COUNTING UPPER INTERACTIONS IN DYCK PATHS

YVAN LE BORGNE

Abstract. A Dyck word w of size n is a shuffle of n copies of the word $x\overline{x}$. An upper interaction in w is an occurrence of a factor $\overline{x}^k x^k$ where $k \geq 1$. We present different methods to enumerate Dyck words according to the size and the number of upper interactions. The generating function has a rather unusual form: it is the ratio of two q-series where occurs an algebraic term. The first method mainly involves calculation over formal power series. Our next two methods interpret different steps of the previous calculation by manipulations or factorisations of the Dyck words.

Version française. Un mot de Dyck w de taille n est un mélange de n copies du mot $x\overline{x}$. Une interaction supérieure dans w est une occurrence d'un facteur $\overline{x}^k x^k$ où $k \geq 1$. On présente différentes méthodes pour énumérer les mots de Dyck selon leur taille et le nombre d'interactions supérieures. La fonction génératrice, peu usuelle, est un quotient de deux q-séries dans lesquelles apparaît un terme algébrique. La première méthode nécessite principalement des calculs sur des séries formelles. Les deux méthodes suivantes interprètent différentes étapes du calcul précédent par des manipulations ou des factorisations des mots de Dyck.

1. Introduction

A *Dyck word* w is a word over the alphabet $\{x, \overline{x}\}$ that contains as many letters x than $\overline{x}$ and such that any prefix contains at least as many letters x as letters $\overline{x}$. The *size* of w is the number of letters x in w. A *Dyck path* is a walk in the plane, starting from the origin, made up of *rises*, steps $(1,1)$, and *falls*, steps $(1,-1)$, that remains above the horizontal axis and finishes on it. Figure 1 gives an example of a Dyck path of size 12. The Dyck path related to the Dyck word w is the walk obtained by replacing in w a letter x by a rise, and a letter $\overline{x}$ by a fall. In the rest of the paper we identify the two notions.

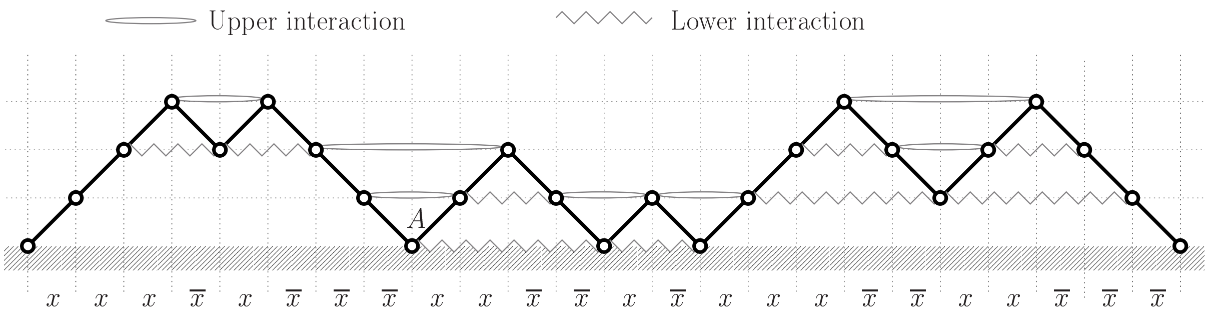

Figure 1. A Dyck path and its upper and lower interactions

Date: 27th November 2004.

Key words and phrases. Dyck paths, Catalytic parameters, Slice functional equations, Linearization of q-algebraic equations, Basic hypergeometric series, Cycle lemma, Heaps models.

An *upper interaction*, respectively a *lower interaction*, in the Dyck word w is an occurrence of a factor $\overline{x}^k x^k$, respectively $x^k \overline{x}^k$, for any $k \geq 1$. The example of Figure 1 contains 7 upper interactions and 9 lower interactions. Lower interactions are easy to take into account in the enumeration because of the usual decomposition of Dyck paths which splits them at the second vertex on the horizontal axis, the vertex A on the Figure 1. Lower interactions are included in one of the two subwalks. Denise and Simion [6] already used this fact to enumerate Dyck paths according to the size and the number of lower interactions. By constrast, there are upper interactions above the vertex A that belong to the two subwalks. It explains why upper interactions seemingly do not satisfy an algebraic decomposition and are more difficult to take into account.

This paper present three methods to find an expression for the generating function

$$A(t,u) = \sum_{\text{non-empty Dyck path } w} t^n u^k \tag{1}$$

where n is the size of w and k the number of upper interactions.

In Section 2, the first method, inspired by a work of Bousquet-Mélou and Rechnitzer [3], consists in building Dyck words by inserting a factor $x^i \overline{x}^i$ after the last letter x. This leads to a functionnal equation that can be solved through calculations over formal power series involving four main steps: an iteration, the kernel method [2], a division and the use of a relation between roots of a polynomial. The resulting generating function has a rather unusual form: it is a ratio of q-series,with $q = tu$, where occurs an algebraic term σ. Our aim is now to understand better these calculations over formal power series by direct manipulations or factorizations of Dyck paths.

In the next method in Section 3, we consider *Dyck paths with small valleys* that are Dyck words that avoid the factor $\overline{x}\overline{x}xx$. An *ad hoc* valuation of the *valleys*, the factors $\overline{x}x$, allows us to recover the generating function $A(t,u)$. We can recursively split these words to obtain a q-algebraic equation that is solvable after a change of unknown function. The choice of this change is crucial in the resolution. A first possibility, inspired by a paper of Janse Van Rensburg [7], leads to a q-linear equation. This equation admits as solution a basic hypergeometric series computable with the algorithm proposed by Abramov, Paule et Petkovšek [1]. We present a second change of unknown function, seemingly a brother of the previous one, that, in our case, requires less calculation to conclude.

