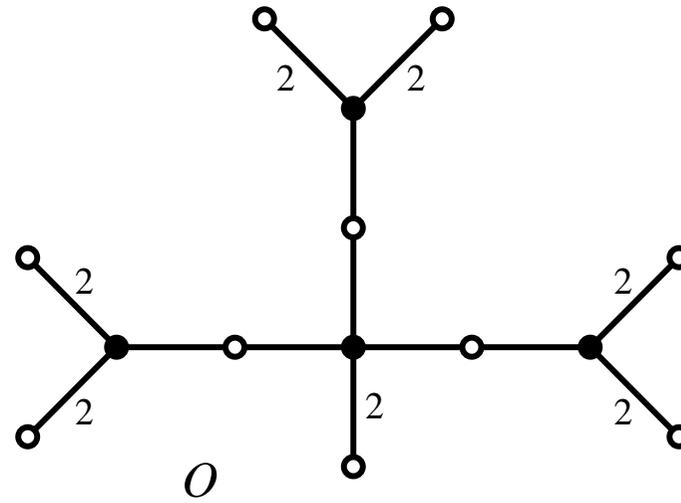
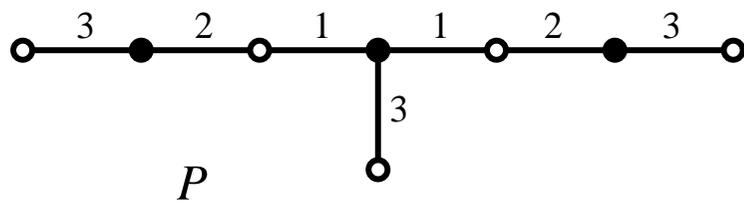
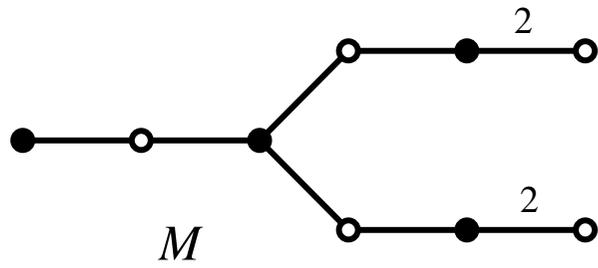
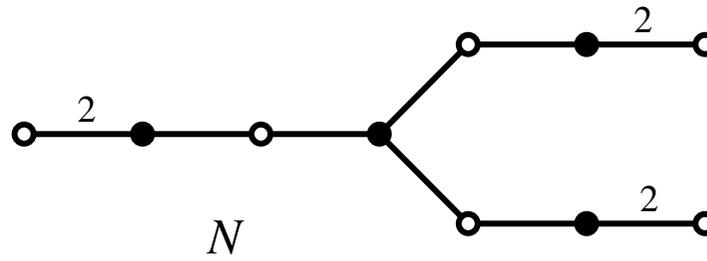
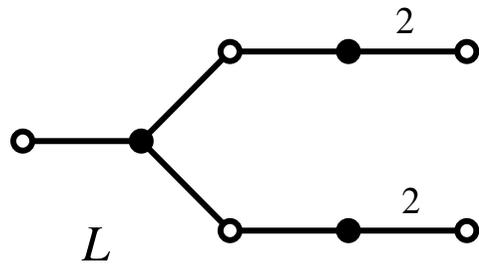
