# Enumeration of three-dimensional convex polygons


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**Abstract**

A self-avoiding polygon (SAP) on a graph is an elementary cycle. Counting SAP on the hypercubic lattice $\mathbb{Z}^d$, with $d \geq 2$, is a well-known unsolved problem, which is studied both for its combinatorial and probabilistic interest and its connections with statistical mechanics. Of course, polygons on $\mathbb{Z}^d$ are defined up to a translation, and the relevant statistic is their perimeter.

A SAP on $\mathbb{Z}^d$ is said to be *convex* if its perimeter is 'minimal', that is, is exactly twice the sum of the side lengths of the smallest hyper-rectangle containing it. In 1984, Delest and Viennot enumerated convex SAP on the square lattice, but no result was available in higher dimension.

We present an elementary approach to enumerate convex SAP in any dimension. We first obtain a new proof of Delest and Viennot's result, which explains combinatorially the form of the generating function. We then compute the generating function for convex SAP on the cubic lattice. In dimension larger than 3, the details of the calculations become very cumbersome. However, our method suggests that the generating function for convex SAP on $\mathbb{Z}^d$ is always a quotient of *differentially finite* power series.


## 1 Introduction

The study of self-avoiding walks (SAW) and self-avoiding polygons (SAP) on a lattice (Fig. 1) is an important and difficult problem which has obvious combinatorial interest and several remarkable connections with statistical mechanics: polygons and walks have become standard models for polymers in dilute solution, and the enumeration of self-avoiding walks is equivalent to the solution of the $n$-vector model, in the limit $n \to 0$. However, despite the pioneering work of Hammersley [19] and others, rigorous results remain elusive. It is strongly believed that the number $a_n$ of SAW (resp. SAP) of length (resp. perimeter) $n$ on a given lattice behaves like $\mu^n n^{\gamma-1}$ (with logarithmic terms in dimension 4. For SAP in $\mathbb{Z}^d$ the perimeter $n$ must of course be even). What has been proved in general is that there exists a constant $\mu$ such that $a_n^{1/n} \to \mu$ when $n \to \infty$ (see [20] and references therein). It is also known that the *connective constant* $\mu$ is the same for walks and polygons on a given lattice. The *universality hypothesis* asserts that the *critical exponent* $\gamma$ only depends on the dimensionality of the lattice. Some nonrigorous methods, confirmed by numerical studies, suggest that $\mu = (2+\sqrt{2})^{1/2}$ for the honeycomb lattice, the critical exponents being $\gamma_w = 43/32$ and $\gamma_p = -3/2$ for walks and polygons respectively for all two-dimensional lattices. Moreover, it has been recently proved than $\gamma = 1$ in dimension greater than 4. A good description of the state of the art in this field is given by Hughes [22] and by Madras and Slade [28].

Obviously, we are very far from proving *exact* results on the number of self-avoiding walks or self-avoiding polygons. This has led to the study of restricted classes of SAP (and, to a smaller

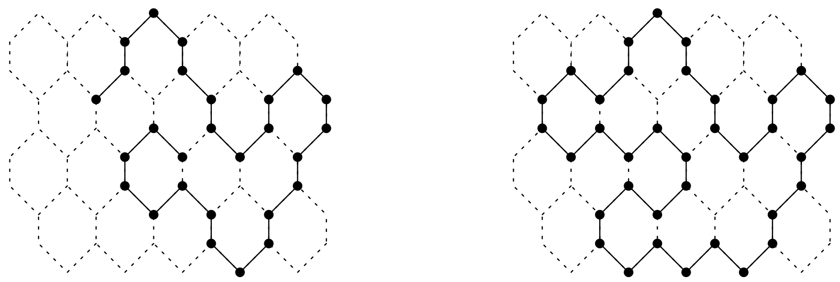

Figure 1: A self-avoiding walk and a self-avoiding polygon on the honeycomb lattice.

extent, SAW), mostly in 2 dimensions. Remarkably, all the classes of SAP that have been enumerated satisfy a *convexity* property and have an algebraic perimeter generating function. Convexity is usually thought of as a property of the *polyomino* (finite connected set of cells) delimited by the polygon. For instance, a polyomino on the square lattice is said to be *vertically convex* if its intersection with any vertical line is connected (Fig. 2). But convexity can also be defined directly on the corresponding polygon: a SAP on the square lattice is *vertically convex* if the following (equivalent) properties are satisfied:

• its intersection with any vertical line has at most two connected components,

• its intersection with any vertical half-space is connected,

• the number of horizontal edges is minimal, that is, it is exactly twice the width of the smallest bounding rectangle of the polygon.

Horizontal convexity is defined is a similar manner. A polyomino (or a self-avoiding polygon) is *convex* if it is vertically and horizontally convex.

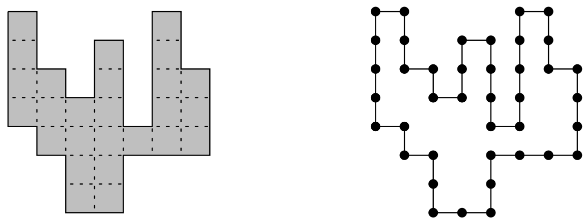

Figure 2: A vertically convex polyomino on the square lattice and the corresponding SAP.

Virtually any reasonable class of 2-dimensional SAP satisfying a convexity property can be enumerated by the perimeter (see for instance [6, 24, 25] for convex SAP and [4, 5, 13] for vertically convex SAP). The most recent results in this field deal with *area* generating functions [3], but this topic has no immediate counterpart in dimension 3 and above.

The definition of convexity given above can readily be extended to self-avoiding polygons on the hypercubic lattice $\mathbb{Z}^d$. A self-avoiding polygon will be said to be *convex* if it satisfies the following (equivalent) properties:

• its intersection with any hyperplane perpendicular to one of the coordinate axes has at most two connected components,

• its intersection with any half-space delimited by such a hyperplane is connected,

• its perimeter is *minimal*, that is, it is exactly twice the sum of the side lengths of its smallest

bounding hyper-rectangle.
*Staircase* self-avoiding polygons form a simple subset of convex SAP which has already been enumerated [18]. To our knowledge, this is the only known result in dimension $> 2$. In this paper we study convex SAP, and a newly defined class of polygons (which includes staircase polygons), called *unimodal* SAP. On the square lattice, they correspond to the so-called *directed convex polyominoes.* Fig. 3 shows some examples. Precise definitions will be given in the next section.

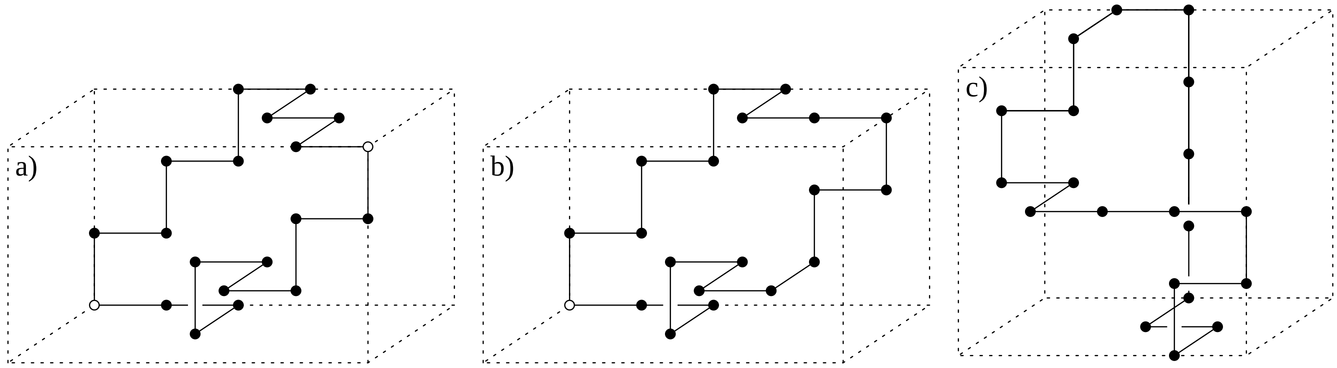


Figure 3: (a) A staircase SAP, with open circles showing the vertices of minimal and maximal coordinates of the bounding hyper-rectangle, which must be vertices of the polygon. (b) A unimodal SAP, showing the vertex of minimal coordinates of the bounding hyper-rectangle, which must be a vertex of the polygon. (c) A convex SAP, whose perimeter is equal to the 'perimeter' of the minimum bounding rectangle.

We present in this paper an approach that permits, at least in principle, the enumeration of convex SAP on any hypercubic lattice. In practice, our work permits the enumeration of unimodal SAP on any hypercubic lattice and provides closed form expressions for the generating functions for convex SAP on the square and cubic lattices. All the generating functions we obtain are quotients of *differentially finite* power series. As mentioned above, the result for 2-dimensional convex SAP is not new. However, our method gives an interesting combinatorial interpretation of the form of the solution. Our three-dimensional result is new and extends the (short) list of exactly solved 3-dimensional models: in this connection, let us cite the Zamolodchikov model [1], directed animals on the cubic lattice with next-nearest-neighbour bonds [8], and a variety of results on plane partitions, starting with MacMahon [27].

Our method is new, that is, is not an extension of any of the methods previously used to count convex SAP on the square lattice. It underlines the fact that counting convex SAP on $\mathbb{Z}^d$ is, essentially, a 1-dimensional problem: we encode *not necessarily self-avoiding* polygons by words on a finite alphabet, and then count the words that correspond to self-avoiding polygons via an inclusion-exclusion argument. This ingredient of the method, although very classical in combinatorics, has never been used previously in polygon enumeration. Most 2D results previously obtained use either correspondences with algebraic languages, or variations on the Temperley methodology, or various decompositions of polyominoes. Indeed, the inclusion-exclusion principle cannot arise naturally if polygons are thought of in terms of surfaces (polyominoes) rather than paths on a lattice. However, our approach has certainly been inspired by a standard argument for counting 2D staircase SAP, which is explained in full detail in [14] and [16], and was probably used by Pólya [29]. In this argument, one considers staircase polygons that are not necessarily self-avoiding, and factors them into self-avoiding staircase polygons. See [12] for a clever variation of this argument and [18] for an extension to higher dimensions. Our approach can also be related to [13], where it is shown how the enumeration of vertically convex SAP can be simplified by considering a slightly more general class of objects.

Our paper is built as follows: in Section 2, we define the main notions (polygons on $\mathbb{Z}^d$, convexity, etc.) in terms of *words* on the alphabet $\{1, 2, \ldots, d\} \cup \{\bar{1}, \bar{2}, \ldots, \bar{d}\}$. We also enumerate three simple families of convex polygons and introduce certain differentially finite power series in terms of which we will state our results. Section 3 presents the exact enumeration of unimodal SAP, while the last section is devoted to the enumeration of convex SAP, with exact formulae in dimension 2 and 3.

# 2 Definitions and first results

## 2.1 Polygons and loops

Let $d \geq 1$, and let us consider the lattice $\mathbb{Z}^d$, with its canonical basis $(e_1, \ldots, e_d)$. An *oriented rooted polygon* of perimeter $2n$ is a $2n$-tuple $(s_1, \ldots, s_{2n})$ of vertices of $\mathbb{Z}^d$ such that $s_i$ and $s_{i+1}$ are neighbours for $1 \leq i \leq 2n$ (with $s_{2n+1} = s_1$). Note that such a polygon is not necessarily self-avoiding. An *oriented polygon* is an oriented rooted polygon defined up to a cyclic permutation of its vertices: that is, the polygon $(s_1, \ldots, s_{2n})$ is equal to the polygon $(s_2, \ldots, s_{2n}, s_1)$. We say that the *rooted* polygon $(s_1, \ldots, s_{2n})$ *represents* the polygon $(s_1, \ldots, s_{2n})$ (unless specified, all the polygons we consider are oriented; hence the word "oriented" will often be omitted). A (rooted or unrooted) polygon is *self-avoiding* if $s_i \neq s_j$ for $1 \leq i \neq j \leq 2n$. A non-empty self-avoiding polygon (resp. rooted self-avoiding polygon) will also be called a *loop* (resp. *rooted loop*). Note that our definition implies that, if $s_1$ and $s_2$ are neighbours, then the polygon $(s_1, s_2)$ is self-avoiding; this might seem a bit strange, as this polygon visits the same edge of the lattice twice. However, it is the only "strange" self-avoiding polygon, and this is a convenient convention, as we later show.