In Section 4 we consider more precisely the valuation of each valley: it is the sum of a constant term and a term that depends geometrically on the *height* of the valley. We expand this sum in each valley to consider *bicolored paths with small valleys* where the valuation of a valley is either the constant term for a *white valley*, or the "geometric" term for a *black valley.* Paths with only white valleys give a combinatorial interpretation of the algebraic term σ that comes from the kernel method in Section 2. Paths with only black valleys are in bijection with certain heaps of segments [4] thus their generating function, a ratio of q-series, is an instance of Viennot's heaps inversion lemma [9]. The iteration in the calculation of Section 2 seems to correspond to the calculation of trivial heaps of segments. Moreover, we have a combinatorial interpretation of the first change of unknown function in Section 3. Finally we consider bicolored paths with small valleys where any black valley is *isolated*, that is, occurs in a factor $xx\overline{x}x\overline{x}\overline{x}$. A partition of these paths leads to a bijection with other heaps of segments. The inversion lemma gives a ratio of q-series where appears the algebraic term σ. Moreover the relation between roots of a polynomial, used in Section 2, gives rise to a term $(q)_n(qt\sigma^2)_n$

that admits here a combinatorial interpretation. Hence, by considering the three above sets of bicolored paths, we are able to interpret combinatorially the four main steps of the calculation over formal power series. It may be possible to merge these interpretations to obtain a combinatorial interpretation of the generating function of Dyck paths counted according to the size and the number of upper interactions.

2. A catalytic parameter for a "slice" functional equation

In [3], Bousquet-Mélou and Rechnitzer use a factorization of partially directed walks. We adapt their work to the case of Dyck paths. The *length of the last descent* of a Dyck word is the number of letters $\overline{x}$ after the last letter x. We define the generating function of non-empty Dyck paths counted according to the size, the number of upper interactions and the length of the last descent by:

$$B(s) \equiv B(t,u;s) = \sum_{\text{non-empty Dyck path } w} t^n u^k s^j.$$

where n is the size of w, k the number of upper interactions and j the length of the last descent. Our aim is to compute $A(t,u) = B(t,u;1)$ but we need to know the additionnal parameter to write an equation linking $B(1)$, $B(s)$ and $B(uts)$. Zeilberger [10] calls this kind of parameter a *catalytic parameter*. The form of the decomposition of Dyck paths that we use also reminds Temperley-like decompositions, used by certain physicists [8].

Lemma 1. *The generating function $B(s)$ of Dyck paths counted according to the size, the number of upper interactions and the length of the last descent satisfies*

$$\begin{aligned} B(s) \quad = \quad & \frac{ts}{1-ts} + \frac{ut}{1-ut}\frac{ts\,(B(s)-B(uts))}{1-ts} \\ & + \frac{ut}{1-ut}\left(\frac{s\,(B(1)-B(s))}{1-s} - \frac{uts\,(B(1)-B(uts))}{1-uts}\right). \end{aligned} \tag{2}$$

Proof. (sketch) A *peak* is a vertex next to a rise and before a fall. We split the set of Dyck paths into three disjoint subsets illustrated on Figure 2: the paths with at most one peak, the paths where the last peak is strictly higher than the previous one, the paths where the last peak is below the previous one.

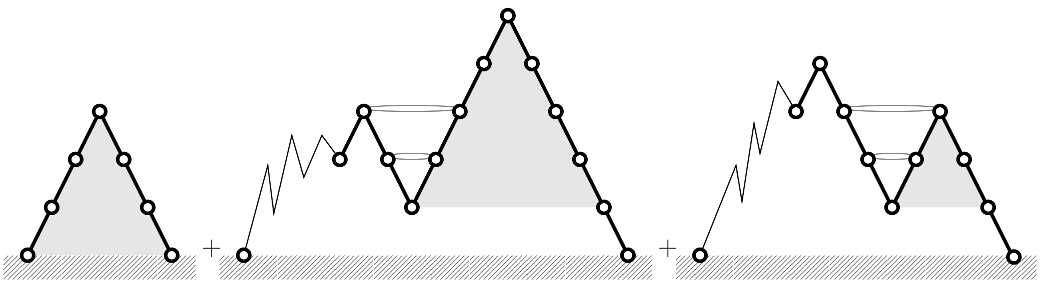

Figure 2. The "slice" decomposition of Dyck paths

The generating function of each subset is obtained by adding a factor $x^k\overline{x}^k$: to the empty path for the first subset or to any non-empty Dyck path otherwise. The length of the last descent is sufficient to determine the number of ways one can extend a Dyck path and the number of additionnal upper interactions in each case. Figure 2 gives extensions leading to each subset. The summation of these extensions over all Dyck paths gives rise to different evaluations of $B(s)$. The generating function of each subset gives one of the three terms in the right-hand side of Equation (2). □

The solution of Equation (2) requires an iteration to remove $B(uts)$ and the kernel method to remove $B(s)$. Like in [3], we obtain for $B(1) = A(t, u)$ a ratio of two q-series in which occurs an algebraic term.