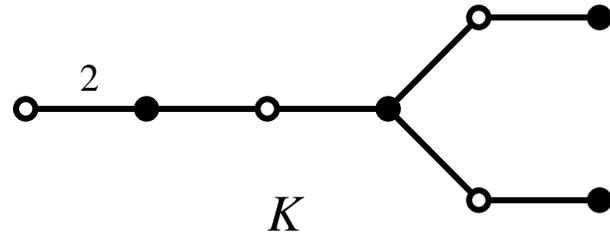
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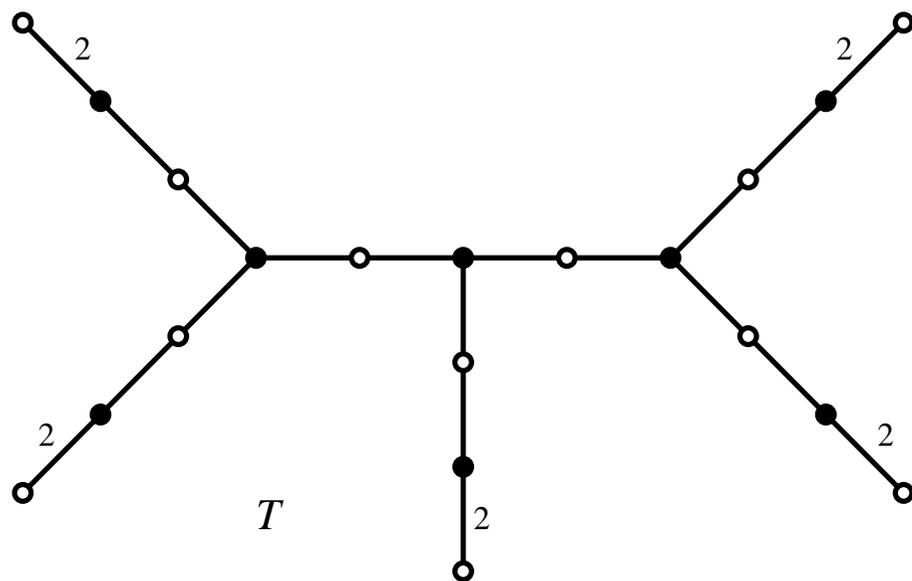
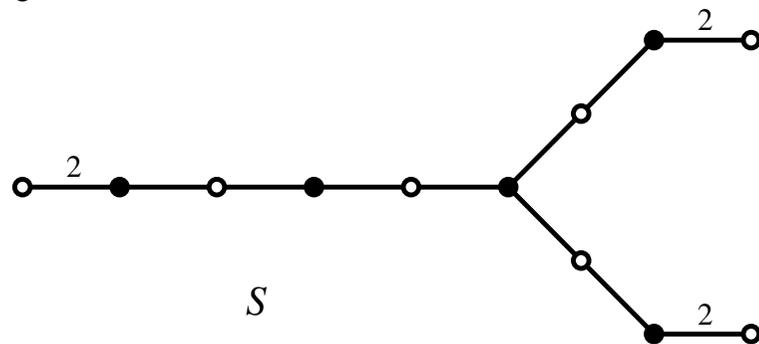
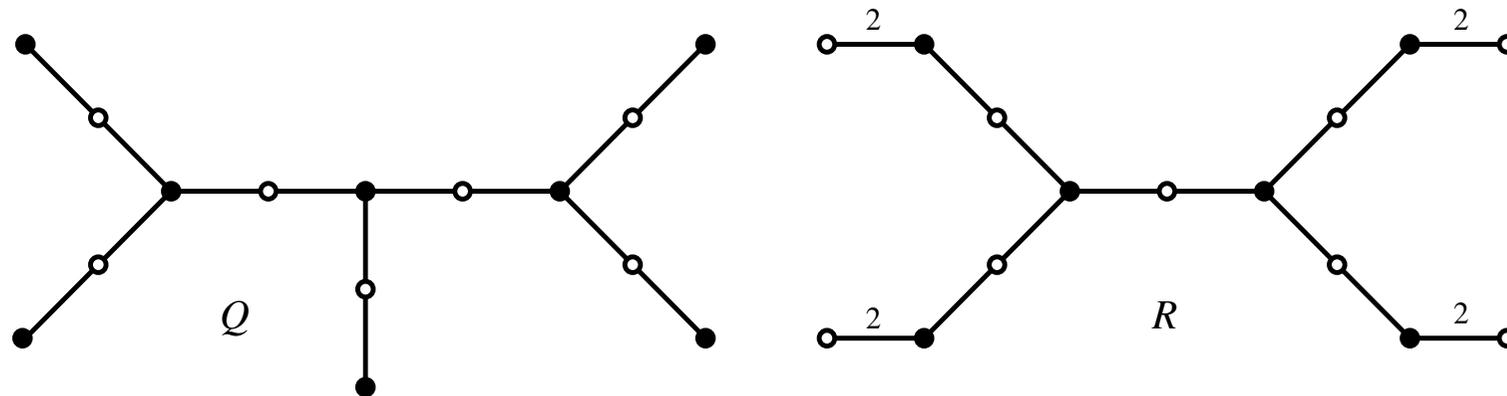