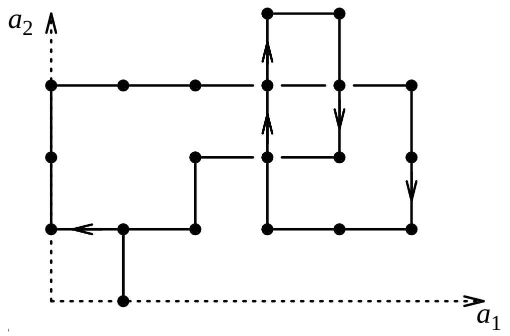


Figure 4: A polygon of dimension 2 on $\mathbb{Z}^2$: $2211111\bar{2}\bar{2}\bar{1}\bar{1}2221\bar{2}\bar{2}\bar{1}\bar{1}\bar{2}\bar{1}\bar{2}\bar{2}\bar{1}$.

As we are interested in their enumeration, polygons must actually be defined up to a translation of $\mathbb{Z}^d$. Therefore, a rooted polygon of perimeter $2n$ will often be regarded as a word $u = u_1 u_2 \ldots u_{2n}$ on the alphabet $\mathcal{A} = \{1, 2, \ldots, d\} \cup \{\bar{1}, \bar{2}, \ldots, \bar{d}\}$, meaning that, if $u_i = k$ (resp. $\bar{k}$), then one goes from the vertex $s_i$ to the vertex $s_{i+1}$ by taking a unitary step along the unit vector $e_k$ (resp. $-e_k$). Note that the number of occurrences of $k$ in $u$, denoted $|u|_k$, is equal to the number of occurrences of $\bar{k}$ in $u$. Conversely, any word $u$ on $\mathcal{A}$ that satisfies $|u|_k = |u|_{\bar{k}}$ for $1 \leq k \leq d$ is a rooted polygon. The *length* $|u|$ of a word $u$ is its number of letters. The *dimension* of a polygon $u$ is the dimension of its convex hull. Equivalently, it is the number of $k \in \{1, \ldots, d\}$ such that $|u|_k > 0$ (see Fig. 4).

We now define three important classes of polygons, illustrated by Fig. 5: the *staircase, unimodal*, and *convex* polygons. Unoriented, self-avoiding examples are given in Fig. 3.

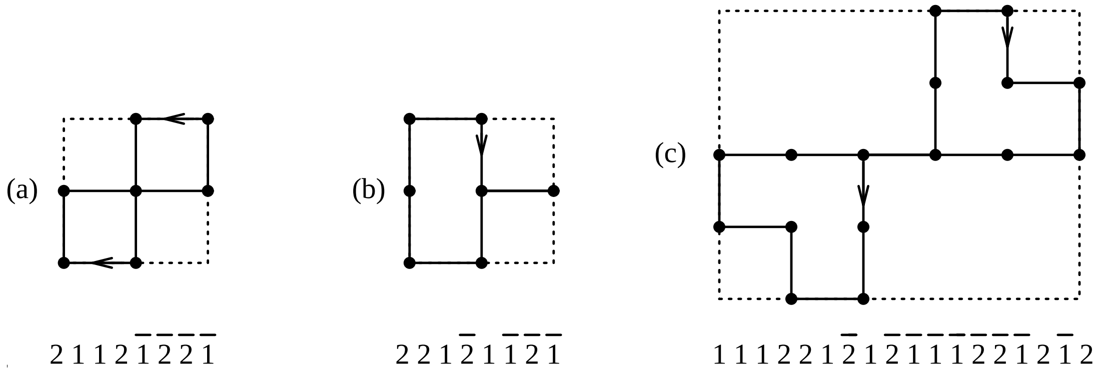


Figure 5: (a) A staircase polygon with $v = 2112$ and $w = \bar{1}\bar{2}\bar{2}\bar{1}$. (b) A unimodal polygon. It is unimodal in direction 1, as it can be encoded $vw$, with $v = 221\bar{2}1$ and $w = \bar{1}\bar{2}\bar{1}$, and in direction 2 as shown by the encoding $v = 221$ and $w = \bar{2}1\bar{1}\bar{2}\bar{1}$. (c) A convex polygon. The rooted polygon $111221\bar{2}1\bar{2}\bar{1}\bar{1}\bar{1}\bar{2}\bar{2}\bar{1}2\bar{1}2$ is unimodal in direction 1, but not in direction 2. Another choice of root gives a rooted polygon unimodal in direction 2 but not 1.

A rooted polygon is a rooted *staircase* polygon if it factors as $vw$, where $v$ (resp. $w$) is a word on $\{1, \ldots, d\}$ (resp. $\{\bar{1}, \ldots \bar{d}\}$). This implies in particular that $|v| = |w|$. A polygon is a *staircase polygon* if it can be represented by a rooted staircase polygon (which is unique when it exists) (Fig. 5.(a)).

Let $k \leq d$. A rooted polygon is *unimodal in direction* $k$ if it can be written $vw$ (where now both $v$ and $w$ are words on $\{1, \ldots, d\} \cup \{\bar{1}, \ldots, \bar{d}\}$), with $|v|_{\bar{k}} = |w|_k = 0$. It is *unimodal* if it is unimodal in all directions. A polygon is unimodal if it can be represented by a unimodal rooted polygon (which is unique when it exists) (Fig. 5.(b)).

A polygon is *convex in direction* $k$ if it can be represented by a rooted polygon that is unimodal in direction $k$. It is *convex* if it is convex in all directions: for each $k$, there exists a rooted version of the polygon (usually depending on $k$) that is unimodal in direction $k$ (for unimodal polygons, the *same* rooted polygon works for all directions). Equivalently, its perimeter is minimal, that is, is exactly twice the sum of the side lengths of its smallest bounding hyper-rectangle (Fig. 5.(c)). Two other, more geometrical characterizations of convexity have been given in the introduction. A unimodal polygon is naturally convex. More precisely, a unimodal polygon is a convex polygon that contains the vertex of minimal coordinates of its smallest bounding hyper-rectangle. Similarly, a staircase polygon is a unimodal polygon that contains the vertex of maximal coordinates of its smallest bounding hyper-rectangle (Figs. 3 and 5).

Let $I \subset \{1, 2, \ldots, d\}$, and let $\ell$ be a rooted loop of dimension $|I|$ on the alphabet $I \cup \bar{I}$, with $\bar{I} = \{\bar{k} : k \in I\}$. Let $u$ be a (rooted or unrooted) polygon. If there exist two words $v$ and $w$ such that $u = v\ell w$, we say that $\ell$ is a *loop* of $u$, and that $u$ *has a loop* in $I$. For instance, the third polygon of Fig. 5 has two loops in $\{1, 2\}$: $221\bar{2}1\bar{2}\bar{1}\bar{1}$ and $\bar{2}\bar{2}\bar{1}2\bar{1}211$.

Our method for counting convex SAP can be roughly described as follows: we will enumerate all convex polygons, and then remove, via an inclusion-exclusion argument, those having several loops. In this process, we will use the fact that, if $\ell_1, \ldots, \ell_i$ are $i$ loops of a convex polygon $u$, then removing all the letters of $u$ occurring in these $i$ loops leaves another polygon. This is only true because two loops of a convex polygon never *overlap*, that is, never have a letter (an edge) in common. This is not true for general polygons: the polygon of Fig. 4 has two overlapping loops, namely $11\bar{2}\bar{2}\bar{1}\bar{1}22$ and $221\bar{2}\bar{2}\bar{1}$. Indeed, if one removes these two loops, one does not obtain a polygon.

### 2.2 First enumerative results

In this subsection we first enumerate rooted *unimodal* polygons on $Z^d$ *having no* 1*-dimensional loop.* We then obtain the corresponding result for *convex* polygons. From this, we derive an asymptotic expression for the number of such polygons, which gives an upper bound on the number of convex SAP.

Let $\mathcal{P}$ be a set of (rooted or unrooted) polygons. The *multi-perimeter generating function* for the elements of $\mathcal{P}$ is

$$\sum_{u\in\mathcal{P}} x_1^{|u|_1}\ldots x_d^{|u|_d},$$

and the *perimeter generating function* is

$$\sum_{u\in\mathcal{P}} t^{|u|_1+\cdot+|u|_d}=\sum_{u\in\mathcal{P}} t^{|u|/2}.$$

**Notations.** For $I\subset\{1,\ldots,d\}$, we define an operator $E_I$ that acts on Laurent series in the variables $x_1,\ldots,x_d$ with real coefficients. Let $f(x_1,\ldots,x_d)=\sum a_{n_1,\ldots,n_d}x_1^{n_1}\ldots x_d^{n_d}$. Then

$$E_I(f(x_1,\ldots,x_d))=\begin{cases} f(x_1,\ldots,x_d) & \text{if } I=\emptyset,\\ \sum a_{n_1,\ldots,n_{i-1},2n_i,n_{i+1},\ldots,n_d}x_1^{n_1}\ldots x_d^{n_d} & \text{if } I=\{i\},\\ E_{I\setminus\{i\}}\left[E_{\{i\}}(f(x_1,\ldots,x_d))\right] & \text{if } |I|>1 \text{ and } i\in I.\end{cases}$$

Equivalently,

$$E_{\{i\}}(f(x_1,\ldots,x_d))=\frac{1}{2}\left[f(x_1,\ldots,x_{i-1},\sqrt{x_i},x_{i+1},\ldots,x_d)+f(x_1,\ldots,x_{i-1},-\sqrt{x_i},x_{i+1},\ldots,x_d)\right].$$

A similar formula could be written for a general set $I$, but as the sum of $2^{|I|}$ different terms. We denote by $\bar{f}(t)$ the series $f(t,\ldots,t)$. We define $\bar{E}_I(f(x_1,\ldots,x_d))$ to be $\bar{g}(t)$, where $g(x_1,\ldots,x_d)=E_I(f(x_1,\ldots,x_d))$. If $1\le i\le j$, the interval $\{i,i+1,\ldots,j\}$ is denoted $[i,j]$.

We now give two simple lemmas, which illustrate two ingredients of this paper: the classical inclusion-exclusion principle and the use of the operators $E_I$. The second lemma provides a very good asymptotic upper bound on the number of convex loops.

**Lemma 2.1** — *The multi-perimeter generating function for rooted unimodal polygons of $\mathbb{Z}^d$ having no one-dimensional loop is*

$$E_{[1,d]}\left[\frac{(1-x_1)(1-x_2)\ldots(1-x_d)}{1-x_1-x_2-\cdots-x_d}\right].$$

**Proof.** In a rooted unimodal polygon, for $k\in\{1,\ldots,d\}$, all occurrences of the letter $k$ precede all occurrences of $\bar{k}$. Hence, the rooted unimodal polygons (including the empty polygon) are in one-to-one correspondence with the words $u$ on the alphabet $\{1,2,\ldots,d\}$ such that $|u|_k$ is even for all $k\le d$. Thus their multi-perimeter generating function is

$$E_{[1,d]}\left[\frac{1}{1-x_1-x_2-\cdots-x_d}\right].$$

More generally, given a set $I\subset[1,d]$, the multi-perimeter generating function for rooted unimodal polygons having a factor $k\bar{k}$ for all $k\in I$ is

$$S_I=E_{[1,d]}\left[\frac{\prod_{k\in I}x_k}{1-x_1-x_2-\cdots-x_d}\right].$$

For $k \leq d$, a unimodal rooted polygon has at most one occurrence of $k\bar{k}$. Hence the inclusion-exclusion principle implies that the multi-perimeter generating function for unimodal rooted polygons having no one-dimensional loop is

$$\sum_{I \subset [1,d]} (-1)^{|I|} S_I,$$

which concludes the proof.

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We can prove a similar result for convex polygons.