Proposition 2. *The generating function of Dyck paths counted according to the size and the number of upper interactions is*

$$A(t,u) = B(1) = -\frac{t\sum_{n\geq 0}\left(\frac{(q-t)\sigma}{1-q}\right)^n \frac{q^{\binom{n+2}{2}-1}}{(q)_n(qt\sigma^2)_n}}{q\sum_{n\geq 0}\left(\frac{(q-t)\sigma}{1-q}\right)^n \frac{q^{\binom{n+2}{2}-1}}{(q)_n(qt\sigma^2)_n}\frac{1-tq^n\sigma}{(1-q^n\sigma)(1-q^{n+1}\sigma)}} \tag{3}$$

where $q = ut$, $(x)_n = \prod_{k=0}^{n-1}(1-q^k x)$ *and* $\sigma = \dfrac{1+t-2q-\sqrt{(1-t)(1-t-4q+4q^2)}}{2t(q-1)}$.

Proof. (sketch) Equation (2) can be rewritten as

$$B(s) = a(s) + b(s)B(1) + c(s)B(qs) + d(s)B(s). \tag{4}$$

We iterate this equation, using its evaluation at $s = q^k s$ to recursively replace the term $B(q^{k+1}s)$ by an expression in terms of $B(q^{k+2}s)$. We show that this process converges, in the sense of formal power series, toward the relation

$$(1-d(s))B(s) = \sum_{n\geq 0}\prod_{k=0}^{n-1}\frac{c(q^k s)}{1-d(q^{k+1}s)}\left(a(q^n s) + b(q^n s)B(1)\right) \tag{5}$$

where the unknown functions $B(q^k s)$, $k \geq 1$, have disappeared. The kernel method consists in choosing $s = \sigma$ such that $1 - d(\sigma) = 0$. Thus σ is one of the roots of the polynomial, deduced from $1 - d(s)$, called the *kernel*:

$$1 + \left(\frac{q-t}{1-q} - 1\right)\sigma + t\sigma^2 = 0. \tag{6}$$

We choose σ to be the unique root of this polynomial that is a formal power series in t and replace s by σ in (5). After the annulation for $s = \sigma$ of the left side of Equation (5), it remains one division to deduce the expression of $B(1)$. The fact that the product of the two roots of the kernel is $1/t$ enables us to rewrite the products of $c(q^k\sigma)/(1 - d(q^{k+1}\sigma))$ as terms where appears $(q)_n(qt\sigma^2)_n$. □

3. A catalytic parameter for a q-algebraic equation

To be able to write a q-algebraic equation whose solution leads to $A(t, u)$ we consider a subset of Dyck paths, the paths with small valleys, and another catalytic parameter. The solution of the q-algebraic equation uses a change of unknown function in the spirit of the works of Brak and Prellberg [5] and Janse Van Rensburg [7].

3.1. Dyck paths with small valleys

A *valley* in a Dyck path is a vertex next to a fall and before a rise. *Dyck paths with small valleys* are Dyck paths that avoid the factor $\overline{x}\overline{x}xx$. We define an *ad hoc* valuation of such paths: there is a weight t on each rise and a weight

$$V_{val}(k) = \frac{q(1-q^{k+1}y)}{t(1-q)} \tag{7}$$

on a valley at height k. The generating function according of Dyck paths with small valleys to these weights is

$$C(y) \equiv C(t,u;y) = \sum_{\text{non-empty Dyck path with small valleys } w} t^n \prod_{k\geq 0} V_{val}(k)^{v_k} \tag{8}$$

where n is the size of w and v_k the number of valleys at height k in w.

Lemma 3. *The generating function $A(t,u)$ of Dyck paths, defined by (1), and the generating function $C(t,u;y)$ of Dyck paths with small valleys satisfy*

$$A(t,u) = C(t,u;1). \tag{9}$$

Proof. We group Dyck paths into sets of paths with the same sequence of heights of the peaks. In each set S there is a single path w_S of minimal size and we use it as the representative of the set. This path is also the single path with small valleys in S. All paths in S are obtained by "digging" the valleys of w_S, that is rewriting recursively factors $\overline{x}x$ of w_S in $\overline{x}\overline{x}xx$ while the path remains above the horizontal axis. In w_S there are as many upper interactions as valleys. Moreover, each rewriting $\overline{x}x \longrightarrow \overline{x}\overline{x}xx$ increases the size and the number of upper interactions by one. Thus the generating function of paths of S according to size and the number of upper interactions corresponds to the weigth of w_S where a rise is weigthed t and a valley at height k,

$$u + ut^2 + \ldots + u^{k+1}t^k = \frac{q(1-q^{k+1})}{t(1-q)} = V_{val}(k)_{|y=1}.$$

We recognize here the valuation of valleys in (8) when $y=1$. The summation over all the sets S, that is over the paths with small valleys, leads to (9). □

The variable y that occurs in the weight of valleys is another example of a catalytic variable since it allows to write a q-algebraic equation for paths with small valleys:

Lemma 4. *The generating function of non-empty paths with small valleys satisfies*

$$C(y) = t + t\left(1 + q\frac{1-qy}{1-q}\right)C(qy) + q\frac{1-qy}{1-q}C(y) + \left(q\frac{1-qy}{1-q}\right)^2 C(qy)C(y). \tag{10}$$

Proof. We split a path with small valleys at the second vertex on the axis, called A in Figure 3. There are five cases due to the avoiding of the factor $\overline{x}\overline{x}xx$ especially around the vertex A.

