The site-perimeter of bargraphs

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Abstract

The site-perimeter enumeration of polyominoes that are both column- and row-convex is a well understood problem that always yields algebraic generating functions. Counting more general families of polyominoes is a far more difficult problem. Here we enumerate (by their site-perimeter) the simplest family of polyominoes that are not fully convex — bargraphs. The generating function we obtain is of a type that, to our knowledge, has never been encountered so far in the combinatorics literature: a q-series into which an algebraic series has been substituted.

1 Introduction

A *polyomino* is a finite connected union of cells on a regular planar lattice. The only lattice considered in this paper is the square lattice (see Figure 1). The enumeration of polyominoes is a longstanding "elementary" combinatorial problem that has some motivations in physics, for example in the study of branched polymers [17] and percolation [31, 19, 8]. However, although this problem has been intensively studied for more than 40 years [18, 22, 32], exact results concerning general polyominoes have remained elusive.

By far the most fruitful avenue of research has been the examination and solution of large subclasses of polyominoes. This research has focussed primarily on counting them according to their most basic geometric properties — area and perimeter. The area of a polyomino is the number of cells it contains, while its perimeter is the number of edges that simultaneously are incident on a cell inside the polyomino and a cell outside. One sometimes distinguishes between the *vertical* and *horizontal* perimeters. Many natural families of *column-convex* polyominoes[1] have been enumerated according to both these parameters simultaneously (see [3, 4] and references therein), while a number of other models have been counted according to area alone [12] or perimeter alone [7].

Fewer results concern the *site-perimeter* of polyominoes, that is, the number of nearest-neighbour vacant cells (see Figure 1). This parameter is of considerable interest to physicists

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[1] A polyomino is *column-convex* if its intersection with any vertical line is connected.

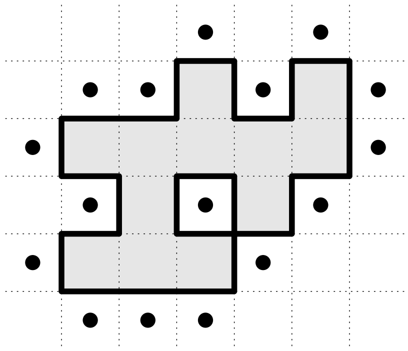

Figure 1: An example of a polyomino with an area of 12, a horizontal perimeter of 14, a vertical perimeter of 12 and a site-perimeter of 16.

and probabilists since it plays an important role in the study of percolation models (see [19, 31] and references therein); more precisely, the probability that the origin of the lattice belongs to a given (site) percolation cluster is a simple function of both the area and site-perimeter of the cluster.

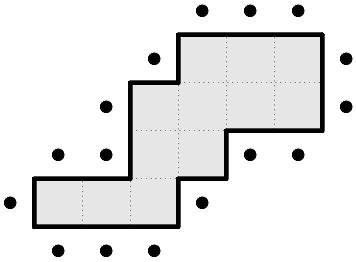

Figure 2: A staircase polyomino and its site-perimeter.

To our knowledge, the first non-trivial site-perimeter generating function that was computed dealt with *staircase polyominoes* [11]: a staircase polyomino is a polyomino whose perimeter consists of two directed paths (containing only north and east steps) that intersect only at the extreme north-east and south-west points (Figure 2). The site-perimeter generating function for these polyominoes was proved to be a simple quadratic function.

Proposition 1 ([11]). *Let a_n be the number of staircase polyominoes with site-perimeter n. Then the site-perimeter generating function for staircase polyominoes is*

$$\sum_{n\geq 0} a_n p^n = \frac{p^2}{2}\left(1 - p^2 - 2p^3 + p^4 - (1+p-p^2)\sqrt{(1+p+p^2)(1-3p+p^2)}\right).$$

Consequently the number of staircase polyominoes of site-perimeter n grows asymptotically as:

$$C\left(\frac{3+\sqrt{5}}{2}\right)^n n^{-3/2},$$

for some positive constant C.

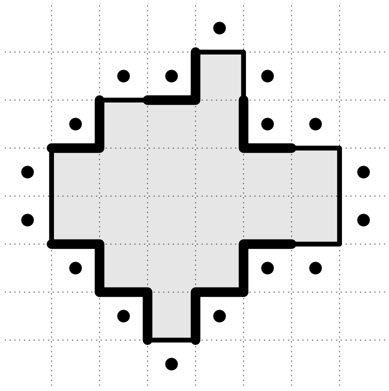

Figure 3: A polyomino is convex if each row and each column consists of a single connected component. The site-perimeter of a such a polyomino is equal to the perimeter minus the number of *inside corners*. The above polyomino has 7 inside corners, a perimeter of 24 and a site-perimeter of 17.

The method used in [11] for counting staircase polyominoes by their site-perimeter was quite close to the method used for the usual perimeter, and it probably became clear at that time to several authors that both enumeration problems are very similar as long as one deals with polyominoes that are both row- and column-convex (each row and each column is a single connected component — see Figure 3). In particular, the site-perimeter generating function for these classes is always algebraic, as is their perimeter generating function. This is because the site-perimeter of any convex polyomino is simply the perimeter minus the number of *inside corners* and it is relatively easy to extend existing perimeter solutions of families of convex polyominoes to include the number of inside corners [13, 24]. Unfortunately, this approach breaks down for non-convex polyominoes, and no family of non-convex polyominoes has yet been counted according to its site-perimeter. For the sake of completeness, let us mention, however, that directed column-convex polyominoes and directed diagonally-convex polyominoes have been counted according to their *directed* site-perimeter — being the number of nearest-neighbour vacant cells that lie to the *north* or *east* of cells of the polyomino [9, 14, 10, 16, 26]. In both cases, the associated generating function is algebraic. Directed site-perimeter plays the same role in *directed percolation* as site-perimeter plays in percolation.

The purpose of this article is to extend the list of site-perimeter results to the simplest family of non-convex polyominoes — *bargraphs.*

Definition 1. A *bargraph* is a column-convex polyomino, such that its lower edge lies on the horizontal axis. It is uniquely defined by the heights of its columns; see Figure 4.

The main result of this paper is the following closed form expression for the site-perimeter generating function of bargraphs.

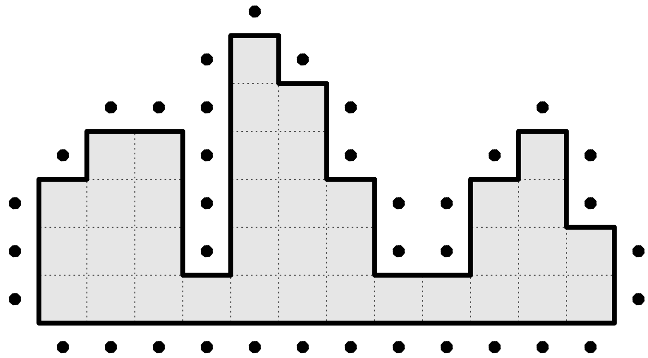

Figure 4: A bargraph polyomino with site-perimeter indicated.

Theorem 2. *Let b_n be the number of bargraphs with site-perimeter n. Let $\sigma \equiv \sigma(p)$ be the following algebraic power series in p:*

$$\sigma(p) = \frac{1 + 2p^3 - p^4 - p^5 - \sqrt{(1 + 2p^3 - p^4 - p^5)^2 - 4p^2(1 + p - p^2)^2}}{2p^2(1 + p - p^2)}.$$