**Lemma 2.2** — *Let $d \geq 2$. The multi-perimeter generating function for convex polygons of $\mathbb{Z}^d$ that have no one-dimensional loop and are of dimension exactly $d$ is*

$$E_{[1,d]}\left[(d-1)! \frac{x_1 x_2 \dots x_d (1-x_1)^2 (1-x_2)^2 \dots (1-x_d)^2}{(1-x_1-x_2-\cdots-x_d)^d}\right].$$

**Proof.** In what follows, by the 'first' occurrence of a letter $k$ in an *unrooted* convex polygon $u = u_1 \dots u_{2n}$ (which is a *cyclic* word), we mean the letter $u_i = k$ such that, if

$$j = \min\{i' \in [1, 2n-1] : \ u_{i-i'} \in \{k, \bar{k}\}\},$$

then $u_{i-j} = \bar{k}$.

We represent a convex polygon $u$ by the rooted polygon $v$ beginning with the first occurrence of 1 in $u$ (as in Fig. 5.(c)). This puts the convex polygons of dimension $d$ in one-to-one correspondence with the words $v = v_1 v_2 \dots v_{2n}$ on the alphabet $\{1, 2, \dots, d\}$ such that

- $v_1 = 1$,
- for $2 \leq k \leq d$, an occurrence of $k$ in $v$ (corresponding to the first occurrence of $\bar{k}$ in $u$) is distinguished,
- $|v|_k$ is even for all $k \leq d$.

For instance, the polygon of Fig. 5.(c) is encoded by the word $v = 111221\bar{2}12111221212$. Thus the multi-perimeter generating function for convex polygons is

$$E_{[1,d]}\left[x_d \frac{\partial}{\partial x_d}\left(\dots\left(x_2 \frac{\partial}{\partial x_2}\left(\frac{x_1}{1-x_1-x_2-\cdots-x_d}\right)\right)\right)\right] = E_{[1,d]}\left[\frac{(d-1)! \ x_1 x_2 \dots x_d}{(1-x_1-x_2-\cdots-x_d)^d}\right].$$

More generally, given two subsets $I$ and $J$ of $\{1, \dots, d\}$, the multi-perimeter generating function for convex polygons having a factor $k\bar{k}$ for all $k \in I$ and a factor $\bar{k}k$ for all $k \in J$ is

$$S_{I,J} = E_{[1,d]}\left[\frac{(d-1)! \ x_1 x_2 \dots x_d \prod_{k \in I} x_k \prod_{k \in J} x_k}{(1-x_1-x_2-\cdots-x_d)^d}\right].$$

For $k \leq d$, a convex polygon has at most one occurrence of $k\bar{k}$ and one occurrence of $\bar{k}k$. We conclude the proof using the inclusion-exclusion principle, which implies that the multi-perimeter generating function for convex polygons having no one-dimensional loop is

$$\sum_{I,J \subset [1,d]} (-1)^{|I|+|J|} S_{I,J}.$$

This concludes the proof.

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An asymptotic result can be derived from the exact formula of Lemma 2.2.

**Corollary 2.3** — *The number of $d$-dimensional convex polygons on $\mathbb{Z}^d$ of perimeter $2n$ having no 1-dimensional loop is asymptotically equal to*

$$\frac{(d-1)^{2d}}{d^{3d}} d^{2n} n^{d-1} (1 + O(1/n)).$$

*This gives an asymptotic upper bound for the number $c_n^{(d)}$ of $d$-dimensional convex loops of perimeter $2n$:*

$$\limsup_{n\to\infty} \frac{c_n^{(d)} d^{3d}}{(d-1)^{2d} d^{2n} n^{d-1}} \leq 1.$$

**Proof.** Let $G(t) = \sum a_n t^n$ be the perimeter generating function for $d$-dimensional convex polygons having no one-dimensional loop, given by Lemma 2.2:

$$G(t) = \bar{E}_{[1,d]} \left[ (d-1)! \frac{x_1 x_2 \dots x_d (1-x_1)^2 (1-x_2)^2 \dots (1-x_d)^2}{(1 - x_1 - x_2 - \cdots - x_d)^d} \right].$$

It is a rational function. The dominant singularity is $1/d^2$. Hence the coefficients $a_n$ have the same asymptotic behaviour as the coefficients $b_n$ defined by

$$\sum b_n t^n = \frac{(d-1)!}{2^d} \left[ \left( \frac{u(1-u)^2}{1-du} \right)^d + \left( \frac{-u(1+u)^2}{1+du} \right)^d \right],$$

with $u = \sqrt{t}$. We obtain

$$b_n \sim \frac{(d-1)^{2d} d^{2n} n^{d-1}}{d^{3d}},$$

which gives the stated result.

■

**Remark.** The number of $d$-dimensional convex loops is also asymptotic to $(d-1)^{2d} d^{2n} n^{d-1} / d^{3d}$. This can be proved [32] via a probablistic argument similar to that used in [10]. In other words, *a random $d$-dimensional convex polygon of large perimeter having no 1-dimensional loop is almost surely self-avoiding.*

### 2.3 The series $Z_\lambda$

Our results will be expressed in terms of a family of multi-variable formal power series indexed by integer partitions and denoted $Z_\lambda$. In this subsection, we discuss several aspects of these series: definition, closed formulae, differential equations satisfied by their one-variable specializations, asymptotic behaviour of the coefficients... We also express the generating function for staircase SAP in terms of these series.

Let $\lambda = (\lambda_1, \dots, \lambda_m)$ be a partition of an integer $n$. That is, $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_m| > 0$ and $n = |\lambda| = \lambda_1 + \cdots + \lambda_m$. The $\lambda_k$ are called the *parts* of $\lambda$ and the *length* $\ell(\lambda)$ of $\lambda$ is $m$. We will sometimes write $\lambda = 1^{m_1} 2^{m_2} \dots n^{m_n}$ if $\lambda$ has $m_i$ parts equal to $i$ for $1 \leq i \leq n$. The *canonical set-partition of* $[1, n]$ associated with $\lambda$ is $\{I_1, \dots, I_m\}$ where, for $j \leq m$, $I_j$ is the following interval:

$$I_j = [\lambda_1 + \ldots + \lambda_{j-1} + 1, \lambda_1 + \ldots + \lambda_j]. \tag{1}$$

The formal power series $Z_\lambda$ is defined by

$$Z_\lambda(x_1,\ldots,x_n) = \sum_{(a_1,\ldots,a_n)\in\mathbb{N}^n} \binom{m+2\sum_{i=1}^{n} a_i}{\left(1+2\sum_{i\in I_j} a_i\right)_{1\le j\le m}} \prod_{j=1}^{m} \binom{\sum_{i\in I_j} a_i}{(a_i)_{i\in I_j}}^2 \prod_{i=1}^{n} x_i^{a_i} \qquad (2)$$

where $\binom{\sum_{i\in I} b_i}{(b_i)_{i\in I}}$ is the multinomial coefficient $(\sum b_i)!/\prod(b_i)!$. By convention, $Z_\emptyset = 1$. This series is *D-finite* [26], meaning that its derivatives $(\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_n)^{\alpha_n} Z_\lambda$, for $\alpha_1,\ldots,\alpha_n \ge 0$, span a finite dimensional vector space on the field $\mathbb{R}(x_1,\ldots,x_n)$ of rational functions in $x_1,\ldots,x_n$.

The parts of size one do not raise difficulties in the evaluation of $Z_\lambda$: a simple summation proves that, if $\lambda' = (\lambda_1,\ldots,\lambda_m,1)$, then

$$Z_{\lambda'}(x_1,\ldots,x_n,x_{n+1}) = E_{\{n+1\}}\left[\frac{1}{x_{n+1}(1-x_{n+1})^{m+1}} Z_\lambda\left(\frac{x_1}{(1-x_{n+1})^2},\ldots,\frac{x_n}{(1-x_{n+1})^2}\right)\right].$$

The Lagrange inversion formula can be used to prove that the series $Z_2(x_1,x_2)$ is algebraic [2]:

$$Z_2(x_1,x_2) = \sum_{(a_1,a_2)\in\mathbb{N}^2} \binom{a_1+a_2}{a_1,a_2}^2 x_1^{a_1}x_2^{a_2} = \frac{1}{\sqrt{1-2x_1-2x_2-2x_1x_2+x_1^2+x_2^2}}. \qquad (3)$$

When $\lambda$ is reduced to a single part $n$, the coefficients of $Z_\lambda$ are just squares of multinomial coefficients:

$$Z_n(x_1,\ldots,x_n) = \sum_{(a_1,\ldots,a_n)\in\mathbb{N}^n} \binom{a_1+\cdots+a_n}{a_1,\ldots,a_n}^2 \prod_{i=1}^{n} x_i^{a_i}.$$

This simpler series arise naturally in the study of staircase loops, as shown in [18].

**Proposition 2.4** — *The multi-perimeter generating function for (oriented) staircase loops on* $\mathbb{Z}^d$ *is*

$$S_d(x_1,\ldots,x_d) = 1 - \frac{1}{Z_d(x_1,\ldots,x_d)}.$$

**Proof.** $Z_d(x_1,\ldots,x_d)$ is exactly the generating function for staircase polygons on $\mathbb{Z}^d$. A staircase polygon is either empty, or factors in a unique way as $u\ell v$, where $\ell$ is a staircase *loop* and $uv$ a new staircase polygon. This gives

$$Z_d(x_1,\ldots,x_d) = 1 + S_d(x_1,\ldots,x_d)Z_d(x_1,\ldots,x_d),$$

which leads to the result. This factorization generalizes the "standard argument for counting 2D staircase SAP" mentioned in the introduction.

■

We will also consider the series $\bar{Z}_\lambda(t) = Z_\lambda(t,\ldots,t)$, which is also D-finite (in one variable) [31]. For instance, $\bar{Z}_2(t)$ is, according to (3), equal to $(1-4t)^{-1/2}$, but it can alternatively be characterized by the initial condition $\bar{Z}_2(0) = 1$ and the differential equation

$$(1-4t)\bar{Z}_2'(t) - 2\bar{Z}_2(t) = 0.$$

Similarly, the series $\bar{Z}_3(t)$ is characterized by $\bar{Z}_3(0) = 1$, $\bar{Z}_3'(0) = 3$ and

$$t(1-t)(1-9t)\bar{Z}_3''(t) + (1-20t+27t^2)\bar{Z}_3'(t) - 3(1-3t)\bar{Z}_3(t) = 0.$$

The series $\bar Z_4(t)$ is characterized by $\bar Z_4(0)=1$, $\bar Z_4'(0)=4$, $\bar Z_4''(0)=56$ and

$$t^2(1-4t)(1-16t)\bar Z_4'''(t)+3t(1-30t+128t^2)\bar Z_4''(t)+(1-68t+448t^2)\bar Z_4'(t)-4(1-16t)\bar Z_4(t)=0.$$

These differential equations can be guessed using the 'gfun' package of Maple [30]. In order to prove them, or equivalently, to prove the linear recurrence relations satisfied by their coefficients, it is convenient to use *Bessel generating functions* [11]. Let $z_n^{(d)}$ denote the coefficient of $t^n$ in $\bar Z_d(t)$. Then the Bessel generating function $Y_d(t)$ for these coefficients is

$$Y_d(t)=\sum_{n\geq 0} z_n^{(d)}\frac{t^n}{n!^2}=J_0(t)^d$$

where $J_0(t)$ is the (modified) Bessel function $\sum_{n\geq 0} t^n/n!^2$. One then introduces the series $J_1(t)=\sum_{n\geq 1} t^n/(n-1)!/n!$. The vector $(J_0,J_1)$ satisfies

$$\begin{pmatrix} J_0'(t)\\ J_1'(t)\end{pmatrix}=\begin{pmatrix} 0 & 1/t\\ 1 & 0\end{pmatrix}\begin{pmatrix} J_0(t)\\ J_1(t)\end{pmatrix},$$

from which it is easy to derive the differential equation satisfied by the vector $(J_0^d, J_0^{d-1}J_1,\ldots,J_1^d)$. This gives in turn the differential equation satisfied by $J_0(t)^d=Y_d(t)$.