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Third result (F. Pakovich, A. Z.): Belyi functions for all unitrees are computed.

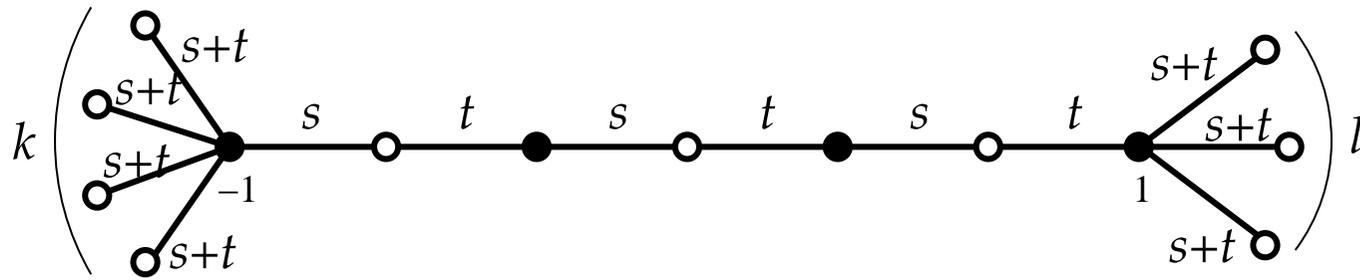
For individual trees the computation may be difficult but a verification is trivial.

For infinite series the situation is significantly more complicated.

- One has to compute a lot of examples;
- to guess a general pattern;
- instead of verification one needs a **proof**,

and every stage is non-trivial.

An example follows. . .



$$m_1 = l(s + t) + t$$

$$m_2 = k(s + t) + s$$

$$p = \text{number of black vertices of degree } s + t$$

$$q = \text{number of white vertices (all of them are of degree } s + t)$$

$$a = l + t/(s + t)$$

$$b = k + s/(s + t)$$

$$P = \left(\frac{x - 1}{2}\right)^{m_1} \cdot \left(\frac{x + 1}{2}\right)^{m_2} \cdot J_p(a, b, x)^{s+t}$$

$$Q = J_q(-a, -b, x)^{s+t}$$

Here J_p and J_q are Jacobi polynomials of degrees p and q .

Notice the negative parameters $-a$ and $-b$ in J_q .

Classical orthogonal Jacobi polynomials $J_m(a, b, x)$ of degree m with parameters a and b are defined for $a, b > -1$.

It turns out that they are polynomials not only in x but also in a, b . Therefore, they can be extended to arbitrary complex parameters.

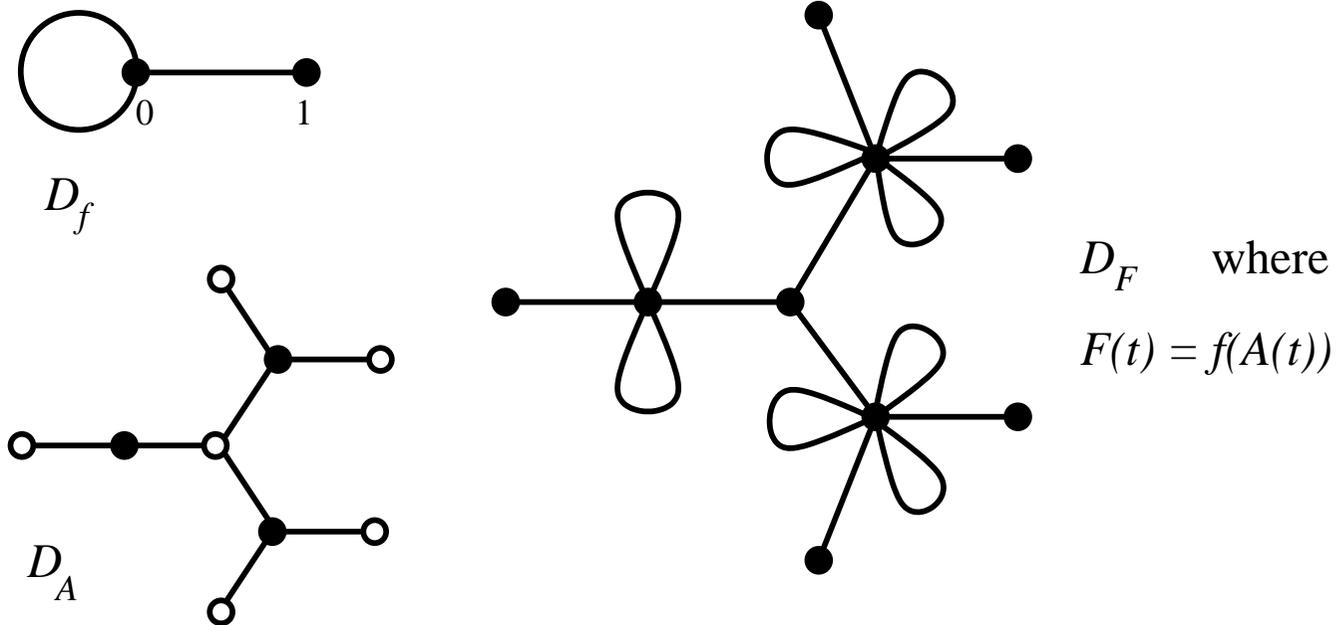
The formulas on the previous slide contain