Then the site-perimeter generating function of bargraphs is

$$\sum_{n \geq 0} b_n p^n = \frac{-p^3 \displaystyle\sum_{n \geq 0} \frac{\sigma^n p^{\binom{n+5}{2}}}{(p)_n (\sigma^2 p^3)_n (1 + p - p^2)^n}}{\displaystyle\sum_{n \geq 0} \frac{\sigma^n p^{\binom{n+5}{2}}}{(p)_n (\sigma^2 p^3)_n (1 + p - p^2)^n} \times \frac{(1 - \sigma p^{n+1})(1 - \sigma p^{n+2}) + \sigma^2 p^{2n+4}(1 - p)}{(1 - \sigma p^n)(1 - \sigma p^{n+1})}},$$

where we use $(a)_n$ to denote the product $(1 - a)(1 - ap) \dots (1 - ap^{n-1})$. The number of bargraphs with site-perimeter n grows asymptotically like

$$C p_c^{-n} n^{-3/2}$$

for some positive constant C, where $p_c = 0.45002...$ is the smallest positive solution of

$$1 - 2p - 2p^2 + 4p^3 - p^4 - p^5 = 0.$$

The contrast between Proposition 1 and our central result above demonstrates that moving from staircase polyominoes to bargraphs is far from trivial. We obtain a confirmation of the difficulty of the bargraph model by submitting it to the "Enting-Guttmann test": we compute the generating functions for polyominoes of fixed width n, for small values of n (these series are rational functions of p), and pay special attention to their denominator [21, 20]. For staircase polyominoes, the nth denominator is found to be $(1 - p^2)^{2n-1}$, while for bargraphs, the first denominators display more and more cyclotomic factors:

$$
\begin{aligned}
D_1 &= (1-p^2)^1 \\
D_2 &= (1-p^2)^2 \\
D_3 &= (1-p^2)^3(1+p+p^2)^1 \\
D_4 &= (1-p^2)^4(1+p+p^2)^2 \\
D_5 &= (1-p^2)^5(1+p+p^2)^3(1+p^2)^1 \\
D_6 &= (1-p^2)^6(1+p+p^2)^4(1+p^2)^2 \\
D_7 &= (1-p^2)^7(1+p+p^2)^5(1+p^2)^3(1+p+p^2+p^3+p^4)^1 \\
D_8 &= (1-p^2)^8(1+p+p^2)^6(1+p^2)^4(1+p+p^2+p^3+p^4)^2.
\end{aligned} \tag{1}
$$

This pattern suggests that the width and site-perimeter generating function for bargraphs is not *D-finite* (and in particular, not algebraic), a result that we prove in Section 5 (precise definitions will be given below). By contrast, the width and site-perimeter generating function for staircase polyominoes will be seen to be algebraic.

In the next section we shall examine the combinatorial constructions used to enumerate column-convex polyominoes according to their perimeter and/or area and demonstrate why they are difficult to generalise to include site-perimeter. In Section 3 we present a new construction that overcomes the difficulties described in Section 2. This construction leads to a linear functional equation that is more difficult to solve than those that have appeared previously in the polyomino literature; in Section 4 we solve it using a combination of known techniques. In passing, we also give a new proof of, and refine, the staircase polyomino result of Proposition 1.

Finally, in Section 5 we use our expression for the site-perimeter generating function to investigate the asymptotic growth of the number of bargraphs with site-perimeter n. We also prove that the generating function of bargraphs counted by their width and site-perimeter is not differentiably finite.

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Throughout the paper, we shall use the following standard notations: $\mathbb{C}[x_1, \ldots, x_n]$ denotes the set of polynomials in the variables x_i with complex coefficients, while $\mathbb{C}(x_1, \ldots, x_n)$ denotes the set of rational functions of the x_i. A formal power series $F(x_1, \ldots, x_n) \equiv F(\mathbf{x})$ with complex coefficients is said to be *algebraic* if there exists a non-trivial polynomial P in $n+1$ variables, with complex coefficients, such that $P(F, x_1, \ldots, x_n) = 0$. It is said to be *differentiably finite* (or *D-finite*) if it and its partial derivatives span a finite dimensional vector space over $\mathbb{C}(x_1, \ldots, x_n)$. Equivalently, for each variable x_i, there exists a nontrivial differential equation of the form:

$$P_d(\mathbf{x})\frac{\partial^d}{\partial x_i^d}F(\mathbf{x}) + \cdots + P_0(\mathbf{x})F(\mathbf{x}) = 0.$$

with $P_j \in \mathbb{C}[x_1, \ldots, x_n]$ for all j. It can be shown that any algebraic function is D-finite [23].

In the work that follows we will consider generating functions that count sets of polyominoes according to their right height (being the number of cells in their rightmost column), horizontal half-perimeter, vertical half-perimeter and site-perimeter; with these quantities being conjugate to the generating variables s, x, y and p respectively.

2 Two classical combinatorial constructions

Broadly speaking, there are two different methods that have been used to count bargraphs and other families of column-convex polyominoes [4, 3] by their area and/or perimeter:

- a *wasp-waist* factorisation which consists of splitting a bargraph into two smaller bargraphs at an especially thin point, and
- a *column-by-column* construction, which is often referred to as the Temperley method.

Both of these methods run into difficulties when we try to extend them to include site-perimeter; in both cases it is a similar configuration that causes the difficulties — a configuration that does not occur in convex polyominoes. While it is not obvious that the wasp-waist method can be made to work at all, we shall derive from a variation of the column-by-column construction a functional equation for the site-perimeter generating function of bargraphs; however this functional equation is more complicated to solve than those described in [3].

2.1 Wasp-waist factorisation

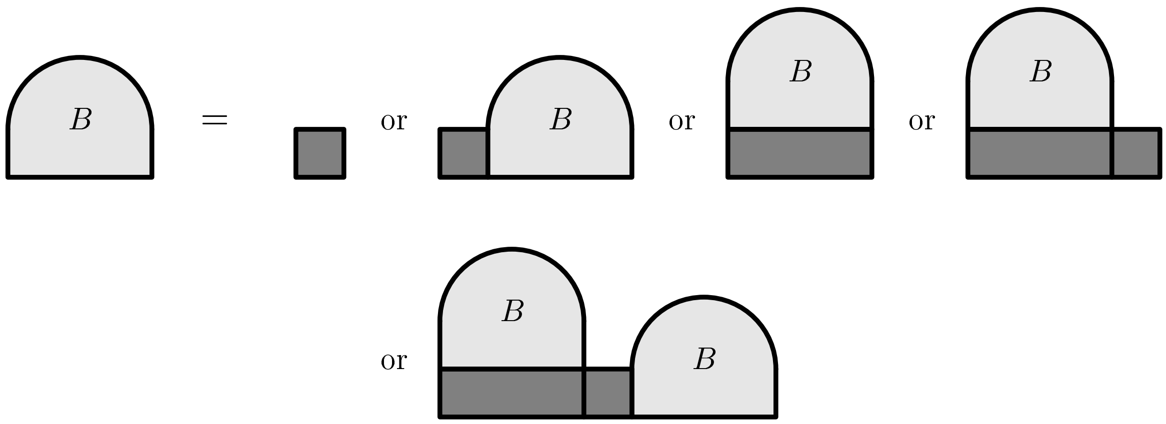

Figure 5: Wasp-waist factorisation of bargraphs.

The idea of the wasp-waist method is that any bargraph can be split into one or two smaller bargraphs at a point at which it is very thin (like the waist of a wasp). See Figure 5. Take a bargraph and find the leftmost column of height 1 (if there is one), then the bargraph is:

1. a single cell, or

2. a single cell attached to a bargraph (if the first column is of height 1), or
3. a bargraph whose bottom row has been duplicated (*i.e.* there is no column of height 1), or
4. a bargraph whose bottom row has been duplicated, with a single cell attached to its right (if only the last column is of height 1), or
5. a bargraph whose bottom row has been duplicated, connected to a single cell, and then connected to another bargraph.

Let $B(x, y)$ be the generating function of bargraphs enumerated by the number of columns (which is also the horizontal half-perimeter), and vertical half-perimeter. The above factorisation directly translates into an algebraic equation defining $B(x, y)$:

$$B(x, y) = xy + (x + y + xy)B(x, y) + xB(x, y)^2.$$

This is readily solved to give the half-perimeter generating function [15]:

$$B(x, y) = \frac{1}{2x}\left(1 - x - y - xy - \sqrt{(1 - x - y - xy)^2 - 4x^2y}\right)$$

This factorisation can be extended to give a functional equation for the perimeter and area generating function [25, 27]. However, trying to include the site-perimeter is problematic: Consider gluing together two columns, one of height a, the other of height b, according to the fifth case of the above factorisation (see Figure 6). The first column has site perimeter $2a + 2$, the second has site-perimeter $2b + 2$, while the resulting three-column polyomino has site-perimeter $a + b + \max(a, b) + 4$. In other words, the site perimeter of the product is not an affine combination of the heights (or the site-perimeters) of the components. Hence we cannot make simple substitutions in the generating functions to obtain the correct site-perimeter. Though it *may* be possible to make the wasp-waist method work, it is certainly not a simple generalisation of the perimeter solution. Note that the solution given in [11] for the site-perimeter of staircase polyominoes is essentially based on a wasp-waist factorisation.