Similar constructions can be used to find the differential equations satisfied by the other series $\bar Z_\lambda(t)$. For instance, $\bar Z_{2^2}(t)$ is characterized by $\bar Z_{2^2}(0)=2$, $\bar Z_{2^2}'(0)=16$, $\bar Z_{2^2}''(0)=304$, $\bar Z_{2^2}'''(0)=9984$, and

$$6t^2(1-4t)(1-16t)\bar Z_{2^2}^{(4)}(t)+t(33-1024t+4448t^2+512t^3)\bar Z_{2^2}'''(t)+2(15-1006t+6912t^2+1792t^3)\bar Z_{2^2}''(t)$$
$$-4(189-2912t-1312t^2)\bar Z_{2^2}'(t)+48(31+24t)\bar Z_{2^2}(t)=0.$$

Note that when $\lambda=2^m=(2,2,\ldots,2)$, the series $\bar Z_\lambda(t)$ reduces to

$$\bar Z_\lambda(t)=\sum_{n\geq 0}\left[t^n\sum_{b_1+\cdots+b_m=n}\binom{m+2n}{1+2b_1,\ldots,1+2b_m}\prod_{i=1}^m\binom{2b_i}{b_i}\right].$$

The singularities of $\bar Z_\lambda(t)$ and its behaviour around these singularities can be derived from the differential equation it satisfies using classical techniques [21]. From Flajolet and Odlyzko's *singularity analysis* [15], we can obtain asymptotic expansions for the coefficients of $Z_\lambda$. In particular, it was proved in [18] that, if $\bar Z_d(t)=\sum_n z_n^{(d)}t^n$, then

$$z_n^{(d)}=\frac{d^{d/2+2n}}{(4\pi n)^{(d-1)/2}}(1+O(n^{-1})).$$

The smallest singularity of $\bar Z_d$ is $1/d^2$, and around this point,

$$\bar Z_d(t)=A_d+\begin{cases} B_d(1-d^2t)^{(d-3)/2}(1+o(1)) & \text{if } d \text{ is even,}\\ B_d(1-d^2t)^{(d-3)/2}\ln(1-d^2t)(1+o(1)) & \text{otherwise,}\end{cases}\tag{4}$$

for some constants $A_d$ and $B_d$.

We conclude this section with a discussion on the "nature" of the series $\bar S_d$, $\bar U_d$ and $\bar C_d$, which count by the perimeter staircase, unimodal and convex loops respectively. Proposition 2.4 implies that $\bar S_d$ is such that $\bar Z_d(t)\bar S_d$ is D-finite. More precisely, $\bar S_d$ is algebraic (resp. D-finite) if and only if $1/\bar{Z}_d(t)$ is algebraic (resp. D-finite). The asymptotic form of $\bar{Z}_d(t)$ given by (4) proves that $\bar{S}_d$ cannot be algebraic if $d$ is odd. We can expect the series $\bar{U}_d$ and $\bar{C}_d$ to be at least "as complicated" as $1/\bar{Z}_d(t)$. Actually, we will see that $\bar{U}_2$ and $\bar{C}_2$ are, like $1/\bar{Z}_2(t)$, algebraic, and that $\bar{Z}_3(t)\bar{U}_3$ and $\bar{Z}_3(t)\bar{C}_3$ are D-finite. These properties will be generalized to some extent to higher dimensions. In particular, we will see that $\bar{U}_d$ is always a quotient of D-finite series. This implies that $\bar{U}_d$ is *differentially algebraic* [7]. Finally, our general method to count convex SAP suggests that $\bar{C}_d$ is also a quotient of D-finite series.

For a further study of the series $\bar{Z}_d(t)$, including their connections with lattice Green functions, Heun functions and elliptic integrals, see [9, 18, 23].

## 3 Unimodal self-avoiding polygons

In this section we show how the multi-perimeter generating function $U_d(x_1, \ldots, x_d)$ for unimodal loops of dimension $d$ lying in $\mathbb{Z}^d$ can be computed by induction on $d$ in terms of the series $Z_\lambda$, for $|\lambda| \leq d$. We also specialise to the perimeter generating function. From now on, we fix an integer $d \geq 2$.

As a unimodal polygon is represented by a *unique* unimodal *rooted* polygon, we only need to handle rooted polygons, i.e., words. Our method is based on the following two remarks:

- a loop $\ell$ of a rooted unimodal polygon $u\ell v$ is unimodal;
- if a rooted unimodal polygon $u\ell_1 v\ell_2 w$ has a loop $\ell_1$ in $I_1 \subset [1,d]$ and a loop $\ell_2$ in $I_2$, then $I_1 \cap I_2 = \emptyset$.

Thus the distribution of loops of a unimodal rooted polygon will be described by an element $\{I_1, I_2, \ldots, I_m\}$ of the following set:

$$\mathcal{S} = \{\{I_1, I_2, \ldots, I_m\} : m \geq 0,\ \emptyset \neq I_i \subset [1,d] \ \text{ and } \ I_i \cap I_j = \emptyset \ \text{ if } \ i \neq j\}. \tag{5}$$

In what follows, we usually assume that the sets $I_i$ are indexed in such a way that $|I_1| \geq \cdots \geq |I_m|$.

For $I = \{I_1, \ldots, I_m\} \in \mathcal{S}$, let $\mathcal{P}_I$ be the set of unimodal rooted polygons having a loop in $I_i$ for $1 \leq i \leq m$, and let $P_I$ be their multi-perimeter generating function. The inclusion-exclusion principle implies that

$$\sum_{I \in \mathcal{S}} (-1)^{|I|} P_I = 1, \tag{6}$$

the "1" in the right-hand side standing for the empty polygon, which has no loop.

Clearly, the series $P_{\{\{1\}\}}$ and $P_{\{\{2\}\}}$ are equal up to an exchange of variables $x_1$ and $x_2$. More generally, if $I = \{I_1, I_2, \ldots, I_m\}$, then the series $P_I$ depends only on the integer partition $(|I_1|, \ldots, |I_m|)$, up to a permutation of the variables. Thus, let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a partition of an integer $n$ not greater than $d$ and let $I = \{I_1, \ldots, I_m\}$ be the canonical set-partition of $[1,n]$ associated with $\lambda$, defined by (1). As $I$ belongs to $\mathcal{S}$, we can define $P_\lambda(x_1, \ldots, x_d)$ to be the series $P_I$. Then, if $J = (J_1, \ldots, J_m) \in \mathcal{S}$ and $(|J_1|, \ldots, |J_m|) = \lambda$, we have

$$P_J = P_\lambda\left((x_i)_{i \in J_1}, \ldots, (x_i)_{i \in J_m}, (x_i)_{i \in J_0}\right) \tag{7}$$

where $J_0 = [1,d] \setminus \bigcup_{1 \leq j \leq m} J_j$. Hence, in order to apply Eq. (6), we only need to compute the series $P_\lambda(x_1, \ldots, x_d)$ for all $\lambda$ such that $|\lambda| \leq d$. The following proposition explains how to express $P_\lambda$ in terms of the series $Z_\lambda$ and $U_{\lambda_j}$, $j \leq m$.

**Proposition 3.1** — *Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a partition of an integer $n$ not greater than $d$, and let $\{I_1, \ldots, I_m\}$ be the set-partition defined by* (1). *Then*

$$P_\lambda(x_1, \ldots, x_d) = E_{[n+1,d]}\left[\frac{1}{D^{m+1}} Z_\lambda\left(\frac{x_1}{D^2}, \ldots, \frac{x_n}{D^2}\right)\right] \prod_{j=1}^{m} U_{\lambda_j}\left((x_i)_{i \in I_j}\right)$$

*where* $D = 1 - \sum_{i=n+1}^{d} x_i$ *and* $Z_\lambda(x_1, \ldots, x_n)$ *is defined by* (2).

**Remark.** In particular, $P_d(x_1, \ldots, x_d) = Z_d(x_1, \ldots, x_d) U_d(x_1, \ldots, x_d)$. Moreover, if $|\lambda| \le d$ and $\lambda \ne (d)$, then all the parts of $\lambda$ are smaller than $d$. Therefore this proposition, combined with Eqs. (6) and (7), allows us to compute the series $U_d(x_1, \ldots, x_d)$ by induction on $d$ in terms of the series $Z_\lambda$, for $|\lambda| \le d$.

**Proof.** Let $u$ be a rooted polygon of $\mathcal{P}_I$. For $1 \le j \le m$, $u$ has a (unique) loop $\ell_j$ in $I_j$. Let $\phi(u)$ be the word obtained by replacing in $u$ the whole loop $\ell_j$ by a single new letter $L_j$, for all $j \le m$. Let $(a_1, \ldots, a_d) \in \mathbb{N}^d$. We wish to enumerate the polygons $u$ of $\mathcal{P}_I$ such that $|\phi(u)|_i = a_i$ for $1 \le i \le d$. The corresponding words $\phi(u)$ are the words $v$ formed by the shuffle of $(m+1)$ words $u_0, u_1, \ldots, u_m$ of the following type:

- $u_0$ is a rooted unimodal polygon of perimeter $2\sum_{i \in I_0} a_i$ on the alphabet $I_0 \cup \bar{I}_0$, with $I_0 = [n+1, d]$;
- for $j \ge 1$, $u_j = v_j L_j w_j$ where $|v_j| = |w_j|$ and $v_j w_j$ is staircase polygon of perimeter $2\sum_{i \in I_j} a_i$ on the alphabet $I_j \cup \bar{I}_j$.

Then $u$ is obtained from $v$ by replacing the letter $L_j$ by a unimodal rooted loop of dimension $|I_j| = \lambda_j$ on the alphabet $I_j \cup \bar{I}_j$, for $1 \le j \le m$.

Thus the multi-perimeter generating function for polygons $u$ of $\mathcal{P}_I$ such that $|\phi(u)|_i = a_i$ for $1 \le i \le n$ is

$$\binom{m + 2\sum_{i=1}^{d} a_i}{2\sum_{i \in I_0} a_i,\ \left(1 + 2\sum_{i \in I_j} a_i\right)_{1 \le j \le m}} \binom{2\sum_{i \in I_0} a_i}{(2a_i)_{i \in I_0}} \prod_{j=1}^{m} \binom{\sum_{i \in I_j} a_i}{(a_i)_{i \in I_j}}^2 \prod_{j=1}^{m} U_{\lambda_j}\left((x_i)_{i \in I_j}\right) \prod_{i=1}^{d} x_i^{a_i}.$$

In this expression, the first binomial coefficient corresponds to the shuffle of the words $u_0, \ldots, u_m$; the next one corresponds to the construction of the word $u_0$, and the last $m$ correspond to the construction of the words $v_j w_j$, for $1 \le j \le m$. We conclude the proof by summing on $a_{n+1}, \ldots, a_d$, using

$$\sum_{(a_1, \ldots, a_p) \in \mathbb{N}^p} \binom{M + a_1 + \cdots + a_p}{M, a_1, \ldots, a_p} x_1^{a_1} \ldots x_p^{a_p} = \frac{1}{(1 - \sum_{1 \le i \le p} x_i)^{M+1}}.$$

∎

We can proceed a little further by summing the series $(-1)^{|I|} P_I$ corresponding to sets $I$ that only differ by their singletons. This is equivalent to counting polygons having no 1-dimensional loop. Let $\mathcal{S}'$ be the subset of $\mathcal{S}$ formed with the elements $I = \{I_1, \ldots, I_m\}$ such that $|I_j| > 1$ for all $j$. For $I \in \mathcal{S}'$, let $\mathcal{P}'_I$ be the set of unimodal rooted polygons *having no one-dimensional loop*, but having a loop in $I_j$ for $1 \le j \le m$, and let $P'_I$ be their multi-perimeter generating function. The inclusion-exclusion principle gives

$$\sum_{I \in \mathcal{S}'} (-1)^{|I|} P'_I = 1. \qquad (8)$$

If $\lambda$ is a partition of an integer $n$ not greater than $d$, with no part equal to 1, we define $P'_\lambda(x_1,\ldots,x_d)$ to be the series $P'_I$, where $I$ is the set-partition defined by (1). As above, if $J=(J_1,\ldots,J_m)\in\mathcal{S}'$ and $(|J_1|,\ldots,|J_m|)=\lambda$, we have

$$P'_J = P'_\lambda\left((x_i)_{i\in J_1},\ldots,(x_i)_{i\in J_m},(x_i)_{i\in J_0}\right). \tag{9}$$

A mixture of the arguments of Lemma 2.1 and Proposition 3.1 allows us to compute $P'_\lambda$, thus leading to the main proposition of this section.