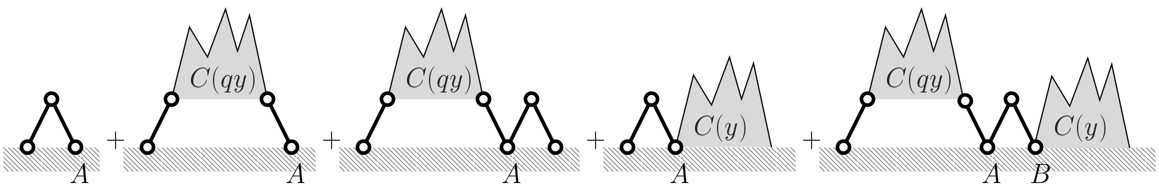

FIGURE 3. Decomposition of paths with small valleys

If w is a path with small valleys of weight $W(t,u,y)$, then $xw\overline{x}$ is a path of weight $tW(t,u,qy)$. This fact explains why we use the catalytic variable y in the weight of paths with small valleys. As an example the valuation of the fifth case is

$$tC(qy)\frac{q(1-qy)}{t(1-q)}t\frac{q(1-qy)}{t(1-q)}C(y)$$

since there are two valleys, A and B. □

3.2. Linearization

The resolution of the q-algebraic equation (10) begins with a change of unknown function: we look for solutions of the form

$$C(y) = \frac{J(qy)}{\alpha J(y) + \beta(y) J(qy)} \tag{11}$$

where α is independent of y, $\beta(y)$ a rational function in y and $J(y)$ a formal power series in y such that $J(0) = 1$. A series $H(y) = \sum_{n\geq 0} h_n y^n$ is a *basic hypergeometric series* if there is a rational function $F(X)$ such that $h_{n+1}/h_n = F(q^n)$ for all $n \in \mathbb{N}$.

Proposition 5. *The unique formal power series in t that satisfies the q-algebraic Equation (10) is*

$$C(y) = \frac{(t-q)t\sigma \sum_{n\geq 0} \left(\frac{(q-t)\sigma}{1-q}\right)^n \frac{q^{\binom{n+2}{2}-1}}{(q)_n (qt\sigma^2)_n} y^n}{\sum_{n\geq 0} \left(\frac{(q-t)\sigma}{1-q}\right)^n \frac{q^{\binom{n+1}{2}}(q^n t\sigma - q - q^{2n}t^2\sigma^2)}{(q)_n (qt\sigma^2)_n} y^n} \tag{12}$$

where again $q \equiv ut$ and σ is defined as in Proposition 3.

Proof. We consider Equation (10) where $C(y)$ is replaced by its expression (11). We reduce it to a single rational function R in y, $J(y)$, $J(qy)$ and $J(q^2y)$). The numerator N of R is a linear combination of $J(y)J(qy)$, $J(qy)^2$, $J(q^2y)J(qy)$ and $J(q^2y)J(y)$. We choose $\beta(y) = -\dfrac{1-qy}{1-q}$ to remove the term $J(q^2y)J(y)$, thus we can factorize $J(qy)$ in N. The other factor of N vanishes if and only if yhe following q-linear equation holds:

$$t\alpha^2 J(y) - \left(1 + \frac{(t-q)(1-qy)}{1-q}\right)\alpha J(qy) + J(q^2y) = 0. \tag{13}$$

The evaluation at $y = 0$ of Equation (13) implies that α is one of the two roots of a polynomial that is the kernel (6) in the proof of Proposition 3. We define $\alpha = 1/(t\sigma)$. We will explain later why we choose this root rather than σ. The change of unknown function defined by this analysis is

$$C(y) = \frac{t\sigma J(qy)}{J(y) - \frac{1-qy}{1-q} t\sigma J(qy)}, \tag{14}$$

and it leads to the q-linear equation (13) where $\alpha = 1/(t\sigma)$. Since (13) is of degree 1 in y and by definition $J(y) = 1 + \sum_{n\geq 1} j_n y^n$, the extraction of the coefficient of y^{n+1} in (13) gives a relation between j_n and j_{n+1} which is

$$j_{n+1} = \frac{(q-t)\sigma q^{n+1}}{(1-q)(1-q^{n+1})(1-q^{n+1}t\sigma^2)} j_n.$$

Thus $J(y)$ is the basic hypergeometric series

$$J(y) = \sum_{n\geq 0} \left(\frac{(q-t)\sigma}{1-q}\right)^n \frac{q^{\binom{n+1}{2}}}{(q)_n (qt\sigma^2)_n} y^n.$$

We plug this expression in (14) and we recognize the expression (12). $\square$

Remark 6. We may also compute $C(y)$ using the same method as in Section 2. The additionnal variable y does not deeply modify the calculations. The change of unknown function (14) was actually conjectured at the sight of this formula. It also explains why we choose $\alpha = 1/(t\sigma)$ instead of σ.

Remark 7. In [7], Janse Van Rensburg suggests a change of unknown function that we generalize in

$$C(y) = \frac{\alpha J(qy) + \beta(y) J(y)}{\gamma(y) J(y)} \tag{15}$$

where α is independent of y and $\beta(y)$ and $\gamma(y)$ polynomials in y. This change also leads to a q-linear equation (L) that admits a basic hypergeometric series as solution. But the resolution of (L) requires the algorithm of Abramov, Paule and Petkovšek [1] and much more calculation in our case.