$$J_p(a, b, x) \quad \text{and} \quad J_q(-a, -b, x) \quad \text{with} \quad a, b > 1$$

so that J_q is a generalized Jacobi polynomial.

How can one guess this pattern is a mystery (Nicolas Magot, 1997, for a particular case).

Example: A composition of two unitrees. It is defined over \mathbb{Q} though it is not itself a unitree.



$$f = -\frac{64x^3(x-1)}{8x+1}, \quad A = \frac{1}{5^5} \cdot (t^2+4)^3(3t+8)^2.$$

VII. Groups

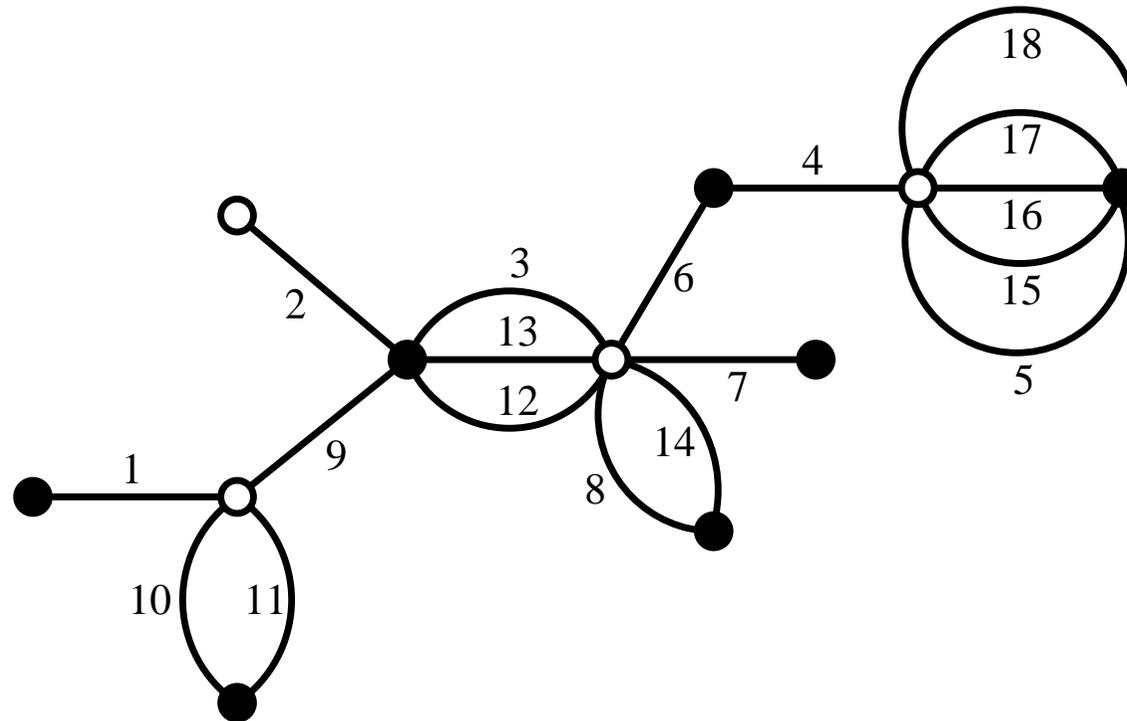
Every rational function $f : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ realizes a **ramified covering** of the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by itself.