**Proposition 3.2** — *The multi-perimeter generating function $U_d(x_1,\ldots,x_d)$ for $d$-dimensional oriented unimodal self-avoiding polygons on $\mathbb{Z}^d$ can be computed by induction on $d$ by combining Eqs. (8) and (9) with the following result:*

*Let $\lambda=(\lambda_1,\ldots,\lambda_m)$ be a partition of an integer $n$ not greater than $d$, having no part equal to 1, and let $\{I_1,\ldots,I_m\}$ be defined by (1). Then*

$$P'_\lambda(x_1,\ldots,x_d) = E_{[n+1,d]}\left[\frac{\prod_{i=n+1}^d(1-x_i)}{D^{m+1}} Z_\lambda\left(\frac{x_1}{D^2},\ldots,\frac{x_n}{D^2}\right)\right]\prod_{j=1}^m U_{\lambda_j}\left((x_i)_{i\in I_j}\right)$$

*where $D=1-\sum_{i=n+1}^d x_i$ and $Z_\lambda(x_1,\ldots,x_n)$ is defined by (2).*

**Example 1.** For $d=2$, we simply have

$$\mathcal{S}'=\{\emptyset,\{\{1,2\}\}\},$$

thus we only need to compute $P'_\emptyset(x_1,x_2)$ and $P'_2(x_1,x_2)$. Proposition 3.2 gives

$$P'_\emptyset(x_1,x_2)=E_{[1,2]}\left[\frac{(1-x_1)(1-x_2)}{1-x_1-x_2}\right]=\frac{1-2x_1-2x_2+x_1^2+x_2^2}{1-2x_1-2x_2-2x_1x_2+x_1^2+x_2^2}$$

and

$$P'_2(x_1,x_2)=Z_2(x_1,x_2)U_2(x_1,x_2).$$

Using Eqs. (8) and (3), we obtain the multi-perimeter generating function for oriented unimodal self-avoiding polygons of dimension 2:

$$U_2(x_1,x_2)=2\frac{x_1x_2}{\sqrt{1-2x_1-2x_2-2x_1x_2+x_1^2+x_2^2}}. \tag{10}$$

This result was first obtained by Lin and Chang by a different method [25]. See also [2] for a third proof.

**Example 2.** For $d=3$, we have

$$\mathcal{S}'=\{\emptyset,\{\{1,2\}\},\{\{1,3\}\},\{\{2,3\}\},\{\{1,2,3\}\}\},$$

thus we need to compute $P'_\emptyset(x_1,x_2,x_3)$, $P'_2(x_1,x_2,x_3)$ and $P'_3(x_1,x_2,x_3)$. Proposition 3.2 gives

$$P'_\emptyset(x_1,x_2,x_3)=E_{[1,3]}\left[\frac{(1-x_1)(1-x_2)(1-x_3)}{1-x_1-x_2-x_3}\right],$$

$$P'_2(x_1,x_2,x_3) = E_{\{3\}}\left[\frac{1}{1-x_3} Z_2\left(\frac{x_1}{(1-x_3)^2}, \frac{x_2}{(1-x_3)^2}\right)\right] U_2(x_1,x_2)$$

and

$$P'_3(x_1,x_2,x_3) = Z_3(x_1,x_2,x_3)U_3(x_1,x_2,x_3).$$

Using Eqs. (3), (8) and (10), the series $Z_3U_3$ can be expressed as an *algebraic* series in $x_1$, $x_2$ and $x_3$.

Finally, further reduction is possible by restricting our study to the *perimeter* generating function $\bar{U}_d$ for unimodal loops. For $I \in \mathcal{S}'$, let us denote by $\bar{P}'_I$ the perimeter generating function for polygons lying in $\mathcal{P}'_I$. Now $\bar{P}'_I$ depends only on the partition $\lambda = (|I_1|, \ldots, |I_m|)$ and will be simply denoted $\bar{P}'_\lambda$. Let $\Lambda_d$ be the set of partitions of integers not greater than $d$, having no part equal to 1. For $\lambda = (\lambda_1, \ldots, \lambda_m) = 2^{m_2} \ldots d^{m_d} \in \Lambda_d$, the number of elements $I = \{I_1, \ldots, I_m\} \in \mathcal{S}'$ such that $(|I_1|, \ldots, |I_m|) = \lambda$ is

$$\frac{d!}{(d-|\lambda|)! \prod \lambda_i! \prod m_i!}.$$

Rewriting Eq. (8) and Proposition 3.2 leads to the following result.

**Proposition 3.3** — *The perimeter generating function $\bar{U}_d$ for d-dimensional oriented unimodal self-avoiding polygons on $\mathbb{Z}^d$ can be computed by induction on d from the following identity:*

$$\sum_{\lambda \in \Lambda_d} \left[\frac{(-1)^{\ell(\lambda)} d!}{(d-|\lambda|)! \prod \lambda_i! \prod m_i!} \bar{E}_{[|\lambda|+1,d]}\left[\frac{\prod_{i=|\lambda|+1}^d (1-x_i)}{D_\lambda^{m+1}} \bar{Z}_\lambda\left(\frac{t}{D_\lambda^2}\right)\right] \prod_{j=1}^m \bar{U}_{\lambda_j}\right] = 1,$$

*with $D_\lambda = 1 - \sum_{i=|\lambda|+1}^{d} x_i$. In particular, $\bar{U}_d$ is a quotient of D-finite series.*

**Example 3.** For $d = 3$, we have $\Lambda_3 = \{\emptyset, 2, 3\}$. Our proposition gives

$$\bar{E}_{[1,3]}\left[\frac{(1-x_1)(1-x_2)(1-x_3)}{1-x_1-x_2-x_3}\right] - 3\bar{E}_{\{3\}}\left[\frac{1}{1-x_3}\bar{Z}_2\left(\frac{t}{(1-x_3)^2}\right)\right]\bar{U}_2 - \bar{Z}_3(t)\bar{U}_3 = 1,$$

that is, with $u = \sqrt{t}$,

$$\bar{Z}_3(t)\bar{U}_3 = \frac{6t^2}{1-9t} - \frac{3t^2}{\sqrt{1-4t}}\left[[(1-3u)(1+u)]^{-1/2} + [(1+3u)(1-u)]^{-1/2}\right]. \tag{11}$$

The logarithm in the asymptotic behaviour of $\bar{Z}_3(t)$ (see (4)) shows that $1/\bar{Z}_3(t)$, and hence $\bar{U}_3$, is not D-finite. However, $\bar{Z}_3(t)\bar{U}_3$ is algebraic of degree 8.

**Example 4.** For $d = 4$, we have $\Lambda_4 = \{\emptyset, 2, 3, 2^2, 4\}$. Proposition 3.3 gives

$$\bar{E}_{[1,4]}\left[\frac{(1-x_1)(1-x_2)(1-x_3)(1-x_4)}{1-x_1-x_2-x_3-x_4}\right] - 6\bar{E}_{[3,4]}\left[\frac{(1-x_3)(1-x_4)}{(1-x_3-x_4)^2}\bar{Z}_2\left(\frac{t}{(1-x_3-x_4)^2}\right)\right]\bar{U}_2$$
$$-4\bar{E}_4\left[\frac{1}{1-x_4}\bar{Z}_3\left(\frac{t}{(1-x_4)^2}\right)\right]\bar{U}_3 + 3\bar{Z}_{2^2}(t)\bar{U}_2^2 - \bar{Z}_4(t)\bar{U}_4 = 1,$$

that is,

$$\bar{Z}_4(t)\bar{U}_4 = \frac{6t^2(1+5t-12t^2)}{(1-4t)(1-16t)} - \frac{3t^2}{\sqrt{1-4t}}\left[\frac{(1-u)^2}{(1-2u)\sqrt{1-4u}} + \frac{(1+u)^2}{(1+2u)\sqrt{1+4u}}\right]$$
$$-2\left[\frac{1}{1-u}\bar{Z}_3\left(\frac{t}{(1-u)^2}\right) + \frac{1}{1+u}\bar{Z}_3\left(\frac{t}{(1+u)^2}\right)\right]\bar{U}_3 + \frac{12t^4}{1-4t}\bar{Z}_{2^2}(t).$$

This equation, combined with (11), permits to compute $\bar{U}_4$ and shows that $\bar{Z}_3(t)\bar{Z}_4(t)\bar{U}_4$ is D-finite.

# 4 Convex polygons

In this section we first develop the general theory, in which the generating function for convex SAP can be expressed in terms of simpler generating functions which overcount the number of convex SAP. This overcounting is corrected by use of the inclusion-exclusion principle, whereby further classes of graph need to be enumerated. For two-dimensional convex SAP there is only one class of graph needed for correction, but for three-dimensional convex SAP there are eight classes.

Thus we first obtain the multiperimeter generating function for convex SAP in two dimensions. We also compute the generating function for 2-dimensional rooted convex SAP that are unimodal in precisely one direction. This is required for one of the "correction" classes of 3-dimensional convex polygons.

A similar calculation is then carried out for convex SAP in three dimensions. The generating functions for the eight different classes of "correction" polygons are obtained. Then, these are combined to give the required generating function for three-dimensional oriented convex self-avoiding polygons.

## 4.1 Generalities

Our study of convex polygons is in the same spirit as the preceding study of unimodal polygons. However, the situation is now more complicated, as

- a loop of a convex polygon is convex, but not necessarily unimodal,
- a convex polygon $u\ell_1 v\ell_2 w$ may have a loop $\ell_1$ in $I_1$ and a loop $\ell_2$ in $I_2$ with $I_1 \cap I_2 \neq \emptyset$.

For instance, the polygon of Fig. 5.(c) has two loops in $\{1,2\}$. Thus, whereas the distribution of loops of a unimodal polygon was described by an element of the set $\mathcal{S}$ defined by (5), the distribution of loops of a convex polygon will be described by a slightly more complex object.

Let us fix an integer $d \geq 2$. We first extend the definition of unimodality to "negative" directions: for $k \leq d$, a rooted polygon is said to be unimodal in direction $\bar{k}$ if it can be written $vw$, with $|v|_k = |w|_{\bar{k}} = 0$. If $k \in [1,d]$, we define $\bar{\bar{k}}$ to be $k$, and if $I \subset [1,d] \cup [\bar{1},\bar{d}]$, we denote $\bar{I} = \{\bar{k} : k \in I\}$.

Let $(J,K)$ be an ordered pair of sets such that $J \subset [1,d] \cup [\bar{1},\bar{d}]$, $K \subset [1,d]$ and $J \cap \bar{J} = J \cap K = J \cap \bar{K} = \emptyset$. A rooted convex loop is *of type* $(J,K)$ if it is a loop of dimension $|J \cup K|$ on the alphabet $J \cup \bar{J} \cup K \cup \bar{K}$, unimodal in direction $j$ for all $j \in J$, but neither unimodal in $k$ nor in $\bar{k}$ for $k \in K$. Our method for counting convex loops will be based on the following two remarks:

- a loop of a convex polygon is convex;
- if a convex polygon $u\ell_1 v\ell_2 w$ has a loop $\ell_1$ of type $(J_1,K_1)$ and a loop $\ell_2$ of type $(J_2,K_2)$, then $J_1 \cap J_2 = J_1 \cap \bar{K}_2 = J_1 \cap K_2 = J_2 \cap K_1 = J_2 \cap \bar{K}_1 = K_1 \cap K_2 = \emptyset$.