4. Toward a combinatorial interpretation

The weight $V_{val}(k)$ of a small valley is the sum of a constant term $q/(t(1-q))$ and a term $-q^{k+2}y/(t(1-q))$ that depends geometrically on the height k. We expand these sums in the paths with small valleys to define the *bicolored paths (with small valleys)*: they are paths with small valleys where the valleys are either white or black. The weight of a *white valley* is $q/(t(1-q))$, the weight of a *black valley* lying at height k is $-q^{k+2}y/(t(1-q))$ and the weight of a rise remains t. By definition, the generating function of these bicolored paths is also $C(y)$.

We study three subsets of the bicolored paths. The *white paths* are the bicolored paths where all valleys are white. The *black paths* are the bicolored paths where all valleys are black. The *bicolored black-isolated paths* are the bicolored paths where all black valleys belong to a factor $xx\overline{x}x\overline{x}\overline{x}$. Each of these subsets yields a combinatorial interpretation of some of the four main steps of the calculation that gives Proposition 2.

4.1. The algebraic term

The function $\sigma(t,u)$ in Proposition 2 can be written as $\sum_{n\geq 0} p_n(u)t^n$ where $p_n(u)$ are polynomials whose coefficients are nonnegative integers. Morever $\sigma(t,1)$ is the generating function of Dyck paths according to the size. These facts suggest the existence of combinatorial interpretations of σ. The first one is the generating function of Dyck paths according to the size, counted by t, and the number of *lower* interactions, counted by u. This result was already obtained by Denise and Simion [6].

Proposition 8. *The generating function $D(t,u)$ of Dyck paths counted according to size and the number of* lower *interactions satisfies*

$$D(t,u) = 1 + \frac{ut}{1-ut} D(t,u) + t\left(D(t,u) - \frac{1}{1-ut}\right) D(t,u) \tag{16}$$

thus $D(t,u) = \sigma$.

We can group Dyck paths into sets with the same sequence of heights of the valleys. The smallest path of each set is a path that avoid $xx\overline{x}\overline{x}$, that is *a path with small peaks*. As for valleys and upper interactions, we can define $D(t,u)$ by a summation over paths with small peaks where a peak is weighted $q/(t(1-q))$ and a rise t.

The white paths correspond to the case $y = 0$ for Dyck paths with small valleys. For $y = 0$, the q-algebraic equation (10) becomes an algebraic equation whose solution is the generating function of the white paths. This generating function is only "almost" equal to $\sigma(t,u)$. A *double-peak* in a Dyck path is a factor $xx\overline{x}\overline{x}$. To recover exactly σ, we consider the *white paths starting with a double-peak.*

Proposition 9. *There exists a bijection f between paths with small peaks and white paths starting with a double-peak such that the weight of $f(w)$ is exactly the weight of w multiplied by t^2. The generating function of the white paths starting with a double-peak is $t^2\sigma$.*

Proof. (sketch) Let w be a non-empty path with small peaks. Let A be the first vertex of w, B the first vertex of maximal height, C the last vertex and D the vertex preceeding B. Using these vertices, we factor w in $w_{AD}xw_{BC}$, where w_{PQ} denotes the subpath of w between the vertices P and Q. We denote by g the morphism on the words over the alphabet $\{x,\overline{x}\}$ defined by $g(x)=\overline{x}$ and $g(\overline{x})=x$. We define $f(w)=xx\overline{x}\overline{x}g(v_{BC})\overline{x}g(v_{AD})$.

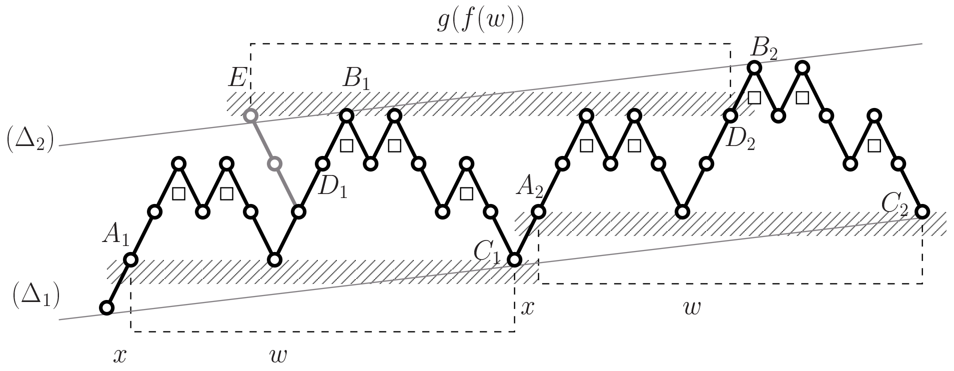

FIGURE 4. A bijection between two interpretations of σ

Figure 4 gives on an example a geometrical definition of this bijection f. It is supposed to convince us that f satisfies all claimed properties. We consider the path $p=xwxw$ and the line (Δ_1) below this path containing exactly three vertices of the path p on it. Then we consider the line (Δ_2) above this path, parallel to (Δ_1), with exactly two vertices B_1 and B_2 of p. Between the vertices E and D_2, we recognize the image of $f(w)$ by g. A path with small peaks w becomes a path with small valleys and the weights $q/(t(1-q))$ remain on the same vertices. $\square$

Remark 10. It is also possible to compute directly the generating function of white paths starting with a double-peak using an algebraic decomposition of these paths.