2.2 Column-by-column construction

In this subsection we first demonstrate how a column-by-column construction for staircase polyominoes leads to a functional equation that defines their site-perimeter generating function. We then explain where the central difficulty lies when we try to extend this approach to bargraphs. In the next section, we shall show how one can circumvent this difficulty.

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Let $S(s; x, y, p)$ be the generating function of staircase polyominoes, enumerated according to the number of cells in their rightmost column, the horizontal and vertical half-perimeters and the site-perimeter (conjugate to the variables s, x, y and p respectively). We will frequently write $S(s)$ as short-hand for $S(s; x, y, p)$.

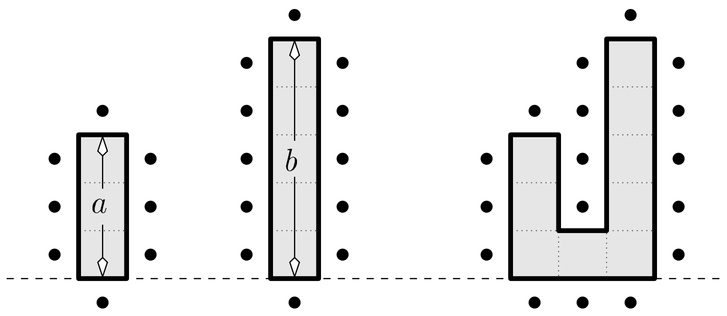

Figure 6: The problematic case of the wasp-waist factorisation of bargraphs. The site-perimeter of the final bargraph components involves the maximum of the heights of the components.

The techniques described in [3] are readily generalised to the enumeration of staircase polyominoes according to their perimeter and site-perimeter. Consider Figure 7. Each staircase polyomino either consists of a single column (case 1) or can be constructed by appending a column of cells to a smaller staircase polyomino in one of four ways (cases 2–5).

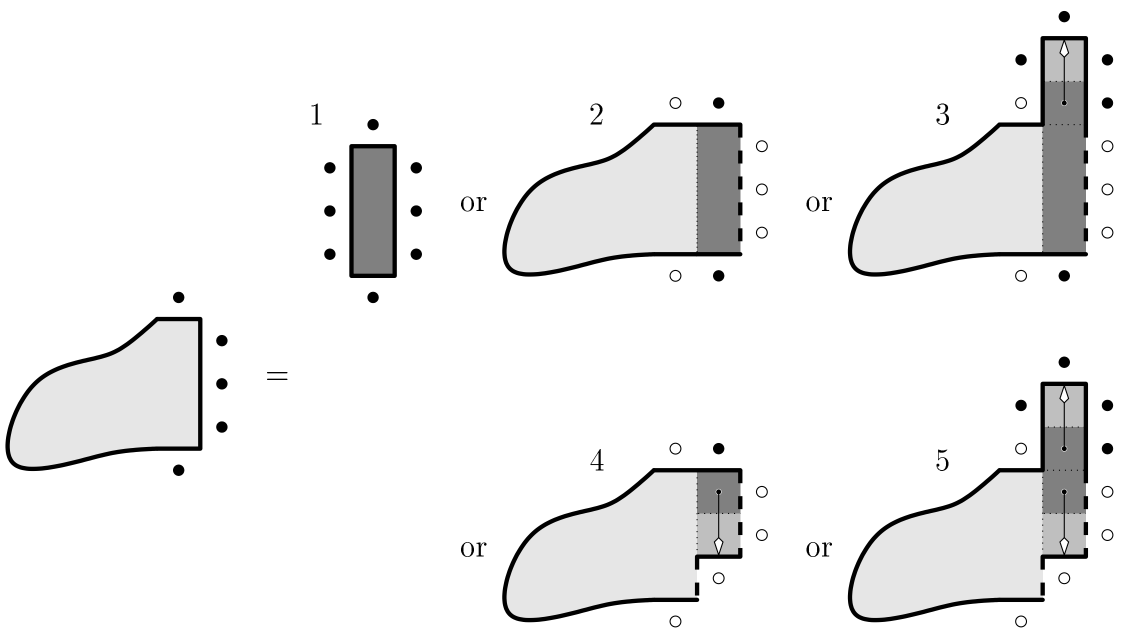

Figure 7: Constructing staircase polyominoes column-by-column. Solid circles indicate new site-perimeter, while hollow circles indicate existing site-perimeter. Similarly dashed lines indicate existing vertical perimeter.

Each of these five cases can be translated into an expression in terms of the generating function.

1. a single column has generating function

$$\frac{xysp^4}{1-syp^2},$$

2. duplicating the last column of the polyomino gives

$$xp^2S(s),$$

3. appending a new column such that the upper edge of the new column is strictly higher, while the lower edge has the same height gives

$$\frac{xysp^3}{1-syp^2}S(s)$$

4. appending a new column such that the lower edge is strictly higher, while the upper edge has the same height gives

$$\frac{xp}{1-s}\big(sS(1)-S(s)\big).$$

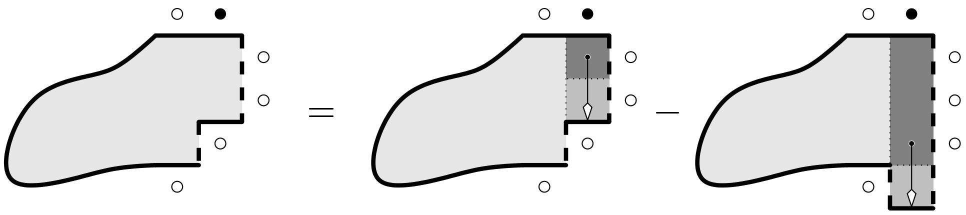

Figure 8: When constructing case 4, care must be taken not to over-count some configurations that have already been counted by case 2, nor to count configurations that are not staircase polyominoes.

This case requires more care than the previous cases. When appending the new column, we require that its lower edge be *strictly* higher than that of the previous column. To ensure that only such configurations are counted we subtract off those configurations where the lower edge does not satisfy this condition (see Figure 8) — these configurations (being careful of their perimeter and site-perimeter) are counted by $xpS(s)/(1-s)$. The final case involves a similar construction:

5. appending a new column such that the lower and upper edges are strictly higher gives

$$\frac{xysp^2}{(1-syp^2)(1-s)}\big(sS(1)-S(s)\big).$$

We note that in each of these cases we only need to know the height of the rightmost column (conjugate to the generating variable s) in order to append correctly the new column.

By adding each of these contributions together we obtain the following:

Proposition 3. *The generating function of staircase polyominoes satisfies the following functional equation:*

$$S(s) = \frac{xsyp^4}{1-syp^2} + \frac{xsp(1+syp-syp^2)}{(1-s)(1-syp^2)}S(1) - \frac{xp(1-p+sp)(1+spy-syp^2)}{(1-s)(1-syp^2)}S(s),$$

where we have written $S(s)$ in place of $S(s;x,y,p)$.

The variable s is said to be *catalytic* since, even though we are not especially interested in the corresponding parameter (the height of the rightmost column) we definitely need s to write a functional equation. The terminology is due to Zeilberger [33].

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Clearly, one can also construct bargraphs column by column, and check how their site-perimeter is modified when adding a column. Consider a three column bargraph reduced to a *well*: that is, its columns have heights i, j and k respectively, with $i > j < k$. (An example is given by Figure 6, with $i = a$, $j = 1$ and $k = b$.) This bargraph has site-perimeter $i + k + 5 + \max(i, k) - j$. Hence if we wish to construct it by appending a column of height k to a two-column bargraph with column heights i and j, then we need to remember both i and j.

Accordingly, our first attempt to solve the bargraph problem was to introduce two catalytic variables, namely a variable s for the height of the last column, and a variable t for the height of the next-to-last column. In order to make this idea work, we realised that it was necessary to separate bargraphs into two types, depending on the relative heights of their last two columns. This led us to a *system of two coupled functional equations in two catalytic variables.* With some work, we were able to massage this system to a *single functional equation in a single catalytic variable.* Later we saw that this equation could be interpreted as a construction that appends one or two columns at a time, and so could be derived more elegantly and directly; this is the construction we describe below.

3 A specific construction for bargraphs

As noted above, the reason that the site-perimeter of bargraphs is harder to calculate than that of families of column- and row-convex polyominoes is due to the presence of well configurations. To illustrate how this problem may be overcome, we will describe how to add a well to the right of a bargraph.