For instance, the first polygon of Fig. 7 has a loop of type $(\{1,2\},\emptyset)$ and a loop of type $(\{\bar{1}\},\{3\})$. Thus, the distribution of loops of a convex polygon will be described by an element $\{(J_1,K_1),\ldots,(J_m,K_m)\}$ of the following set:

$$\mathcal{T} = \{\{(J_1,K_1),\ldots,(J_m,K_m)\} : m \geq 0, J_i \cup K_i \neq \emptyset,\ J_i \subset [1,d] \cup [\bar{1},\bar{d}]\ ,\ K_i \subset [1,d],$$
$$J_i \cap \bar{J}_i = \emptyset\ \forall i,\ J_i \cap J_j = K_i \cap K_j = \emptyset\ \ \forall i \neq j,\ J_i \cap K_j = J_i \cap \bar{K}_j = \emptyset\ \ \forall (i,j)\}\,.$$

For $I = \{(J_1, K_1), \ldots, (J_m, K_m)\} \in \mathcal{T}$, let $\mathcal{Q}_I$ be the set of $d$-dimensional convex polygons having a loop of type $(J_i, K_i)$ for all $i \le m$, and let $Q_I$ be their multi-perimeter generating function. The inclusion-exclusion principle implies that

$$\sum_{I \in \mathcal{T}} (-1)^{|I|} Q_I = 0. \tag{12}$$

This relation is similar to (6) and could be used to compute the generating function for $d$-dimensional convex loops. It is, however, not very convenient, mainly because none of the $Q_I$ is exactly the generating function for $d$-dimensional convex loops. For that reason, we are instead going to use a variation of (12), in which, as in (8), we also get rid of one-dimensional loops to reduce the number of terms in the summation.

Let $\mathcal{T}'$ be the subset of $\mathcal{T}$ containing the sets $\{(J_1, K_1), \ldots, (J_m, K_m)\}$ such that $1 < |J_i \cup K_i| < d$ for all $i$. For $I \in \mathcal{T}'$, let $\mathcal{Q}'_I$ be the set of $d$-dimensional convex polygons *having no loop of dimension one*, but having a loop of type $(J_i, K_i)$ for all $i \le m$, and let $Q'_I$ be their multi-perimeter generating function. Then the generating function for $d$-dimensional convex polygons having no loop of dimension $< d$ is

$$\sum_{I \in \mathcal{T}'} (-1)^{|I|} Q'_I.$$

In other words, this series is the generating function for convex polygons having only $d$-dimensional loops. It can alternatively be expressed in terms of the series $Z_d$, $U_d$ and $C_d$. First, note that a convex polygon cannot have more than two $d$-dimensional loops. More precisely, a convex polygon $u$ having only loops of dimension $d$ has either only one loop, and hence is a $d$-dimensional convex loop, or has two loops, of type $(J_1, K_1)$ and $(J_2, K_2)$ respectively, with $|J_1 \cup K_1| = |J_2 \cup K_2| = d$. The remarks made above imply that, in this case, we have $K_1 = K_2 = \emptyset$ and $J_2 = \bar{J}_1$ (see Fig. 5.(c) for an example with $J_1 = \{1, 2\}$). There are $2^{d-1}$ possible such pairs $\{J_1, J_2\}$. Moreover, $u$ can be represented by the rooted polygon $v\ell_1 w\ell_2$, where $\ell_i$ is a loop of type $(J_i, \emptyset)$ and $vw$ is basically a staircase polygon on the alphabet $J_1 \cup J_2$: in fact, $|v| = |w|$ and $|v|_{\bar{k}} = |w|_k = 0$ for $k \in J_1$. Since staircase polygons are counted by $Z_d(x_1, \ldots, x_d)$, we have proved that

$$C_d(x_1, \ldots, x_d) + 2^{d-1} Z_d(x_1, \ldots, x_d)(U_d(x_1, \ldots, x_d))^2 = \sum_{I \in \mathcal{T}'} (-1)^{|I|} Q'_I, \tag{13}$$

and this is the formula we will apply to count convex loops on $\mathbb{Z}^2$ and $\mathbb{Z}^3$. Unlike what was done in the previous section, we will not give a closed formula for the series $Q'_I$, valid in any dimension: it will be clear from the computation of $C_3$ that this would require the introduction of series other than the $Z_\lambda$, $U_k$ and $C_k$.

## 4.2 Convex loops of dimension 2

**Proposition 4.1** — *The multi-perimeter generating function for oriented convex self-avoiding polygons of dimension* 2 *is*

$$C_2(x_1, x_2) = E_{[1,2]} \left[ \frac{x_1 x_2 (1 - x_1)^2 (1 - x_2)^2}{(1 - x_1 - x_2)^2} \right] - 2 Z_2(x_1, x_2)(U_2(x_1, x_2))^2,$$

*that is*

$$\frac{2x_1x_2\left[1 - 3x_1 - 3x_2 + 3x_1^2 + 3x_2^2 + 5x_1x_2 - x_1^3 - x_2^3 - x_1^2x_2 - x_1x_2^2 - x_1x_2(x_1 - x_2)^2\right]}{\Delta^2} - \frac{8x_1^2x_2^2}{\Delta^{3/2}}$$

*where* $\Delta = 1 - 2x_1 - 2x_2 - 2x_1x_2 + x_1^2 + x_2^2$.

**Proof.** When $d = 2$, the set $\mathcal{T}'$ is reduced to $\{\emptyset\}$. Hence Eq. (13) reads

$$C_2(x_1, x_2) + 2Z_2(x_1, x_2)(U_2(x_1, x_2))^2 = Q'_\emptyset.$$

The series $Q'_\emptyset$ is the generating function for two-dimensional convex polygons having no one-dimensional loop, which is given by Lemma 2.2.

■

**Remarks**

1. This result was first proved by Lin and Chang [25]. It is a refinement of the perimeter generating function, previously obtained by Delest and Viennot [6]. See also [2] and [24] for alternative proofs of these two results, as well as [17] or [2] for the expansion of the series $C_2(x_1, x_2)$.
2. Our proof provides a nice combinatorial interpretation of each of the two parts of the expression of $C_2(x_1, x_2)$: the rational part counts convex polygons having no one-dimensional loop, and the algebraic part counts those that are not self-avoiding (i.e., have two loops).

We conclude this section with the enumeration of rooted convex loops of type $(\{1\}, \{2\})$. It is fairly clear from (13) that this will be needed for the evaluation of $C_3(x_1, x_2, x_3)$. The proofs of the following proposition and of the results of the next subsection require arguments that are very similar to those of Lemmas 2.1 and 2.2 and will not always be repeated.

**Lemma 4.2** — *The multi-perimeter generating function for 2-dimensional rooted convex loops that are unimodal in direction 1 but not unimodal in 2 or $\bar{2}$ is*

$$C'_2(x_1, x_2) = (1 - x_1) \left( E_{[1,2]} \left[ \frac{x_2(1 - x_1)(1 - x_2)^2}{(1 - x_1 - x_2)^2} \right] - 2E_{[1,2]} \left[ \frac{1 - x_2}{x_1(1 - x_1 - x_2)} \right] U_2(x_1, x_2) \right).$$

**Proof.** It is more natural to compute $C'_2(x_1, x_2)/(1 - x_1)$, which is the generating function for 2-dimensional rooted convex *polygons* that have neither one-dimensional loops nor unimodal two-dimensional loops, but are unimodal in direction 1 (Fig. 6.(a)). The generating function for 2-dimensional rooted convex polygons that have no one-dimensional loop and are unimodal in direction 1 is

$$E_{[1,2]} \left[ \frac{x_2(1 - x_1)(1 - x_2)^2}{(1 - x_1 - x_2)^2} \right].$$

Some of these polygons have a two-dimensional *unimodal* loop, which is either of type $(\{1, 2\}, \emptyset)$ or of type $(\{1, \bar{2}\}, \emptyset)$, the two cases being obviously symmetric. In the first case, the rooted polygon factors as $v\ell w$ where $\ell$ is a special letter representing the loop (Fig. 6.(b)); the word $w\bar{1}1v$ is unimodal in $\bar{1}$ and $\bar{2}$, and has a $\bar{1}1$ loop but no $\bar{2}2$ loop. Thus the generating function for the possible pairs $(v, w)$ is

$$E_{[1,2]} \left[ \frac{1 - x_2}{x_1(1 - x_1 - x_2)} \right],$$

and this concludes the proof.

■

## 4.3 Convex loops of dimension 3

We will apply Eq. (13) with $d = 3$. We first describe the set $\mathcal{T}'$. Let us start with the elements $I = \{(J, K)\}$. They satisfy $|J \cup K| = 2$. We find three types of such elements:

- $J = \{j, k\} \subset \{1, \bar{1}, 2, \bar{2}, 3, \bar{3}\}$ and $K = \emptyset$, with $\bar{k} \neq j \neq k$,

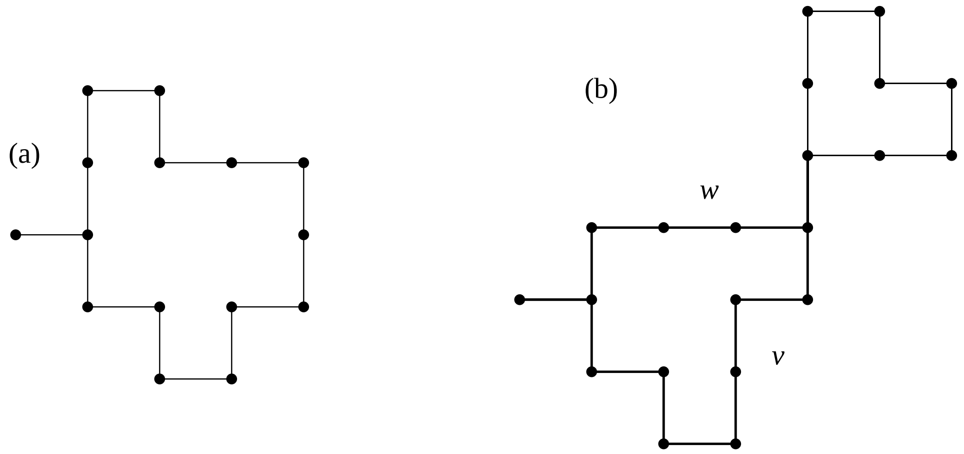


Figure 6: The polygons of Lemma 4.2.

- $J = \{j\} \subset \{1, \bar{1}, 2, \bar{2}, 3, \bar{3}\}$ and $K = \{k\} \subset \{1, 2, 3\}$, with $\bar{k} \neq j \neq k$,
- $J = \emptyset$ and $K = \{j, k\} \subset \{1, 2, 3\}$ with $j \neq k$.

Note that a polygon having a loop of type $(\emptyset, \{j, k\})$ always has a 1-dimensional loop as well. Thus $Q'_I = 0$ in the third case above. More generally, any set $I$ of $\mathcal{T}'$ containing an element $(\emptyset, \{j, k\})$ is such that $Q'_I = 0$. The first two kinds of loop above can be combined to form elements $I$ of $\mathcal{T}'$ in eight different ways, leading to a partition of $\mathcal{T}'$ into eight (relevant) classes. We describe below a representative $I$ of each class. The other elements of the class are obtained via an appropriate permutation of the letters. We also give the cardinality of each class.

- $I = \emptyset$. Cardinality 1.
- $I = \{(\{1, 2\}, \emptyset)\}$. Cardinality 12.
- $I = \{(\{1\}, \{2\})\}$. Cardinality 12.
- $I = \{(\{1, 2\}, \emptyset), (\{\bar{1}, \bar{2}\}, \emptyset)\}$. Cardinality 6.
- $I = \{(\{1, 2\}, \emptyset), (\{\bar{2}, 3\}, \emptyset)\}$. Cardinality 24.
- $I = \{(\{1, 2\}, \emptyset), (\{\bar{1}\}, \{3\})\}$. Cardinality 24.
- $I = \{(\{1\}, \{2\}), (\{\bar{1}\}, \{3\})\}$. Cardinality 6.
- $I = \{(\{1, \bar{2}\}, \emptyset), (\{2, \bar{3}\}, \emptyset), (\{3, \bar{1}\}, \emptyset)\}$. Cardinality 16.