4.2. The ratio of basic hypergeometric series

To enumerate black paths, we use a bijection with heaps of marked segments with an *ad hoc* weight. A *marked segment* $s=([l(s),r(s)],m(s))$ is defined by an interval $[l(s),r(s)]\subseteq\mathbb{N}$ of $l(s)-r(s)+1$ *elements* and a subset $m(s)\subseteq[l(s)+1,r(s)-1]$ of marked elements. If the marked segment is a singleton $([r(s),r(s)],\emptyset)$ then its weight $V_{seg}(s)$ is $-\frac{q^{r(s)+2}y}{(1-q)}$. Otherwise $l(s)<r(s)$ and the weight of the marked segment $s=([l(s),r(s)],m(s))$ is

$$V_{seg}(s)=\left(-\frac{q^{l(s)+2}y}{1-q}\right)^2 t^{r(s)-l(s)}\prod_{k\in m(s)}\left(-\frac{q^{k+2}y}{1-q}\right).$$

A *heap of (marked) segments* of size n is a set $\{(s_i, h_i)\}_{i=1\dots n}$ where s_i is a marked segment and $h_i \in \mathbb{N}$ is its height, such that

- if $h_i = h_j$ then $[l(s_i), r(s_i)] \cap [l(s_j), r(s_j)] = \emptyset$: two segments can not overlap.
- if $h_i > 0$ then there exists $(s_j, h_i - 1)$ such that $[l(s_i), r(s_i)] \cap [l(s_j), r(s_j)] = \emptyset$: a segment that is not on the floor lay on a segment just below.

A heap of segments is *a half-pyramid of (marked) segments* if there is at most one segment s_i on the floor, that is $h_i = 0$, and moreover $l(s_i) = 0$. On the right of Figure 5 there is a half-pyramid of marked segments. The generating function of half-pyramids of segments is defined by

$$F(y) = \sum_{\text{half-pyramid } h} \left(\prod_{s \text{ segment of } h} V_{seg}(s) \right).$$

As in the case of white paths, we consider *black paths starting with a double-peak.*

Proposition 11. *There exists a bijection h between black paths starting with a double-peak and half-pyramids of marked segments. This bijection implies that $t^2 F(y)$ is the generating function of black paths starting with a double-peak.*

Proof. (sketch)

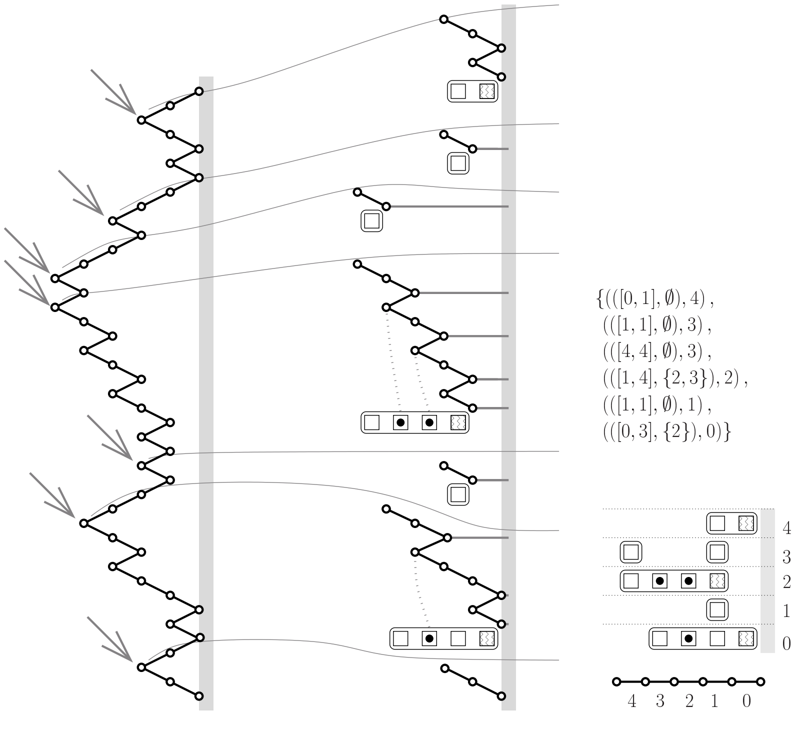

FIGURE 5. Black paths and half-pyramids of marked segments

The bijection is a variant of a previous one between Dyck paths and half-pyramids of segments [4]. We discuss here its specificities. We use the example of Figure 5 to define how a black path w is mapped to a half-pyramid $h(w)$. A *cutting peak* in w, pointed by a grey arrow on Figure 5, is a peak that is not before a factor $\overline{x}xx$. Two consecutive cutting peaks are the endpoints of a *block* b where $l(b)$ is the height of the vertex before the first rise, $r(b) + 1$ is the height of the second cutting peak and

$m(b)$ is the set of all heights of valleys in b. The block b is mapped to the marked segment $s = ([l(b), r(b)], m(b)\backslash\{l(b)\})$. Given a sequence of marked segments, we "let them fall on the floor" to obtain an heap of segments $h(w)$. Moreover, since the black path starts with a double-peak, the vertex before the first rise of the first block b is at height 0, leading to a segment $s = ([0, r(s)], m(s))$. By definition of cutting peaks, all the other segments are carried by a previous segment in the sequence. Thus $h(w)$ is a half-pyramid of marked segments. We claim without proving it here that h is a bijection. Since the weight of the marked segment s was defined to be the weight of rises and valleys in the block b, the weight of $h(w)$ is almost the weight of w: only the weight of the two first rises in w is not take into account in $h(w)$. Thus the generating function of black paths starting with a double-peak is $t^2F(y)$. $\square$