3.1 Appending a well

Take a bargraph B that ends in a column of height n, and consider how two columns of heights b and c (respectively) may be appended onto such a bargraph to form a well (see Figure 9). The column heights must satisfy $n > b > 0$ and $c > b$. It is clear that the vertical half-perimeter increases by $(c - b)$ and the number of columns increases by 2. The increase

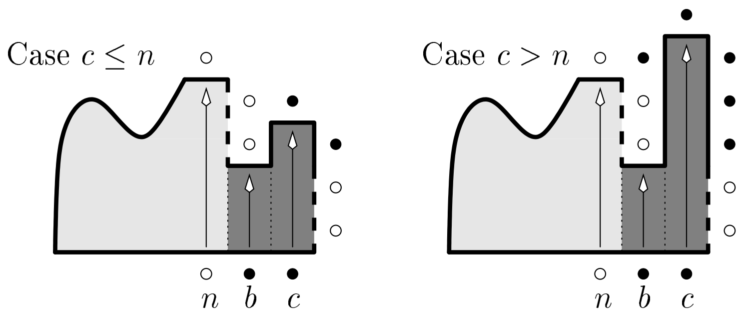

Figure 9: Adding a well to a bargraph.

in the site-perimeter is $(c-b)+\max\{0, c-n\}+3$. Hence, if $M(s;x,y,p) \equiv M(s)$ is the generating function of this single bargraph B (meaning that $M(s)$ is really a monomial), then the generating function $\tilde{M}(s;x,y,p) \equiv \tilde{M}(s)$ for all bargraphs obtained by appending a well to B is

$$\tilde{M}(s) = x^2 M(1;x,y,p) \sum_{b=1}^{n-1} \left(\sum_{c=b+1}^{n} p^3 (yp)^{-b} (syp)^c + \sum_{c \geq n+1} p^{3-n} (yp)^{-b} (syp^2)^c \right).$$

After summing these geometric series, we obtain

$$\begin{aligned} \tilde{M}(s) &= x^2 M(1;x,y,p) \left(\frac{s^2 p^3 q (1-s^{n-1})}{(1-sq)(1-s)} - \frac{s^{n+1} p^3 q^2 (1-p)(1-y^{n-1}p^{n-1})}{(1-q)(1-sq)(1-spq)} \right) \\ &= \frac{x^2 s^2 p^3 q M(1)}{(1-s)(1-sq)} - \frac{x^2 s p^3 q (1-pq) M(s)}{(1-s)(1-q)(1-spq)} + \frac{x^2 s p^3 q (1-p) M(sq)}{(1-q)(1-sq)(1-spq)} \qquad (2) \end{aligned}$$

where $q = yp$. Note that the above expression is a linear transformation of $M(s)$. Hence if $M(s)$ now denotes the generating function for *any class* $\mathcal{M}$ of bargraphs, then the generating function for all bargraphs obtained by adding a well in all possible ways to bargraphs of $\mathcal{M}$ will be given by (2).

3.2 The functional equation for bargraphs

We are now able to describe a new construction of bargraphs, which consists of appending one or two columns at a time; this can then be translated into a functional equation satisfied by their site-perimeter generating function. The most striking difference between the new functional equation below and that derived for staircase polyominoes in Proposition 3, is that this equation contains a new term, namely $B(sq)$ which comes from the well configurations and already appears in Eq. (2). This produces a drastic change in the nature of the functional equation. At first sight, the new equation appears to be very similar to those obtained in [3] for the enumeration of families of directed convex polyominoes according to their perimeter and area. However, a closer examination reveals a major difference: the new equation cannot be simply evaluated at $s = 1$, and this causes the solution described in [3] to fail.

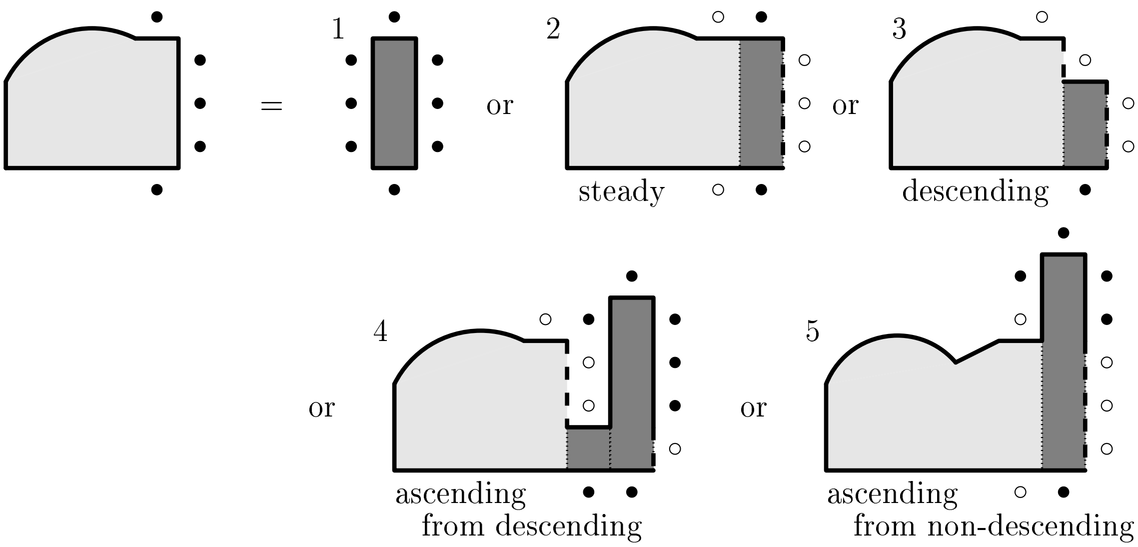

Figure 10: Building bargraphs

Proposition 4. *The generating function of bargraphs satisfies the following functional equation*

$$B(s) = a(s) + b(s)B(1) + c(s)B(sq) + d(s)B(s),$$

where

$$\begin{aligned} a(s) &= \frac{xsqp^3}{1-spq} \\ b(s) &= \frac{xsp\big((1-sq)(1-spq) + xs^2p^2q^2(1-p)\big)}{(1-s)(1-sq)(1-spq)} \\ c(s) &= \frac{x^2sqp^3(1-p)}{(1-q)(1-sq)(1-spq)} \\ d(s) &= -\frac{xp\Big((1-q)(1-p)(1+s^2pq) + sp\big((1-q)(1+pq-2q) + xpq^2(1-p)\big)\Big)}{(1-q)(1-s)(1-sqp)}, \end{aligned}$$

and we have written $q = py$, and $B(s)$ in place of $B(s; x, y, p)$.

Proof. We proceed in much the same manner as for staircase polyominoes and consider how columns may be appended. The construction is made simpler by dividing the set of bargraphs into three different subsets according to the relative heights of the two rightmost columns: we call a bargraph *ascending* if it is either a single column, or the heights of its two rightmost columns are strictly increasing (from left to right). A bargraph is *steady*, if the heights of its two rightmost columns are equal. A bargraph is *descending* if it is neither ascending nor steady. The bargraph construction is in five cases. These are shown in Figure 10, and we describe them below:

1. The generating function for bargraphs consisting of a single column is

$$\frac{xysp^4}{1-syp^2} = \frac{xsqp^3}{1-spq}.$$

2. If a bargraph is *steady* then it can be (uniquely) constructed by duplicating the last column of some bargraph; doing this increases the site perimeter by two and the number of columns by 1 and so gives

$$xp^2 B(s).$$

3. If a bargraph is *descending*, then it can be constructed by appending a shorter column to the right of some bargraph; doing so increases the site-perimeter by one, and gives:

$$\frac{xp}{1-s}\big(sB(1) - B(s)\big).$$

   This case is analogous to Case 4 of the staircase polyomino construction.

While the construction of steady and descending bargraphs is relatively easy, more care is required in the construction of ascending bargraphs. In particular we must consider if we are constructing an ascending bargraph from an ascending, steady or descending bargraph.