Fig. 7 gives examples of polygons lying in $\mathcal{Q}'_I$, for $I = \{(\{1, 2\}, \emptyset), (\{\bar{1}\}, \{3\})\}$, $I = \{(\{1\}, \{2\}), (\{\bar{1}\}, \{3\})\}$ and $I = \{(\{1, \bar{2}\}, \emptyset), (\{2, \bar{3}\}, \emptyset), (\{3, \bar{1}\}, \emptyset)\}$ respectively.

The following lemmas give the series $Q'_I$ for the eight sets $I$ above. The series $Q'_I$ for all $I \in \mathcal{T}'$ can be derived by a permutation of the variables.

**Lemma 4.3** — *Let $I = \emptyset$. Then*

$$Q'_I = E_{[1,3]} \left[ \frac{2x_1x_2x_3(1-x_1)^2(1-x_2)^2(1-x_3)^2}{(1-x_1-x_2-x_3)^3} \right].$$

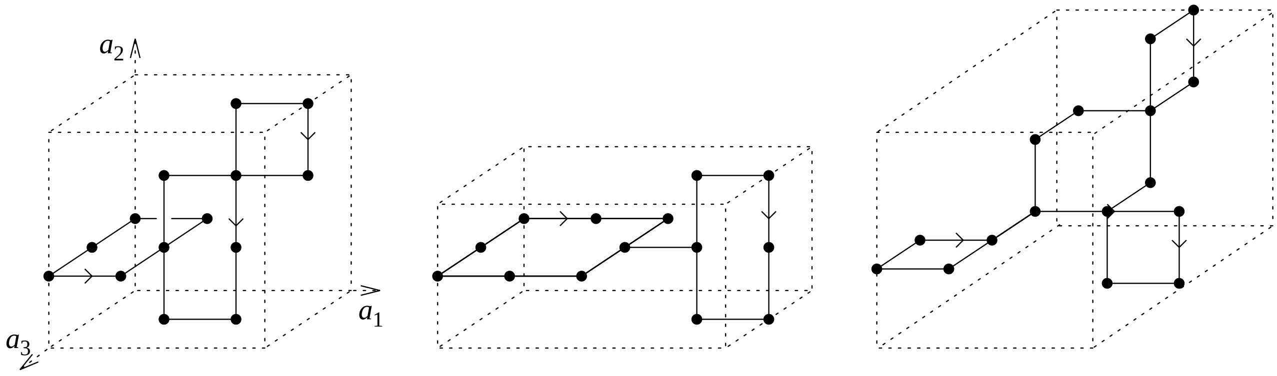


Figure 7: Examples of 3-dimensional convex polygons.

**Proof.** This is Lemma 2.2 in the case $d = 3$.

**Lemma 4.4** — *Let* $I = \{(\{1,2\},\emptyset)\}$. *Then*

$$Q'_I = E_{[1,3]}\left[\frac{x_3(1-x_1)(1-x_2)(1-x_3)^2}{(1-x_1-x_2-x_3)^2}\right] U_2(x_1,x_2).$$

**Proof.** The polygons of $\mathcal{Q}'_I$ can be represented by rooted polygons that factor as $u\ell$ where $\ell$ is a unimodal rooted loop of type $(\{1,2\},\emptyset)$ and $u$ is a convex rooted polygon, unimodal in directions $\bar{1}$ and $\bar{2}$, having no one-dimensional loop, and such that $|u|_3 > 0$.

■

**Lemma 4.5** — *Let* $I = \{(\{1\},\{2\})\}$. *Then*

$$Q'_I = E_{\{1,3\}}\left[\frac{x_3(1-x_1)(1-x_3)^2}{(1-x_1-x_3)^2}\right] C'_2(x_1,x_2)$$

*where* $C'_2$ *is given by Lemma* 4.2.

**Proof.** The polygons of $\mathcal{Q}'_I$ can be represented by rooted polygons that factor as $u\ell$ where $\ell$ is a rooted loop of type $(\{1\},\{2\})$ and $u$ is a convex rooted polygon on the alphabet $\{1,\bar{1},3,\bar{3}\}$, unimodal in direction $\bar{1}$, having no one-dimensional loop, and such that $|u|_3 > 0$.

■

**Lemma 4.6** — *Let* $I = \{(\{1,2\},\emptyset),(\{\bar{1},\bar{2}\},\emptyset)\}$. *Then*

$$Q'_I = E_{\{3\}}\left[x_3(1-x_3)^2\frac{\partial}{\partial x_3}\left[\frac{1}{(1-x_3)^2}Z_2\left(\frac{x_1}{(1-x_3)^2},\frac{x_2}{(1-x_3)^2}\right)\right]\right](U_2(x_1,x_2))^2.$$

**Proof.** This is a nice combination of all the tools previously introduced. The polygons of $\mathcal{Q}'_I$ can be represented by rooted polygons that factor as $u\ell_1 v\ell_2$, where $\ell_1$ (resp. $\ell_2$) is a rooted loop of type $(\{1,2\},\emptyset)$ (resp. $(\{\bar{1},\bar{2}\},\emptyset)$), and $uLv$ is the shuffle of a staircase polygon on $\{1,\bar{1},2,\bar{2}\}$, a special letter $L$ representing the loop $\ell_1$, and a rooted convex polygon on $\{3,\bar{3}\}$, the shuffle having no $3\bar{3}$ or $\bar{3}3$ loops.

■

**Lemma 4.7** — *Let* $I = \{(\{1,2\},\emptyset),(\{\bar{2},3\},\emptyset)\}$. *Then*

$$Q'_I = E_{[1,3]}\left[\frac{(1-x_1)(1-x_3)}{x_2(1-x_1-x_2-x_3)}\right]U_2(x_1,x_2)U_2(x_2,x_3).$$

**Proof.** The polygons of $\mathcal{Q}'_I$ can be represented by rooted polygons that factor as $u\ell_1 v\ell_2$, where $\ell_1$ (resp. $\ell_2$) is a rooted loop of type $(\{1,2\},\emptyset)$ (resp. $(\{\bar{2},3\},\emptyset)$). The rooted polygon $a2\bar{2}v$ is unimodal in 2 and $\bar{3}$, and $vu$ is unimodal in $\bar{1}$. Moreover, $u2\bar{2}v$ contains a $2\bar{2}$ loop, but no $\bar{3}3$ nor $\bar{1}1$ loop.

∎

**Lemma 4.8** — *Let* $I = \{(\{1,2\},\emptyset),(\{\bar{1}\},\{3\})\}$. *Then*

$$Q'_I = E_{[1,2]}\left[\frac{1-x_2}{x_1(1-x_1-x_2)}\right]U_2(x_1,x_2)C'_2(x_1,x_3)$$

*where* $C'_2$ *is given by Lemma* 4.2.

**Proof.** The polygons of $\mathcal{Q}'_I$ can be represented by rooted polygons that factor as $u\ell_1 v\ell_2$, where $\ell_1$ (resp. $\ell_2$) is a rooted loop of type $(\{1,2\},\emptyset)$ (resp. $(\{\bar{1}\},\{3\})$). The rooted polygon $v\bar{1}1u$ lies in $\{1,\bar{1},2,\bar{2}\}$, is unimodal in $\bar{1}$ and $\bar{2}$, has a $\bar{1}1$ loop but no $\bar{2}2$ loop.

∎

**Lemma 4.9** — *Let* $I = \{(\{1\},\{2\}),(\{\bar{1}\},\{3\})\}$. *Then*

$$Q'_I = \frac{1}{1-x_1}C'_2(x_1,x_2)C'_2(x_1,x_3)$$

*where* $C'_2$ *is given by Lemma* 4.2.

**Proof.** See Fig. 7.(b).

∎

**Lemma 4.10** — *Let* $I = \{(\{1,\bar{2}\},\emptyset),(\{2,\bar{3}\},\emptyset),(\{3,\bar{1}\},\emptyset)\}$. *Then*

$$Q'_I = A(x_1,x_2,x_3)(U_2(x_1,x_2,x_3))^3$$

*where*

$$A(x_1,x_2,x_3) = \sum_{(a,b,c)\in\mathbb{N}^3}\left[x_1^a x_2^b x_3^c \sum_{\alpha\le a,\ \beta\le b}\binom{a+b-\beta}{a}\binom{a-\alpha+b}{b}\binom{2a-\alpha+2b-\beta+c+1}{c}\binom{\alpha+\beta+c}{\alpha,\beta,c}\right].$$

**Proof.** This is the most interesting case. It reveals that we need other ingredients than the series $Z_\lambda$ to enumerate convex loops. The polygons of $\mathcal{Q}'_I$ can be represented by rooted polygons that factor as $u\ell_1 v\ell_2 w\ell_3$ where $\ell_1$ (resp. $\ell_2$, $\ell_3$) is a rooted loop of type $(\{1,\bar{2}\},\emptyset)$ (resp. $(\{2,\bar{3}\},\emptyset)$, $(\{3,\bar{1}\},\emptyset)$), $u$ (resp. $v$, $w$) is a word on $\{1,\bar{2},\bar{3}\}$ (resp. $\{\bar{1},2,\bar{3}\}$, $\{\bar{1},\bar{2},3\}$) and

$$|u|_1 = |v|_{\bar{1}} + |w|_{\bar{1}},\ |v|_2 = |w|_{\bar{2}} + |u|_{\bar{2}},\ |w|_3 = |u|_{\bar{3}} + |v|_{\bar{3}}. \tag{14}$$

In the expression of $A(x_1,x_2,x_3)$ given above, $a$ (resp. $b$, $c$) denotes the number of 1 (resp. 2, 3) in $u$ (resp. $v$, $w$), and $\alpha$ (resp. $\beta$) denotes the number of occurrences of $\bar{1}$ (resp. $\bar{2}$) in $w$.

∎

We could use the eight lemmas above to express the series $Q'_I$ for all $I \in \mathcal{T}'$, and then apply Eq. (13) to compute the generating function $C_3(x_1, x_2, x_3)$ for convex loops. For the sake of simplicity, we will only compute explicitely the *perimeter* generating function $\bar{C}_3$. It involves the series $A(t,t,t)$, which turns out to have a simple expression.

**Lemma 4.11** — *Let $\bar{A}(t) = A(t,t,t)$. Then*

$$\bar{A}(t) = \sum_{n\geq 0} t^n \left( \sum_{p=0}^{n} 2^p \binom{n+p}{p} (n+1-p) \right) = \frac{1}{2(1+t)^2} \left[ \frac{5-4t}{\sqrt{1-8t}} - 3 \right].$$

**Proof.** The series $\bar{A}(t)$ counts the triples of words $(u,v,w)$ satisfying the conditions stated in the proof of Lemma 4.10, and in particular Eq. (14). Equivalently, the coefficient of $t^n$ in $A(t)$ is the number of triples of words $(u,v,w)$ on the alphabet $\{x, x_1, x_2\}$ such that

$$|u|_x = |vw|_{x_1}, \quad |w|_x = |uv|_{x_2} \quad \text{and} \quad |uvw|_x = |uvw|_{x_1} + |uvw|_{x_2} = n. \tag{15}$$

Let $(u,v,w)$ be such a triple, and let us write $v = x_2^j \tilde{v} x_1^i$, where $\tilde{v}$ does not start with $x_2$ nor ends with $x_1$. Let $p = n - i - j \geq 0$. We have

$$|u|_x = |\tilde{v}w|_{x_1} + i, \quad |w|_x = |u\tilde{v}|_{x_2} + j, \quad |u\tilde{v}w|_x = n \quad \text{and} \quad |u\tilde{v}w|_{x_1} + |u\tilde{v}w|_{x_2} = p.$$

Conversely, let $z$ be a word on $\{x, x_1, x_2\}$ such that $|z|_x = n$ and $|z|_{x_1} + |z|_{x_2} = p$. Let $u$ be the longest prefix of $z$ such that $z = u\tilde{u}$ with $|u|_x = |\tilde{u}|_{x_1} + i$. Let $w$ be the longest suffix of $z$ such that $z = \tilde{w}w$ with $|w|_x = |\tilde{w}|_{x_2} + j$. We note that $u$ is a prefix of $\tilde{w}$ and we define the word $\tilde{v}$ by $z = u\tilde{v}w$. Then the triple $(u, x_2^j \tilde{v} x_1^i, w)$ satisfies the conditions (15). The decomposition $(u,v,w) \longmapsto (i, j, u\tilde{v}w)$ is one-to-one, and thus

$$\bar{A}(t) = \sum_{n\geq 0} t^n \left( \sum_{p=0}^{n} 2^p \binom{n+p}{p} (n+1-p) \right).$$

In order to prove that this is the expansion of the algebraic series $\tilde{A}(t) = \frac{1}{2(1+t)^2} \left( \frac{5-4t}{\sqrt{1-8t}} - 3 \right)$, we note that

$$\tilde{A}(t) = \frac{S'(t)}{[1 - S(t)]^2} = \frac{d}{dt} \frac{1}{1 - S(t)}$$

where $S(t) = (1 - \sqrt{1-8t})/4$. But the series $S(t)$ satisfies

$$S(t) = \frac{t}{1 - 2S(t)},$$

and the Lagrange inversion formula gives

$$\frac{1}{1 - S(t)} = 1 + \sum_{n\geq 1} \frac{t^n}{n} \left( \sum_{p=0}^{n-1} 2^p \binom{n+p-1}{p} (n-p) \right),$$

which leads to the expected expression of $\tilde{A}(t)$.