For Belyi functions, **the monodromy group is an invariant of the Galois action.**

Ritt's theorem

A covering is a non-trivial composition of coverings of smaller degrees if and only if its monodromy group is imprimitive.

The monodromy group can be found on the basis of the dessin:



$$a = (2, 9, 12, 13, 3)(4, 6)(5, 18, 17, 16, 15)(8, 14)(10, 11)$$

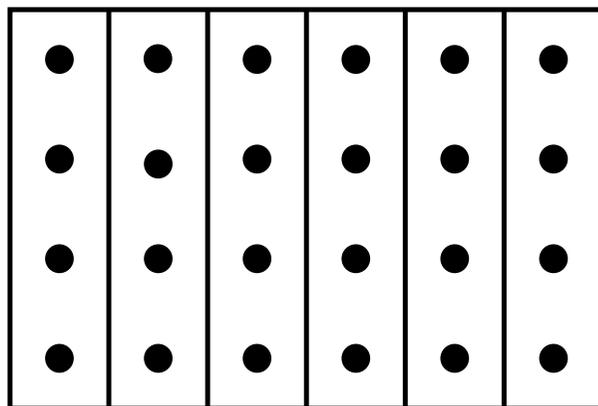
$$b = (1, 10, 11, 9)(3, 13, 12, 8, 14, 7, 6)(4, 5, 15, 16, 17, 18)$$

$$c = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10) = (ab)^{-1}$$

Monodromy group:

$$G = \langle a, b \rangle$$

Definition. A permutation group acting on a set of n points is *imprimitive* if the set can be split in m disjoint blocks of equal size, $1 < m < n$, such that an image a block under the action of every element of the group is once again a block.



If such a partition does not exist, the group is *primitive*.

The groups S_n and A_n are primitive. For a prime p , every group of degree p is primitive.

Primitive groups are not frequent...

Degree	2	3	4	5	6	7	8	9	10	11	12	13	14
Primitive	1	2	2	5	4	7	7	11	9	8	6	9	4
Transitive	1	2	5	5	16	7	50	34	45	8	301	9	63

Degree	15	16	17	18	19	20	21	22	23
Primitive	6	22	10	4	8	4	9	4	7
Transitive	104	1954	10	983	8	1117	164	59	7

Degree	24	25	26	27	28	29	30	31
Primitive	5	28	7	15	14	8	4	12
Transitive	25000	211	96	2392	1854	8	5712	12

Degree	32
Primitive	5
Transitive	2 801 324

Fourth result: (N. Adrianov, A. Z.) Complete classification of primitive monodromy groups of weighted trees:

Proposition. Beside the symmetric and alternating groups S_n and A_n for all n , and the cyclic and dihedral groups C_p and D_{2p} for p prime, there are finitely many primitive monodromy groups of weighted trees.

There are:

- 184 trees (up to a color exchange);
- 85 Galois orbits;
- 34 groups;
- The highest degree of a group from this list is 32.

Weight	Group	Order	Orbits	Trees
5	$AGL_1(5)$	20	1	2
6	$PSL_2(5)$	60	2	2
	$PGL_2(5)$	120	7	7
7	$AGL_1(7)$	42	1	2
	$PSL_3(2)$	168	2	4
8	$A\Gamma L_1(8)$	168	1	4
	$PSL_2(7)$	168	2	2
	$PGL_2(7)$	336	6	7
	$ASL_3(2)$	1 344	6	14
9	$A\Gamma L_1(9)$	144	1	2
	$AGL_2(3)$	432	2	4
	$PSL_2(8)$	504	3	3
	$P\Gamma L_2(8)$	1 512	4	10
10	$PGL_2(9)$	720	3	3
	$P\Gamma L_2(9)$	1 440	2	2
11	$PSL_2(11)$	660	1	2
	M_{11}	7 920	1	2
12	$PGL_2(11)$	1 320	2	4
	M_{11}	7 920	3	10
	M_{12}	95 040	9	20