4. Consider first how to construct an ascending bargraph from a *descending* one. Such a construction gives rise to a well configuration, and it is simpler to consider appending two columns (to form the well) to the end of *any* bargraph, rather than appending a single column to a descending bargraph. This construction is then the one described in the previous subsection (see (2)), and so gives:

$$\frac{x^2s^2p^3qB(1)}{(1-s)(1-sq)} - \frac{x^2sp^3q(1-pq)B(s)}{(1-s)(1-q)(1-spq)} + \frac{x^2sp^3q(1-p)B(sq)}{(1-q)(1-sq)(1-spq)}.$$

5. Finally, we construct an ascending bargraph from a bargraph that is either *steady* or *ascending* in the following manner. Let $R(s; x, y, p)$ be the generating function of bargraphs that are steady or ascending. By analogy with case 3 of the staircase polyomino construction, the generating function for our fifth and last class of bargraphs is then

$$\frac{xsqp^2}{1-spq}R(s).$$

   Since R counts all bargraphs, except the descending ones, case 3 of the current construction gives

$$R(s) = B(s) - \frac{xp}{1-s}\big(sB(1) - B(s)\big)$$

   and hence constructing an ascending bargraph from a non-descending bargraph gives:

$$\frac{xsqp^2}{1-spq}\left(B(s) - \frac{xp}{1-s}\big(sB(1) - B(s)\big)\right).$$

Adding these different contributions together finishes the proof. $\square$

4 Site-perimeter generating functions

In this section we will solve the functional equation of Proposition 4. Our method combines two different techniques that have appeared previously in the combinatorics literature, but which have so far been applied independently. One of them is a simple iteration technique, which aims to “kill” the $B(sq)$ term. It was the key tool in [3]. The other one is the so-called *kernel method* which has been known since the 70’s, and is currently undergoing something of a revival (see the references in [1, 2, 6]). In the following subsection, we see it at work on the equation for staircase polyominoes, and thus we provide a new proof of Proposition 1.

4.1 Staircase polyominoes & the kernel method

Above we derived a functional equation satisfied by the staircase polyomino generating function (Proposition 3). Let us concentrate on the site-perimeter (only) generating function by setting $x = y = 1$. The generating function then satisfies

$$S(s) = \frac{sp^4}{1-sp^2} + \frac{sp(1+sp-sp^2)}{(1-s)(1-sp^2)}S(1) - \frac{p(1-p+sp)(1+sp-sp^2)}{(1-s)(1-sp^2)}S(s),$$

where we write $S(s)$ as shorthand for $S(s;1,1,p)$.

This functional equation contains two (related) unknowns, $S(s)$ and $S(1)$. Perhaps the first thing one might try is to remove the unknown $S(s)$ by setting $s = 1$. Since some of the terms in the equation are singular at $s = 1$, we must first multiply through by $(1-s)$ and then set $s = 1$. This gives

$$0 = \frac{p(1+p-p^2)}{1-p^2}S(1) - \frac{p(1+p-p^2)}{1-p^2}S(1),$$

and so yields a tautology. If one retains x and y then one reaches a similar result.

Indeed this approach *should not* yield a solution, since it would lead to a *rational* solution for $S(1;x,y,p)$. This result would then imply that $S(1;1,1,p)$ and $S(1;x,x,1)$ are both rational contradicting Proposition 1 and other well established results[2].

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Let us start again by collecting all the $S(s)$ terms on the left-hand side of the equation

$$\left(1 + \frac{p(1-p+sp)(1+sp-sp^2)}{(1-s)(1-sp^2)}\right) S(s) =$$
$$\frac{(1+p-p^2)(1-s(1-p+p^2)+s^2p^2)}{(1-s)(1-sp^2)}\ S(s) = \frac{sp^4}{1-sp^2} + \frac{sp(1+sp-sp^2)}{(1-s)(1-sp^2)}S(1). \quad (3)$$

[2]Staircase polyominoes, counted according to their perimeter, are equivalent through a myriad of bijections to a vast number of objects counted by the Catalan numbers (see [30, Exercise 6.19]).

We can remove the unknown $S(s)$ from this equation by choosing a value of s, such that the coefficient in front of $S(s)$, called the *kernel*, is zero. That is to say, we set $s = \sigma$ such that

$$1 - \sigma(1 - p + p^2) + \sigma^2 p^2 = 0.$$

This quadratic equation has two solutions, only one of which, denoted below by σ, is a formal power series in p:

$$\sigma = \frac{1 - p + p^2 - \sqrt{(1 + p + p^2)(1 - 3p + p^2)}}{2p^2} = 1 + p + O(p^2).$$

The other solution is $1/(p^2\sigma)$. Note that $S(\sigma(p); 1, 1, p)$ is a well-defined power series in p. Substituting σ for s into equation (3) then gives

$$0 = \frac{\sigma p^4}{1 - \sigma p^2} + \frac{\sigma p(1 + \sigma p - \sigma p^2)}{(1 - \sigma)(1 - \sigma p^2)} S(1)$$

and so

$$\begin{aligned} S(1) &= \frac{p^3(\sigma - 1)}{1 + \sigma p(1 - p)} \\ &= \frac{p^2}{2}\left(1 - p^2 - 2p^3 + p^4 - (1 + p - p^2)\sqrt{(1 + p + p^2)(1 - 3p + p^2)}\right) \\ &= p^4 + 2p^6 + 2p^7 + 5p^8 + 10p^9 + 21p^{10} + \cdots \end{aligned}$$

which gives Proposition 1 and is in agreement with [11]. The same argument can be used to find the full x, y, p generating function, and we thus obtain a refinement of Proposition 1:

Proposition 5. *The three-variable generating function for staircase polyominoes is*

$$\begin{aligned} S(1; x, y, p) &= \frac{yp^3(\sigma - 1)}{1 + \sigma yp - \sigma yp^2} \\ &= xyp^4 + xy(x + y)p^6 + 2x^2y^2p^7 + xy(x^2 + xy + y^2 + xy^2 + x^2y)p^8 + \cdots \end{aligned}$$

where σ is the unique power series in p satisfying

$$(1 - \sigma)(1 - \sigma yp^2) + xp(1 - p + \sigma p)(1 + \sigma yp - \sigma yp^2) = 0.$$

4.2 Bargraphs

As noted above, the functional equation satisfied by the bargraph generating function appears, at first sight, to be quite similar to those satisfied by the area-perimeter generating functions of families of directed and convex polyominoes found in [3], with the essential difference that its coefficients are singular at $s = 1$. We shall combine the iteration technique of [3] with the kernel method described above.

Again, for simplicity, we shall work with the site-perimeter generating function, since the full generating function can be found using the same steps. The functional equation becomes

$$B(s) = a(s) + b(s)B(1) + c(s)B(sp) + d(s)B(s).$$

where we have written $B(s)$ as shorthand for $B(s;1,1,p)$ and

$$\begin{aligned}
a(s) &= \frac{sp^4}{1-sp^2}\\
b(s) &= \frac{sp\big(1-sp(1+p)+s^2p^3(1+p-p^2)\big)}{(1-s)(1-sp)(1-sp^2)}\\
c(s) &= \frac{sp^4}{(1-sp)(1-sp^2)}\\
d(s) &= -\frac{p\big((1-p)(1+s^2p^2)+sp(1-2p+p^2+p^3)\big)}{(1-s)(1-sp^2)}.
\end{aligned}$$

This equation is obviously more complicated than that found for staircase polyominoes due to the presence of a third unknown, namely $B(sp)$. We remove this unknown by iterating the functional equation.