■

We can now state our main result.

**Proposition 4.12** — *The perimeter generating function for 3-dimensional oriented convex self-avoiding polygons is, with $u = \sqrt{t}$,*

$$\bar{C}_3 = \frac{8t^3(4-51t+198t^2-135t^3)}{(1-9t)^3} + \frac{96t^4(1-6t)}{(1-9t)(1-4t)} - \frac{24t^4(1-t)(8-60t+145t^2-120t^3+16t^4)}{(1-4t)^4}$$
$$+\frac{96t^4}{\sqrt{1-4t}}\left[\frac{(1-t)(1-2t)(2-7t+4t^2)}{(1-4t)^3} - \frac{2(1-6t)}{(1-9t)^2} + \frac{2t^2}{(1+t)^2(1-4t)}\right]$$
$$+\frac{24ut^4}{1-4t}\left[\frac{1-t-2u}{[(1-3u)(1+u)]^{3/2}} - \frac{1-t+2u}{[(1+3u)(1-u)]^{3/2}}\right] - \frac{64t^6(5-4t)}{(1+t)^2(1-4t)^{3/2}\sqrt{1-8t}}$$
$$-\frac{4}{\bar{Z}_3(t)}\left[\frac{6t^2}{1-9t} - \frac{3t^2}{\sqrt{1-4t}}\left[[(1-3u)(1+u)]^{-1/2} + [(1+3u)(1-u)]^{-1/2}\right]\right]^2.$$

**Remarks**

1. The generating function for oriented convex self-avoiding polygons *lying* in $\mathbb{Z}^3$ (but possibly having a smaller dimension) is naturally $\bar{C}_3 + 3\bar{C}_2 + 3\bar{C}_1$ where $\bar{C}_1 = t$ is the generating function for 1-dimensional convex loops.

2. The series $\bar{C}_3$ is of the same nature as $\bar{Z}_3(t)^{-1}$, and in particular, is not D-finite. However, $\bar{Z}_3(t)\bar{C}_3$ is D-finite (but not algebraic). If we write $\bar{C}_3 = A(t) + B(t)/\bar{Z}_3(t)$, where $A(t)$ and $B(t)$ are algebraic series, then $A(t)$ is algebraic of degree 16 and $B(t)$ is algebraic of degree 8. This indicates that the order of the differential equation satisfied by $\bar{Z}_3(t)\bar{C}_3$ must be large.

3. The singularities of $\bar{C}_3$ are $1/9, 1/8, 1/4, 1$ and $-1$. The singularities $1/8$ and $-1$ are unexpected, for they are not singularities of the generating function $\bar{U}_3$ for three-dimensional unimodal loops. Both singularities come from polygons having three 2-dimensional loops studied in Lemma 4.10.

4. The expression of $\bar{C}_3$ provides the following asymptotic form for the number of three-dimensional oriented self-avoiding convex polygons of perimeter $2n$ (see [15]):

$$2^6 3^{2n-9} n^2 \left\{1 + \frac{c}{n} + \frac{c_1}{n\ln n} + \frac{c_2}{n(\ln n)^2} + \cdots + \frac{c_k}{n(\ln n)^k} + o\left(\frac{1}{n(\ln n)^k}\right)\right\}.$$

5. The generating function for the number of three-dimensional *non-oriented* self-avoiding convex polygons is naturally $\bar{C}_3/2$. It can be expanded in $t$, thus providing Table 1. Fig. 8 shows the 16 non-oriented 3-dimensional convex self-avoiding polygons of perimeter 6.

| Perimeter | Number of convex SAP |
|---|---|
| 2 | 0 |
| 4 | 0 |
| 6 | 16 |
| 8 | 180 |
| 10 | 2076 |
| 12 | 24320 |
| 14 | 283800 |
| 16 | 3271452 |
| 18 | 37122924 |
| 20 | 414389148 |
| 22 | 4553285052 |
| 24 | 49308937324 |
| 26 | 527006325324 |
| 28 | 5566577161596 |
| 30 | 58182007306620 |
| 32 | 602427623684460 |
| 34 | 6185373972824508 |
| 36 | 63029968840513068 |
| 38 | 637933411743484236 |
| 40 | 6417057886206389532 |
| 42 | 64191579720592519596 |
| 44 | 638880848814254960988 |
| 46 | 6329262005238693415452 |
| 48 | 62437666862669512654732 |
| 50 | 613550261967727616850492 |

Table 1: The number of convex self-avoiding polygons of small perimeter.

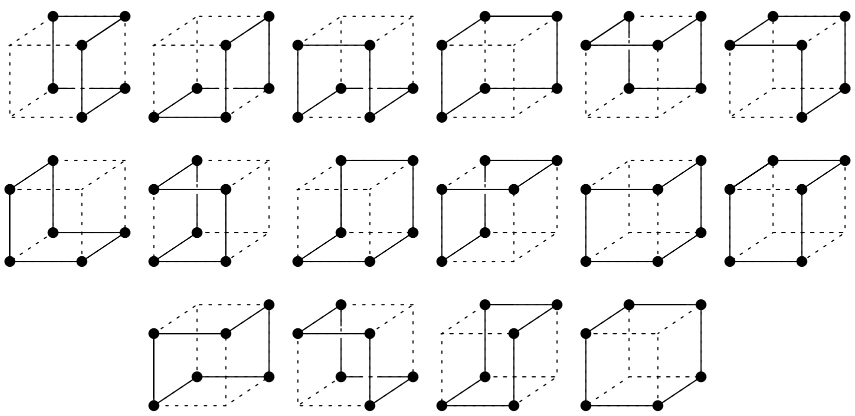

Figure 8: The 3-dimensional convex self-avoiding polygons of perimeter 6.

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## References

[1] R. J. Baxter, Partition function of the three-dimensional Zamolodchikov model, *Phys. Rev. Lett.* **53** (1984) 1795–1798.

[2] M. Bousquet-Mélou, Codage des polyominos convexes et équations pour l'énumération suivant l'aire, *Discrete Appl. Math.* **48** (1994) 21–43.

[3] M. Bousquet-Mélou, A method for the enumeration of various classes of column-convex polygons, *Discrete Math.* **154** (1996) 1–25.

[4] R. Brak, A. J. Guttmann and I. G. Enting, Exact solution of the row-convex polygon perimeter generating function, *J. Phys. A: Math. Gen.* **23** (1990) 2319–2326.

[5] M.-P. Delest, Generating functions for column-convex polyominoes, *J. Combin. Theory Ser. A* **48** (1988) 12–31.

[6] M.-P. Delest and G. Viennot, Algebraic languages and polyominoes enumeration, *Theoret. Comput. Sci.* **34** (1984) 169–206.

[7] J. Denef and L. Lipschitz, Power series solution of algebraic differential equations, *Math. Ann.* **267** (1984) 213–238.

[8] D. Dhar, Exact solution of a directed-site animals-enumeration problem in three dimensions, *Phys. Rev. Lett.* **51** (1983) 853–856.

[9] J. W. Essam, Exact enumeration of parallel walks on directed lattices, *J. Phys. A: Math. Gen.* **26** (1993) L863–L869.

[10] I. G. Enting, A. J. Guttmann, L. B. Richmond and N. C. Wormald, Enumeration of almost-convex polygons on the square lattice, *Random Structures Algorithms* **3** (1992) 445–461.

[11] J.-M. Fédou, Sommes de carrés de multinomiaux, Preprint 1993, `http://www.labri.u-bordeaux.fr/˜fedou`.

[12] J.-M. Fédou and N. Rouillon, Polyominos et $q$-analogues de fonctions de Bessel, une preuve combinatoire, Proceedings of the seventh Conference "Formal power series and algebraic combinatorics", B. Leclerc and J.-Y. Thibon eds., Marne-la-Vallée, France, june 1995.

[13] S. Feretić, A new way of counting the column-convex polyominoes by perimeter, Proceedings of the seventh Conference "Formal power series and algebraic combinatorics", B. Leclerc and J.-Y. Thibon eds., Marne-la-Vallée, France, june 1995.

[14] P. Flajolet, Pólya festoons, Research Report INRIA 1991, Rocquencourt, France.

[15] P. Flajolet and A. Odlyzko, Singularity analysis of generating functions, *SIAM J. Discrete Math.* **3** (1990) 216–240.

[16] I. Gessel, A noncommutative generalization and $q$-analog of the Lagrange inversion formula, *Trans. Amer. Math. Soc.* **257** (1980) 455–481.

[17] I. Gessel, On the number of convex polyominoes, preprint 1990.

[18] A. J. Guttmann and T. Prellberg, Staircase polygons, elliptic integrals, Heun functions and lattice Green functions, *Phys. Rev. E* **47** (1993) R2233–R2236.

[19] J. M. Hammersley, On the rate of convergence to the connective constant of the hypercubical lattice, *Quart. J. Math. Oxford Ser.* (2) **12** (1961) 250–256.

[20] J. M. Hammersley and D. J. A. Welsh, Further results on the rate of convergence to the connective constant of the hypercubic lattice, *Quart. J. Math. Oxford Ser.* (2) **13** (1962) 108–110.

[21] P. Henrici, *Applied and computational complex analysis*, Vol. 2. John Wiley and Sons, New-York, 1977.

[22] B. D. Hughes, *Random walks and random environments. Vol. 1: Random walks*, Clarendon Press, Oxford, 1995.

[23] G. S. Joyce, On the cubic lattice Green functions, *Proc. Roy. Soc. London Ser. A* **445** (1994) 463–477.

[24] D. Kim, The number of convex polyominoes with given perimeter, *Discrete Math.* **70** (1988) 47-51.

[25] K. Y. Lin and S. J. Chang, Rigorous results for the number of convex polygons on the square and honeycomb lattices, *J. Phys. A: Math. Gen.* **21** (1988) 2635–2642.

[26] L. Lipschitz, D-finite power series, *J. Algebra* **122** (1989) 353–373.

[27] P. A. MacMahon, *Combinatory analysis*, 2 vols., Cambridge University Press, Cambridge, 1915–1916 (Reprinted: Chelsea, New-York, 1960).

[28] N. Madras and G. Slade, *The self-avoiding walk*, Probability and its Applications, Birkhauser, Boston, 1993.

[29] G. Pólya, On the number of certain lattice polygons, *J. Combin. Theory* **6** (1969) 102–105.

[30] B. Salvy and P. Zimmermann, GFUN: a Maple package for the manipulation of generating and holonomic functions in one variable, *ACM Trans. Math. Software* **20** (1994) 163–167.

[31] R. P. Stanley, Differentiably finite power series, *Europ. J. Combinatorics* **1** (1980) 175–188.

[32] N. C. Wormald, Private communication.