A *trivial heap* is a heap $\{(s_i, h_i)\}$ where all segments are on the floor, that is $h_i = 0$ for all i. The *alternating generating function of trivial heaps* is defined by

$$T(y) = \sum_{\text{trivial heap } h} \left(\prod_{s \text{ segment of } h} (-1)V_{seg}(s) \right).$$

Lemma 12. *The alternating generating function of trivial heaps $T(y)$ satisfies the q-linear equation*

$$\begin{aligned} T(y) &= T(qy) + \frac{q^2y}{1-q}T(qy) - \frac{tqy}{1-q}\left(T(qy) - T(q^2y)\right) \\ &\quad + t/q^2\left(T(qy) - T(q^2y) - \frac{q^3y}{1-q}T(q^2y)\right). \end{aligned} \tag{17}$$

and

$$T(y) = \sum_{n\geq 0} \left(\frac{q-t}{1-q}\right)^n \frac{q^{\binom{n+1}{2}}}{(q)_n(t/q)_n} y^n.$$

Proof. (sketch) The weight of a segment $s = ([l(s), r(s)], m(s))$ can be distributed to the elements of $[l(s), r(s)]$ as follows:

$$V_{seg}(s) = \left(-\frac{q^{r(s)+2}y}{1-q}\right)\left(-\frac{q^{l(s)+1}y}{1-q}\right) \prod_{k\in m(s)} \left(-\frac{q^{k+1}y}{1-q}\right) \prod_{k\in[l(s)+1, r(s)-1]\backslash m(s)} \frac{t}{q}.$$

Thus there are three kinds of element in $[l(s), r(s)]$: the *first element* $r(s)$, weighted $-\frac{q^{r(s)+2}y}{1-q}$, the *heavy elements* $k \in m(s) \cup \{l(s)\}$, weighted $-\frac{q^{k+1}y}{1-q}$, and the *light elements* $k \in [l(s)+1, r(s)-1]\backslash m(s)$ weighted t/q. That distribution corresponds to the evelation of the valuation of the first black valley in front of the last rise of the block. A singleton contains only its first element. In the other segments $l(s)$ is an heavy element, $r(s)$ the first element and the elements of $[l(s)+1, r(s)-1]$ are either heavy or light. We split the set of trivial heaps S into four disjoint subsets: the set of trivial heaps S_1 where 0 is not an element of a segment, the set of trivial heaps S_2 where there is a singleton $([0,0], \emptyset)$, the set of trivial heaps S_3 where there is a segment $([0, r(s)], m(s))$ whose 1 is a light element and the set of trivial heaps S_4 where there is a segment $([0, r(s)], m(s))$ whose 1 is either an heavy element or the first element. The alternating generating function of each of these subsets gives a term in the right side of Equation (17). The *translated segment* of a segment s is $t = ([l(s)+1, r(s)+1], \{k+1 | k \in m(s)\})$. The weight $V_{seg}(t)$ is $V_{seg}(s)$ where qy has been substitued to y. The subset S_1 is exactly the set obtained by translating all heaps, thus the alternating generating function of this subset is $T(qy)$. Since we are in an abstract, it is left to the reader to note that the alternating generating functions of the sets are $\frac{q^2y}{1-q}T(qy)$ for S_2, $t/q^2\left(T(qy)-(1+\frac{q^3y}{1-q})T(q^2y)\right)$ for S_3 and $-\frac{tqy}{1-q}(T(qy)-T(q^2y))$ for S_4.

Equation (17) is of degree 1 in y. Moreover $T(y)=1$ since only the empty trivial heap is not weighted by a factor y. By a resolution similar to that of Equation (13) we compute the basic hypergeometric series $T(y)$. □

The heap inversion Lemma of Viennot [9] states that the generating function of heaps where all segments on the floor belong to a set of segments A is $H_A(y)/H(y)$ where $H(y)$ is the alternating generating function of trivial heaps and $H_A(y)$ the alternating generating function of trivial heaps where no segment belongs to A. In the case of half-pyramids, A is the set of segments $s=([0,r(s)],m(s))$ and $H_A(y)=T(qy)$. Thus the generating function of half-pyramids is $T(qy)/T(y)$. This fact, Proposition 11 and Lemma 12 lead to the generating funtion of black paths starting with a double peak. In the next proposition we consider all non-empty black paths.

Proposition 13. *The generating function of non-empty black paths satisfies*

$$G(y)=\frac{(1-q)^2}{tq^4y^2}\left(t^2\frac{T(qy)}{T(y)}-t^2+\frac{t^2q^2y}{1-q}\right) \tag{18}$$

and

$$G(y)=\frac{t\displaystyle\sum_{n\geq 0}\frac{(q-t)^{n+1}q^{\binom{n+2}{2}}y^n}{(1-q)^n(q)_n(t/q)_{n+2}}}{q^2\displaystyle\sum_{n\geq 0}\frac{(q-t)^nq^{\binom{n+1}{2}}y^n}{(1-q)^n(q)_n(t/q)_n}} \tag{19}$$

Proof. A black path starting with a double-peak is either the path $xx\overline{x}\overline{x}$, or the path $xx\overline{x}\overline{x}x\overline{x}$ or it starts with the factor $xx\overline{x}\overline{x}x\overline{x}$ and followed by any non-empty black path. On the other hand Proposition 11 states that the generating function of black paths starting with a double-peak is $t^2F(y)$. This leads to the equation

$$t^2F(y)=t^2-t^2\frac{q^2y}{1-q}+t\frac{q^4y^2}{(1-q)^2}G(y). \tag{20}$$

Using the fact that $F(y)=T(qy)/T(y)$, standard calculations lead to (18) and (19). □

Remark 14. Equation (18) is reminiscent of the second change of unknown function (15) used by Janse Van Rensburg to solve a similar question [7]. Moreover the combinatorial interpretation leading to Equation (20) explains why $\gamma(y)$ is the denominator.