We must first alter the form of the equation so as to isolate $B(s)$ and so make it simpler to iterate.

$$B(s) = \alpha(s) + \beta(s)B(1) + \gamma(s)B(sp), \tag{4}$$

where

$$\begin{aligned}
\alpha(s) &= \frac{a(s)}{1-d(s)} &= \frac{sp^4(1-s)}{\eta(s)}\\
\beta(s) &= \frac{b(s)}{1-d(s)} &= \frac{sp\big(1-sp(1+p)+s^2p^3(1+p-p^2)\big)}{\eta(s)(1-sp)}\\
\gamma(s) &= \frac{c(s)}{1-d(s)} &= \frac{sp^4(1-s)}{\eta(s)(1-sp)},
\end{aligned} \tag{5}$$

and

$$\eta(s) = (1+p-p^2)(1+s^2p^2) - s(1+2p^3-p^4-p^5). \tag{6}$$

By setting $s = sp$ in this equation we obtain an expression for $B(sp)$ in terms of $B(1)$ and $B(sp^2)$ which can be substituted into equation (4) to give

$$B(s) = \Big(\alpha(s)+\gamma(s)\alpha(sp)\Big) + \Big(\beta(s)+\gamma(s)\beta(sp)\Big)B(1) + \gamma(s)\gamma(sp)B(sp^2).$$

At first sight this does not appear to have been helpful since it has simply replaced the unknown $B(sp)$ with $B(sp^2)$, however if we press on and iterate this process we obtain

$$\begin{aligned}
B(s) = \left(\sum_{k=0}^{N}\gamma(s)\ldots\gamma(sp^{k-1})\alpha(sp^k)\right) &+ \left(\sum_{k=0}^{N}\gamma(s)\ldots\gamma(sp^{k-1})\beta(sp^k)\right)B(1)\\
&+ \Big(\gamma(s)\ldots\gamma(sp^N)\Big)B(sp^{N+1}).
\end{aligned}$$

We notice that $\gamma(s) = sp^4\Big(1 + o(1)\Big)$ in the space of formal power series in p (with rational coefficients in s), and so

$$\prod_{k=0}^{N} \gamma(sp^k) = s^{N+1} p^{\binom{N+5}{2}-6}\Big(1 + o(1)\Big)$$

which implies that $\prod_{k=0}^{N} \gamma(sp^k)$ converges to the zero function in this space of formal power series. Further, it ensures that the sums

$$\sum_{k=0}^{N} \gamma(s)\ldots\gamma(sp^{N-1})\alpha(sp^N) \qquad \text{and} \qquad \sum_{k=0}^{N} \gamma(s)\ldots\gamma(sp^{N-1})\beta(sp^N)$$

are convergent (as formal power series in p) as $N \to \infty$. Hence by taking the $N \to \infty$ limit in the above equation we obtain:

$$B(s) = \left(\sum_{k\geq 0} \gamma(s)\ldots\gamma(sp^{k-1})\alpha(sp^k)\right) + \left(\sum_{k\geq 0} \gamma(s)\ldots\gamma(sp^{k-1})\beta(sp^k)\right) B(1). \qquad (7)$$

Note that, as a power series in p, $\gamma(s)$ has rational coefficients in s, while all the $\gamma(sp^i)$, for $i > 0$, have polynomial coefficients in s.

Up to this point the application of the iteration method to bargraphs enumerated according to their site-perimeter is very similar to its application to the functional equations in [3]. The next step in that paper was to remove the unknown $B(s)$ by setting $s = 1$ and then solving for $B(1)$. Unfortunately when trying to set $s = 1$ in equation (7) above, we find that

$$\alpha(1) = \gamma(1) = 0 \qquad \text{and} \qquad \beta(1) = 1,$$

so that equation (7) becomes $B(1) = B(1)$, which, though true, is of no help[3]. This is very similar to the situation we observed in the resolution of the staircase polyomino functional equation and we shall again use the kernel method to obtain the solution.

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A careful examination of equation (7) shows that the denominator of each summand contains a factor of $\eta(s)$ and so by multiplying through by $\big(1-d(s)\big) = \eta(s)/(1-s)/(1-sp^2)$ we can rewrite the equation as

$$B(s)\big(1 - d(s)\big) = \left(a(s) + c(s)\sum_{k\geq 1} \gamma(sp)\ldots\gamma(sp^{k-1})\alpha(sp^k)\right)$$
$$+ \left(b(s) + c(s)\sum_{k\geq 1} \gamma(sp)\ldots\gamma(sp^{k-1})\beta(sp^k)\right) B(1).$$

[3]Strictly speaking, one first has to multiply equation (7) by $\eta(s)$ before setting s to 1, in order to be in the space of formal power series in p with *polynomial* coefficients in s. Indeed, the series $\gamma(s)$, when expanded in p, has coefficients that diverge at $s = 1$.

This is somewhat equivalent to equation (3), and we seek to eliminate $B(s)$ by setting $s = \sigma$ such that $\eta(\sigma) = 0$ and so obtain an expression for $B(1)$. As was the case for staircase polyominoes, there is only one solution to this equation that is also a formal power series, namely

$$\sigma = \frac{1 + 2p^3 - p^4 - p^5 - \sqrt{(1 + 2p^3 - p^4 - p^5)^2 - 4p^2(1 + p - p^2)^2}}{2p^2(1 + p - p^2)}.$$

This gives:

$$B(1) = -\frac{a(\sigma) + c(\sigma)\sum_{k\geq 1}\gamma(\sigma p)\dots\gamma(\sigma p^{k-1})\alpha(\sigma p^k)}{b(\sigma) + c(\sigma)\sum_{k\geq 1}\gamma(\sigma p)\dots\gamma(\sigma p^{k-1})\beta(\sigma p^k)}. \tag{8}$$

Substituting this expression into equation (7) gives the full solution $B(s; 1, 1, p)$.

We obtain Theorem 2 by substituting the expressions for $a(s)$, $b(s)$, $\alpha(s)$, $\beta(s)$ and $\gamma(s)$ into the above expression (8) for $B(1)$. The expression is greatly simplified by noting that the kernel $\eta(s)$ may be rewritten as

$$\eta(s) = (1 + p - p^2)(1 - s/\sigma)(1 - s\sigma p^2)$$

so that

$$\eta(\sigma p^n) = (1 + p - p^2)(1 - p^n)(1 - \sigma^2 p^{n+2}).$$

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This technique may also be applied, with a little additional effort, to the full perimeter and site-perimeter generating function: the formal expression (8) still holds, with p now replaced by $q = yp$, the rational functions $\alpha(s)$, $\beta(s)$ and $\gamma(s)$ being related to $a(s)$, $b(s)$, $c(s)$ and $d(s)$ as in (5), where $a(s), b(s), c(s)$ and $d(s)$ are themselves given by Proposition 4.

Theorem 6. *The perimeter and site-perimeter generating function of bargraphs is given by*

$$B(1; x, y, p) = \frac{-p^2 q \sum_{n\geq 0} \frac{x^{2n}\sigma^n p^{3n}(1-p)^n q^{\binom{n+2}{2}}}{(1-q)^n(1+xp-xp^2)^n (q)_n (\sigma^2 pq^2)_n}}{\sum_{n\geq 0} \frac{x^{2n}\sigma^n p^{3n}(1-p)^n q^{\binom{n+2}{2}}}{(1-q)^n(1+xp-xp^2)^n (q)_n (\sigma^2 pq^2)_n} \times \frac{(1-\sigma q^{n+1})(1-\sigma p q^{n+1}) + x\sigma^2 p^2 q^{2n+2}(1-p)}{(1-\sigma q^n)(1-\sigma q^{n+1})}},$$

where σ is the unique power series in p which satisfies

$$(1-q)(1-\sigma)(1-\sigma qp) = \\ - xp\big((1-q)(1-p)(1+\sigma^2 pq) + \sigma p\left((1-q)(1+pq-2q) + xpq^2(1-p)\right)\big).$$

We have written $q = py$ and $(a)_n$ as shorthand for the product $\prod_{k=0}^{n-1}(1 - aq^k)$.

5 Analysis of the generating function

In this section we analyse two aspects of the generating function we have just obtained: the asymptotic behaviour of the number of bargraphs with site-perimeter n, and the nature of the width & site-perimeter generating function.

5.1 Site-perimeter asymptotics

We determine the asymptotic behaviour of the number of bargraphs with site-perimeter n by analysing the singularity structure of the generating function given in Theorem 2. An examination of this series shows that the possible sources of singularities are:

- divergence of the summands in the numerator and denominator,
- divergence of the numerator or denominator,
- a singularity arising from the square-root in $\sigma(p)$,
- poles given by the zeros of the denominator.

It is in fact the case that the dominant singularity is a square-root singularity arising from the square-root singularity in $\sigma(p)$.

Theorem 7. *The site-perimeter generating function for bargraphs has a unique dominant isolated singularity at $p = p_c = 0.45002...$ where*

$$1 - 2p - 2p^2 + 4p^3 - p^4 - p^5 = 0.$$

Further, it is a square root singularity. Consequently, the number of bargraphs with site-perimeter n grows asymptotically like

$$C\mu^n n^{-3/2}$$

where $\mu = 1/p_c = 2.2221....$

Proof. Let $B(p)$ denote the series of Theorem 2. Since this is a formal power series with positive coefficients, one of its dominant singularities lies on the positive p-axis, and coincides with the radius of convergence (Pringsheim's theorem). We will first show that there is no singularity on this axis for $0 \leq p < p_c$ and hence inside the open disc $|p| < p_c$. We will then extend this to $|p| = p_c, p \neq p_c$, and finally show that there is indeed an isolated singularity at $p = p_c$ and that it is a square root singularity.