Weight	Group	Order	Orbits	Trees
13	$\text{PSL}_3(3)$	5 616	3	12
14	$\text{PSL}_2(13)$	1 092	1	1
	$\text{PGL}_2(13)$	2 184	2	4
15	$\text{PSL}_4(2)$	20 160	3	6
16	$\text{A}\Gamma\text{L}_2(4)$	5 760	1	2
	$\text{AGL}_4(2)$	322 560	4	12
17	$\text{PSL}_2(16)$	4 080	1	1
	$\text{PSL}_2(16) \rtimes C_2$	8 160	1	1
20	$\text{PGL}_2(19)$	6 840	1	3
21	$\text{P}\Gamma\text{L}_3(4)$	120 960	1	2
23	M_{23}	10 200 960	1	4
24	M_{24}	244 823 040	5	18
31	$\text{PSL}_5(2)$	9 999 360	1	6
32	$\text{ASL}_5(2)$	319 979 520	1	6
Total	34	—	85*	184

*For certain orbits we are not entirely sure that the “orbit” in question is indeed a single orbit and not a union of several orbits.

Ingredients of the proof:

1. A theorem by Jordan (1873), often attributed to Marggraf, stating that a primitive permutation group of degree n containing a permutation with cycle structure $(n - r, 1^r)$ is $(r + 1)$ -transitive.

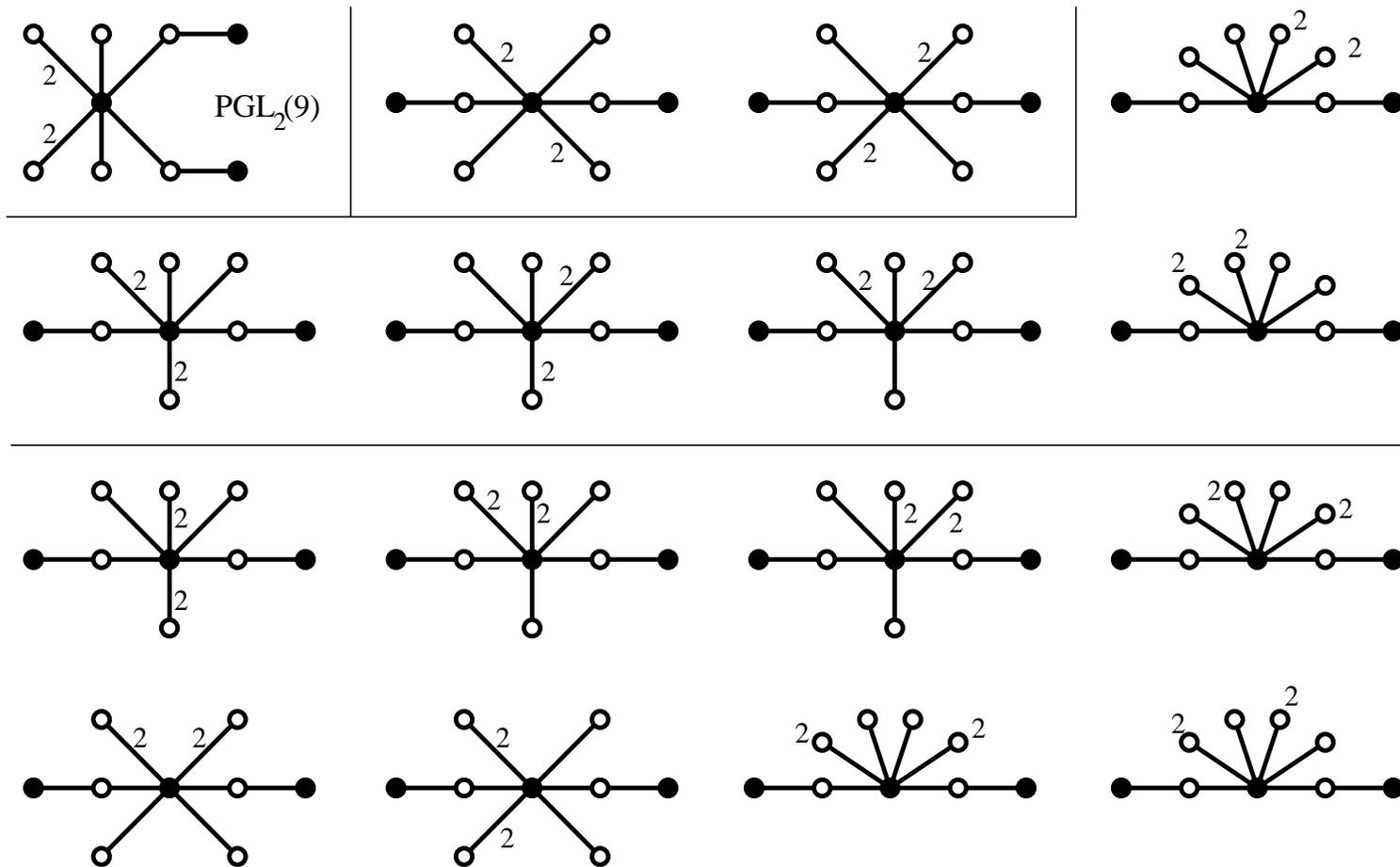
Thanks to the classification of finite simple groups, all multiply transitive groups are known.