4.3. A partial mixing of the two interpretations

Proposition 15. *The generating function of bicolored black-isolated paths starting with a double-peak is*

$$t^2\sigma\frac{\displaystyle\sum_{n\geq 0}\frac{q^{6n}t^n\sigma^n}{(q-1)^{3n}(q)_n(qt\sigma^2)_n}y^n}{\displaystyle\sum_{n\geq 0}\frac{q^{5n}t^n\sigma^n}{(q-1)^{3n}(q)_n(qt\sigma^2)_n}y^n} \tag{21}$$

where $q\equiv ut$ and $\sigma$is defined as in Propositions 2 and 5.

Proof. (sketch) We group bicolored black-isolated paths starting with a double-peak into sets of paths that admit the same sequence of heights of black valleys *and* the same sequence of minimal heights between two black valleys. Let S be one of these sets. There is a single smallest path w in S. In w there is a factor $x\overline{x}^{2+i}x\overline{x}x^{j+2}\overline{x}$ between consecutive black valleys. Figure 6 gives a example of a path w. The generating function of paths in S is obtained by inserting white paths counted by σ before double rise or double fall in w except for a double rise in a factor $x\overline{x}xx$. The path w is mapped to a half-pyramid of segments where the segment $s = [l(s), r(s)]$ is weighted $\frac{q^5 t\sigma}{(q-1)^3} q^{r(s)} (t\sigma)^{r(s)-l(s)}$. Only a factor $t^2\sigma$ at the start of the path is forgotten in this map. On Figure 6, w is mapped to the half-pyramid defined by $s_1, \ldots, s_4$. We compute the alternating generating function of trivial heaps made up of these segments. The heap inversion lemma leads to the generating function (21) where appears the terms $(q)_n(qt\sigma^2)_n$. □

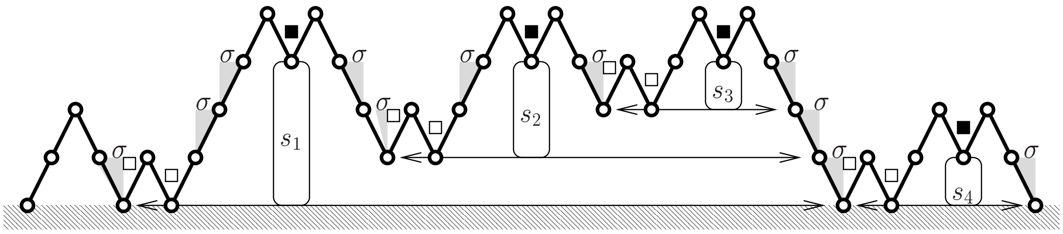

FIGURE 6. A representative of a set of bicolored black-isolated paths

REFERENCES

[1] S. A. Abramov, P. Paule, and M. Petkovšek. q-Hypergeometric solutions of q-difference equations. *Discrete Math.*, 180(1-3):3–22, 1998.

[2] M. Bousquet-Mélou and M. Petkovšek. Walks confined in a quadrant are not always D-finite. *Theoret. Comput. Sci.*, 307(2):257–276, 2003.

[3] M. Bousquet-Mélou and A. Rechnitzer. The site-perimeter of bargraphs. *Adv. in Appl. Math.*, 31(1):86–112, 2003.

[4] M. Bousquet-Mélou and X. G. Viennot. Empilements de segments et q-énumération de polyominos convexes dirigés. *J. Combin. Theory Ser. A*, 60(2):196–224, 1992.

[5] R. Brak and T. Prellberg. Critical exponents from nonlinear functional equations for partially directed cluster models. *J. of Stat. Phys.*, 78(3-4):701–730, 1995.

[6] A. Denise and R. Simion. Two combinatorial statistics on Dyck paths. *Discrete Math.*, 137(1-3):155–176, 1995.

[7] E. J. Janse van Rensburg. Interacting columns: generating functions and scaling exponents. *J. Phys. A*, 33(42):7541–7554, 2000.

[8] H. N. V. Temperley. Combinatorial problems suggested by the statistical mechanics of domains and of rubber-like molecules. *Phys. Rev. (2)*, 103:1–16, 1956.

[9] G. X. Viennot. Heaps of pieces. I. Basic definitions and combinatorial lemmas. In *Combinatoire énumérative*, volume 1234 of *Lecture Notes in Math.*, pages 321–350. Springer, Berlin, 1986.

[10] D. Zeilberger. The umbral transfer-matrix method. I. Foundations. *J. Combin. Theory Ser. A*, 91(1-2):451–463, 2000. In memory of Gian-Carlo Rota.

LaBRI, Université Bordeaux 1, 351 cours de la Libération, 33405 Talence Cedex, France

E-mail address: borgne@labri.fr