The first step is to observe that the series $\sigma(p)$ itself has a unique dominant (square root) singularity at p_c. Then, the algebraic equation that σ satisfies, namely $\eta(\sigma) = 0$ (where $\eta(s)$ is given by (6)) may be rearranged to give

$$\sigma = 1 + p + (\sigma - 1)p^2 + \frac{\sigma p^3(\sigma - 1 + p^2)}{1 - \sigma p^2}.$$

This shows that the series $\sigma(p)$ has nonnegative coefficients. In particular, for $|p| \le p_c$, we have $|\sigma(p)| \le \sigma(|p|)$. Moreover, $\sigma(p)$ increases from 1 to $1/p_c$ as p goes from 0 to p_c, while $p\sigma(p)$ increases from 0 to 1.

Let $\tilde{B}(s,p)$ be the following bivariate series:

$$\tilde{B}(s,p) = \frac{N(s,p)}{D(s,p)}$$

where the numerator is (up to some power of p) the same as the numerator of $B(p)$:

$$N(s,p) = -p^3 \sum_{n\ge 0} \frac{p^{3n} s^n p^{\binom{n+2}{2}}}{(1+p-p^2)^n (p)_n (s^2p^3)_n}$$

while the first two terms of the denominator are rewritten so as to avoid singularities at $s=1$ and $sp=1$:

$$\begin{aligned} D(s,p) \;\; &= \;\; \frac{-(1-p-3p^2+p^3+2p^4+3p^5+2p^6-5p^7-5p^8+3p^9+2p^{10}-p^{11})s}{(1-p)(1+p-p^2)(1-sp^2)(1-s^2p^3)} \\ &\quad + \frac{p^2s^2(1+p-p^2)(p^{10}+p^9-4p^8-3p^7+2p^6+3p^5+p^4-2p^2-p+1)}{(1-p)(1+p-p^2)(1-sp^2)(1-s^2p^3)} \\ &\quad + \sum_{n\ge 2} \frac{p^{3n} s^n p^{\binom{n+2}{2}}}{(1+p-p^2)^n (p)_n (s^2p^3)_n} \times \frac{(1-sp^{n+1})(1-sp^{n+2})+s^2p^{2n+4}(1-p)}{(1-sp^n)(1-sp^{n+1})}. \end{aligned}$$

Given the algebraic equation satisfied by $\sigma(p)$, it is easy to check that $B(p) = \tilde{B}(\sigma(p),p)$.

Clearly, $N(s,p)$ and $D(s,p)$ are convergent as soon as $|p| < 1$ and $|s^2p^3| < 1$. Since both these conditions are satisfied for $|p| \le p_c$ and $s = \sigma(p)$, we are able to conclude that the numerator and denominator of $B(p)$ are convergent on this disk, and analytic on $\{|p| \le p_c, p \ne p_c\}$. Thus the only possible singularities of B on this set are isolated poles given by the zeros of the denominator. Assume such a pole exists, and take one of minimal modulus r. Then by Pringsheim's theorem, r itself is a singularity of B. We shall prove that the denominator of B (or, equivalently, the series $D(\sigma(p),p)$) is always negative for $p \in [0, p_c]$.

On this interval, we have $1+p-p^2 \ge 1$, $p\sigma \le 1$, and for $n \ge 2$, $2n + \binom{n+2}{2} \ge 6n-2$, so that the third part of $D(\sigma(p),p)$ is trivially bounded from above by

$$\sum_{n\ge 2} \frac{p_c^{6n-2}}{(1-p_c)^n(1-p_c)^n} \times \frac{2}{(1-p_c)(1-p_c^2)} = \frac{2p_c^{10}}{(1-p_c)^4(1+p_c)((1-p_c)^2-p_c^6)} = 0.017458... \tag{9}$$

Given that

$$(p^{10}+p^9-4p^8-3p^7+2p^6+3p^5+p^4-2p^2-p+1) > 0,$$

the sum of the first two parts of $D(\sigma(p),p)$ is bounded by

$$\begin{aligned} R(p) &= \frac{-\sigma(1-p-3p^2+p^3+2p^4+3p^5+2p^6-5p^7-5p^8+3p^9+2p^{10}-p^{11})}{(1-p)(1+p-p^2)(1-\sigma p^2)(1-\sigma^2p^3)} \\ &\quad + \frac{p\sigma(1+p-p^2)(p^{10}+p^9-4p^8-3p^7+2p^6+3p^5+p^4-2p^2-p+1)}{(1-p)(1+p-p^2)(1-\sigma p^2)(1-\sigma^2p^3)} \\ &= -\frac{\sigma(1-p^2)(p^{10}+p^9-5p^8-3p^7+3p^6+4p^5+p^4-3p^2-p+1)}{(1+p-p^2)(1-\sigma p^2)(1-\sigma^2p^3)}. \end{aligned}$$

The numerator of this expression is positive for $0 \le p \le p_c$, so that, since $\sigma \ge 1$,

$$R(p) \le -\frac{p^{10}+p^9-5p^8-3p^7+3p^6+4p^5+p^4-3p^2-p+1}{(1+p-p^2)(1-p^3)}.$$

This function of p is monotone increasing and equals $-.05614...$ at p_c. Comparison with (9) shows that the denominator of B is always negative on $[0,p_c]$.

This tells us that the radius of convergence of B is at least p_c, and that $B(p_c)$ is finite. Since $B(p)$ has positive coefficients, it is also finite on the circle $|p| = p_c$. Hence the only possible singularity of B on the disk $|p| \le p_c$ is p_c itself. Now $N(s,p)$ and $D(s,p)$ are holomorphic in a neighborhood of $(\sigma(p_c),p_c) = (1/p_c,p_c)$, and, since $D(1/p_c,p_c) \neq 0$, the series $\tilde{B}(s,p)$ is holomorphic in a neighborhood of $(1/p_c,p_c)$. A local expansion gives, as p approaches p_c:

$$B(p) = \tilde{B}(1/p_c,p_c) + (\sigma(p)-1/p_c)\frac{\partial\tilde{B}}{\partial s}(1/p_c,p_c)(1+o(1)) + (p-p_c)\frac{\partial\tilde{B}}{\partial p}(1/p_c,p_c)(1+o(1))$$

and the announced result follows, since for some positive constant a

$$\sigma(p) = 1/p_c - a\sqrt{1-p/p_c},$$

provided $\frac{\partial\tilde{B}}{\partial s}(1/p_c,p_c) \neq 0$, which can be checked numerically. □

5.2 Nature of the generating function

It is clear that the expressions we have obtained for the bargraph generating functions in Theorems 2 and 6 are substantially more complicated than those of staircase polyominoes (stated in Proposition 1 and 5). The presence of the algebraic series $\sigma(p)$ in the expression also makes these series more complicated than the q-series arising in the area and perimeter generating functions of families of column-convex polyominoes (see [3]).

This section is devoted to proving that the generating function of bargraphs counted by their width and site-perimeter is indeed fundamentally different from that of staircase polyominoes, in that it is not differentiably finite. By contrast, the three-variable generating function $S(1;x,y,p)$ given in Proposition 5, which counts staircase polyominoes by their width, height and site-perimeter, is algebraic, and hence D-finite.

Theorem 8. *The generating function $B(1; x, 1, p)$ which counts bargraphs by their width and site-perimeter is not D-finite. Consequently, the series $B(1; x, y, p)$ and $B(s; x, y, p)$ are not D-finite either.*

The second statement comes from the fact that the specialisations of a D-finite series are D-finite [23]. Our proof that the two-variable power series $B(1; x, 1, p)$ is not D-finite is inspired by a numerical test Guttmann and Enting proposed for examining the "solvability" of models in lattice statistical mechanics [21]. It consists of examining the singularity structure of the coefficients of the power series expanded in one of its variables.

We will show, by iterating the functional equation of Proposition 4, that the coefficient of x^n in the bargraph generating function $B(1; x, 1, p)$ is a rational function of p, whose denominator is a product of cyclotomic polynomials[4]. The first few denominators of these rational functions are given by (1). They suggest that a new cyclotomic polynomial factor appears in the denominator of every second coefficient of x, so that more and more singularities accumulate on the unit circle $|p| = 1$. This starkly contrasts with the staircase polyomino generating function — the denominator of the coefficient of x^n is simply $(1 - p^2)^{2n-1}$. Such an accumulation of singularities indicates that the power series is not D-finite.