2. G. Jones (2012): Classification of primitive groups containing a permutation with cycle structure $(n - r, 1^r)$.

Based on Jones's own results and also on those by P. Müller, W. Feit and others.

3. K. Magaard, S. Shpectorov, G. Wang (2011): Classification of affine groups which admit a planar generation.
4. GAP calculations: all primitive groups up to degree 127 were tested.

An example in which several invariants are involved



Weight $n = 10$, passport $(8^1 1^2, 2^4 1^2, 8^1 1^2)$.

16 trees, four Galois orbits.

The orbit sizes: **1** (group $\text{PGL}_2(9)$), **2** (symmetry),
5 (self-dual), **8** (not self-dual).

The monodromy group of the latter $5 + 8 = 13$ trees is S_{10} .

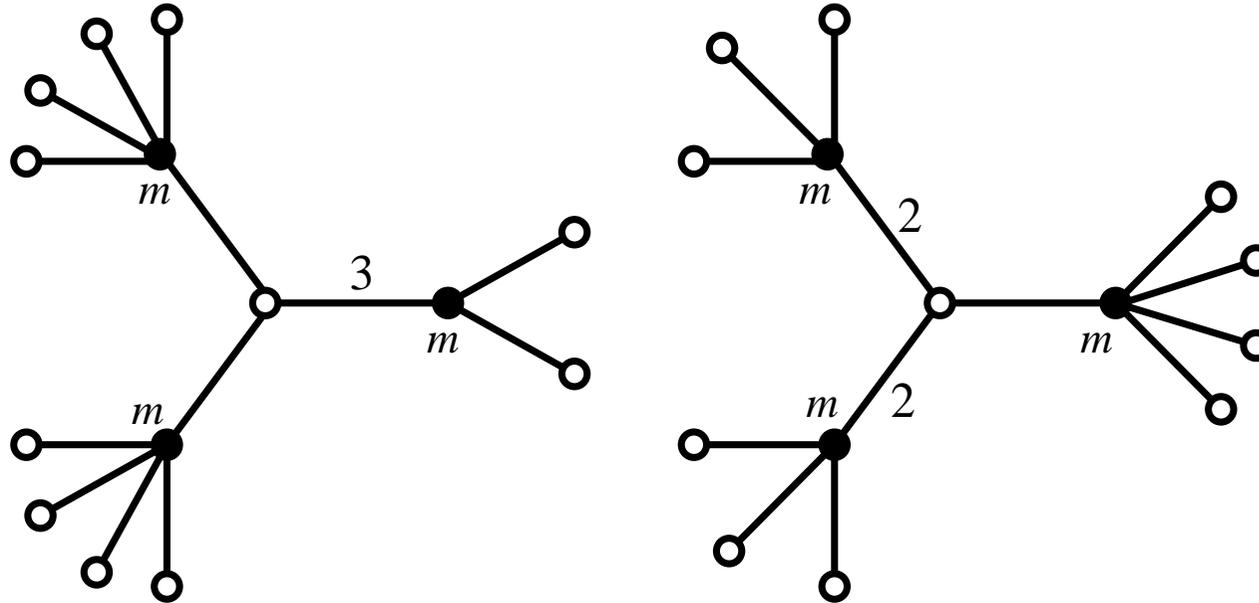
VIII. Diophantine invariants

Sometimes a splitting into more than one Galois orbit is due to numeric relations between vertex degrees. We call such relations **diophantine invariants**.

The theory of dessins d'enfants sets as its goal the search for a full set of combinatorial invariants of the Galois action. Therefore, the existence of diophantine invariants is a kind of “bad news” for the theory.

But there is also good news: everything related to diophantine equations is very beautiful.

An example follows. . .



Weight $3m$, vertex degrees $(m^3, 5^1 1^{3m-5})$.