Proposition 9 ([5]). *Let $F(x, p) = \sum_{n \geq 0} x^n F_n(p)$ be a D-finite series in x with rational coefficients in p. That is to say, there exist polynomials $P_0, \dots, P_d$ such that*

$$P_d(x, p) \frac{\partial^d}{\partial x^d} F(x, p) + \cdots + P_0(x, p) F(x, p) = 0.$$

For $n \geq 0$, let S_n be the set of poles of $F_n(p)$, and let $S = \bigcup_n S_n$. Then S has only a finite number of accumulation points.

Theorem 8 follows by applying the above proposition to the following result.

Proposition 10. *For $n \geq 2$, the coefficient of x^{2n-3} in the bargraph generating function $B(1; x, 1, p)$ is a rational function of p that is singular at any primitive n^{th} root of unity.*

Proof. Let $B_n(s, p) \equiv B_n(s)$ denote the coefficient of x^n in the generating function $B(s; x, 1, p)$. Using the functional equation of Proposition 4, we can compute these coefficients by induction on n. Indeed, we have the initial conditions:

$$B_0(s) = 0, \qquad\qquad B_1(s) = \frac{sp^4}{1 - sp^2},$$

and for $n \geq 1$,

$$B_{n+1}(s) = c_1(s) B_n(1) + c_2(s) B_{n-1}(1) + c_3(s) B_n(s) + c_4(s) B_{n-1}(s) + c_5(s) B_{n-1}(sp), \quad (10)$$

[4]The cyclotomic polynomials $\Psi_d(x)$ are the factors of $(1 - x^n)$, for $n \geq 1$. More precisely, $(1 - x^n) = \prod_{d|n} \Psi_d(x)$.

where

$$
\begin{aligned}
c_1(s) &= \frac{sp}{1-s} \\
c_2(s) &= \frac{s^3p^5(1-p)}{(1-s)(1-sp)(1-sp^2)} \\
c_3(s) &= -\frac{p(1-p)(1+sp-sp^2+s^2p^2)}{(1-s)(1-sp^2)} \\
c_4(s) &= -\frac{sp^5}{(1-s)(1-sp^2)} \\
c_5(s) &= \frac{sp^4}{(1-sp)(1-sp^2)}.
\end{aligned}
$$

The recurrence relation (10) shows that each coefficient $B_n(s)$ can be written as a rational function of s and p, the denominator of which is a product of factors $(1-p^i)$, with $i \geq 1$ and $(1-sp^i)$, with $i \geq 0$. As the series $B_n(1,p)$ is well-defined (it counts bargraphs of width n by their site-perimeter), the denominator of $B_n(s)$ actually does not contain any factor $(1-s)$. Let $\mathbb{C}_n[s,p]$ be the set of polynomials in s and p that may be written as a product

$$\prod_{i=1}^{n}(1-p^i)^{c_i}(1-sp^i)^{d_i}$$

where c_i, d_i are some non-negative integers. Similarly we denote by $\mathbb{C}_n[p]$ the set of polynomials in p that may be written as a product $\prod_{i=1}^{n}(1-p^i)^{c_i}$. We are going to prove that

$$B_{2m-1}(s) = \frac{N_{2m-1}(s)}{(1-sp^{m+1})D_{2m-1}(s)}, \tag{11}$$

$$B_{2m}(s) = \frac{N_{2m}(s)}{(1-p^{m+1})(1-sp^{m+1})^2 D_{2m}(s)}, \tag{12}$$

where $N_m(s)$ and $D_m(s)$ are some polynomials in s and p, with the further restriction that both $D_{2m-1}(s)$ and $D_{2m}(s)$ belong to $\mathbb{C}_m[s,p]$.

Indeed, (11) is true for $m=1$, since $B_1(s) = sp^4/(1-sp^2)$. Then, equation (10) can be used to compute the denominator of $B_2(s)$, which is found to be $(1-p^2)(1-sp^2)^2$, so that (12) also holds for $m=1$.

Let us now assume that $B_{2m-3}(s)$ and $B_{2m-2}(s)$ satisfy the above property, with $m \geq 2$, and apply the recurrence (10) with $n = 2m-2$. We see that the only term that introduces denominator factors that do not belong to $\mathbb{C}_m[s,p]$ is the last one, namely $c_5(s)B_{2m-3}(sp)$. By assumption,

$$B_{2m-3}(sp) = \frac{N_{2m-3}(sp)}{(1-sp^{m+1})D_{2m-3}(sp)}$$

and $D_{2m-3}(sp)$ belongs to $\mathbb{C}_m[s,p]$. Hence we may write $B_{2m-1}(s)$ as required by (11). A similar analysis of the coefficient B_{2m} proves (12).

Let ξ be a primitive $(n+1)$th root of unity, that is, a root of the irreducible polynomial $\Psi_{n+1}(p)$. We want to prove that $B_{2n-1}(1)$ is singular at ξ. We are actually going to prove a stronger result, namely that, for $m = 1 \dots n$, the series $B_{2m-1}(p^{n-m}, p)$ is singular at ξ. Of course, the case $m = n$ then gives the desired result.

We proceed by induction on m with fixed n. As $B_1(s) = sp^4/(1-sp^2)$, the result is true for $m = 1$. Let $m \geq 2$, and assume the result holds for $m-1$. The recurrence relation (10), combined with the forms (11-12), shows that

$$B_{2m-1}(s) = \frac{N(s)}{D(s)} + \frac{sp^4}{(1-sp)(1-sp^2)} B_{2m-3}(sp)$$

where $D(s) \in \mathbb{C}_m[s,p]$. Thus by setting $s = p^{n-m}$ we find:

$$B_{2m-1}(p^{n-m}) = \frac{N(p^{n-m})}{D(p^{n-m})} + \frac{p^{n-m+4}}{(1-p^{n-m+1})(1-p^{n-m+2})} B_{2m-3}(p^{n-m+1})$$

where the polynomial $D(p^{n-m})$ belongs to $\mathbb{C}_n[p]$, and hence cannot vanish at ξ. By the inductive hypothesis, $B_{2m-3}(p^{n-m+1})$ is singular at ξ, and so is $B_{2m-1}(p^{n-m})$. $\square$

◁ ◁ ◇ ▷ ▷

We note that similar accumulations of poles have been observed in the two-variable generating functions of many lattice animal problems [21, 20]. In certain cases, including some animal models related to heaps of dimers [5] and self-avoiding polygons [29, 28], these numerical observations have been sharpened into proofs. Consequently, these generating functions are known to be non-D-finite.

Remark. The non-D-finitedness of $B(1; x, 1, p)$ does not give any information about the nature of the power series $B(1; 1, 1, p)$. One can readily construct multi-variable series that are not D-finite, whose specialisations are D-finite. Consider for example, the series

$$F(x,p) = \sum_{n \geq 1} \frac{x^n p^n}{(1-p^n)(1-p^{n+1})}.$$

By Proposition 9 it is not D-finite in x. However, setting $x = 1$ in this function gives

$$\begin{aligned} F(1,p) &= \sum_{n\geq 1} \frac{p^n}{(1-p^n)(1-p^{n+1})} = \frac{1}{1-p} \sum_{n \geq 1} \left(\frac{p^n}{1-p^n} - \frac{p^{n+1}}{1-p^{n+1}} \right) \\ &= \frac{p}{(1-p)^2}, \end{aligned}$$

which is rational, hence D-finite.

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A Series data

site-perimeter n	bargraph polyominoes	staircase polyominoes	site-perimeter n	bargraph polyominoes	staircase polyominoes
1	0	0	16	752	2838
2	0	0	17	1500	6678
3	0	0	18	3022	15825
4	1	1	19	6107	37734
5	0	0	20	12429	90469
6	2	2	21	25365	217962
7	2	2	22	52042	527418
8	4	5	23	107090	1281250
9	8	10	24	221235	3123603
10	14	21	25	458316	7639784
11	26	46	26	952439	18740795
12	52	102	27	1984262	46096732
13	97	230	28	4144601	113666820
14	193	526	29	8676232	280928470
15	377	1216	30	18202536	695796891

Table 1: The number of bargraphs and staircase polyominoes with site-perimeter n.

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