- either one orbit over a real quadratic field;
- or two orbits over \mathbb{Q} .

Computation gives the field $\mathbb{Q}(\sqrt{D})$ where

$$D = 3(2m - 1)(3m - 2).$$

Question: can $D = 3(2m - 1)(3m - 2)$ be a perfect square?

1. $2m - 1$ and $3m - 2$ are coprime:

$$\begin{aligned}3m - 2 &= 1 \cdot (2m - 1) + (m - 1), \\2m - 1 &= 2 \cdot (m - 1) + 1.\end{aligned}$$

2. Only $2m - 1$ can be divisible by 3 ($3m - 2$ cannot).

3. Hence, $3(2m - 1) = 6m - 3$ and $3m - 2$ must both be squares.

4. Denoting

$$6m - 3 = a^2, \quad 3m - 2 = b^2$$

we get

$$a^2 - 2b^2 = 1.$$

Pell equation! (Plus the condition of a being a multiple of 3.)

Pell's name was attributed to this equation by error...

- Pythagoras (VI before J. C.): the equation $a^2 - 2b^2 = 0$ does not have integral solutions; then what about $a^2 - 2b^2 = 1$?
- Brahmagupta (VII)
- Bhaskara II (XII)
- Narayana Pandit (XIV)
- Brouncker (XVII)
- Fermat, Euler, Lagrange, Abel, ... (XVII–XIX)
- Dirichlet (XIX)
- etc.

There are infinitely many solutions. They are obtained as follows: let us take the smallest solution $(a_0, b_0) = (1, 0)$; then

$$\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}^n \cdot \begin{pmatrix} a_0 \\ b_0 \end{pmatrix}, \quad n = 0, 1, 2, \dots$$

It turns out that for every other solution the parameter a is a multiple of 3.

Recall that $a^2 = 6m - 3$, so that $m = \frac{a^2 + 3}{6}$ where m is the degree of the three black vertices.

First values of the parameter m are

1 634, 1 884 962, 2 175 243 842, ...

Growth exponent: $(3 + 2\sqrt{2})^4 \approx 1154$.

IX. Enumeration

Theorem (A. Z.) Let a_n be the number of **rooted** trees of weight n , and let $f(t) = \sum_{n \geq 0} a_n t^n$ be the generating function of the sequence a_n . Then

$$f(t) = \frac{1 - t - \sqrt{1 - 6t + 5t^2}}{2t}$$

$$= 1 + t + 3t^2 + 10t^3 + 36t^4 + 137t^5 + 543t^6 + 2219t^7 + \dots$$

This function can also be represented as a chain fraction

$$f(t) = \frac{1}{1 - \frac{h}{1 - \frac{h}{1 - \frac{h}{1 - \dots}}}} \quad \text{where} \quad h = \frac{t}{1 - t}.$$

A Catalan-type recurrence for the numbers a_n :

$$a_0 = 1, \quad a_1 = 1, \quad a_{n+1} = a_n + \sum_{k=0}^n a_k a_{n-k} \quad \text{for } n \geq 1.$$

Another recurrence:

$$a_n = \frac{6n-3}{n+1} \cdot a_{n-1} - \frac{5n-10}{n+1} \cdot a_{n-2} \quad \text{for } n \geq 2.$$

Asymptotic: $a_n \sim \frac{1}{2} \sqrt{\frac{5}{\pi}} \cdot 5^n n^{-3/2}.$

In fact, a_n is the [Sequence A002212](#) of the “On-Line Encyclopedia of Integer Sequences”. It has many different interpretations.

Let $b_{m,n}$ be the number of rooted trees of weight n with m edges, and let $h(s, t) = \sum_{m,n \geq 0} b_{m,n} s^m t^n$. Then

$$\begin{aligned} h(s, t) &= \frac{1 - t - \sqrt{1 - (2 + 4s)t + (1 + 4s)t^2}}{2st} \\ &= 1 + st + (s + 2s^2)t^2 + (s + 4s^2 + 5s^3)t^3 \\ &\quad + (s + 6s^2 + 15s^3 + 14s^4)t^4 + \dots \end{aligned}$$

Explicit formula for $b_{m,n}$:

$$b_{m,n} = \binom{n-1}{m-1} \cdot \frac{1}{m+1} \binom{2m}{m}.$$



Studying trees

Thank you !

Merci !

Спасибо !

