Percolation models and animals

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Abstract

We first define site- and bond-percolation models on a general graph. We underline the link between the percolation probability and the enumeration of animals. Next, we focus on certain polynomials introduced as approximants to the percolation probability for several specific graphs. We show that such polynomials can be defined for any graph and explain why they often converge in the algebra of formal power series.

The difference between these polynomials – denoted $P_n(q)$ – and their formal limit $P_\infty(q)$ is characterized by a doubly-indexed family of correction terms $d_{n,k}$. These correction terms play an important role, since they permit to extend the expansion of $P_\infty(q)$ when $P_n(q)$ can no longer be computed. For a given k, the $d_{n,k}$'s are related to "compact" animals, i.e. animals with small site-perimeter for their height. These animals are often easier to enumerate than general animals, and this is why some researchers were able to conjecture formulas for $d_{n,k}$, for k small, over several specified graphs. We prove some of these formulas: for directed site- and bond-percolation on the square lattice, and for directed bond-percolation on the honeycomb lattice.

1 Percolation models

Percolation processes are probabilistic models introduced by Broadbent and Hammersley in 1957 (in their bond version). Percolation processes can model various physical phenomena where a fluid spreads randomly through a medium. This is a general framework, and examples include: molecules penetrating a porous solid, disease infecting a community, etc... Percolation models differ from diffusion models, studied earlier, in that the medium, instead of the fluid, bears the random properties.

Since their introduction, these models have been extensively studied by mathematicians and physicists alike. Several books have been written on this topic (for example [11, 17]) and many papers are published every year (see [7, 16] for survey papers). Like many discrete probabilistic models, percolation models can be completely described in combinatorial terms.

1.1 Site-percolation

Let us consider an *infinite countable oriented graph*, that is, a pair (V, f) where V is an infinite countable set whose elements are called *vertices*, and f is a function from $V \times V$ to $\mathbb{N}$. Let v and w be two vertices. An *edge* going from v to w is a triple (v, w, i) where $1 \leq i \leq f(v, w)$. We require the *outdegree* of any vertex to be finite, that is:

$$\forall v \in V, \qquad \sum_{w \in V} f(v, w) < \infty.$$

Finally, the graph is supposed to have a distinguished vertex O called the *origin*. Note that we can mimic non oriented graphs by requiring that $f(v, w) = f(w, v)$ for all $(v, w) \in V^2$. Hence, our general description includes models of non-directed percolation.

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Let $(X_v)_{v\in V\setminus\{O\}}$ be a (countable) family of independent equidistributed random variables:

$$\forall v \in V \setminus \{O\}, \qquad P(X_v = 0) = q \quad \text{and} \quad P(X_v = 1) = 1 - q \ ,$$

with $0 \leq q \leq 1$. The vertex v is said to be *occupied* if $X_v = 1$ and *empty* otherwise. The status of the origin O is a bit different: we adopt the convention of some authors (e.g. [14]) who consider that it is occupied with probability 1.

Definition 1.1 — *A vertex v is* connected *if there exists a sequence of occupied vertices $(v_0, v_1, \dots, v_n)$ such that $v_0 = O$, $v_n = v$ and, for $1 \leq i \leq n$, $f(v_{i-1}, v_i) > 0$. We say that* the system percolates *if infinitely many vertices are connected.*

One of the main questions is to compute the *percolation probability* $P(q)$, that is the probability that the system percolates. This probability is a non-increasing function of q with $P(0) = 1$ and $P(1) = 0$. The *critical probability* q_c is

$$q_c = \inf\{q \in [0,1],\ P(q) = 0\}.$$

The *critical exponent* β describes the behaviour of $P(q)$ near the critical probability: it is expected that $P(q)$ is equivalent, up to a multiplicative constant, to $(q_c-q)^\beta$. Alas, for almost any relevant graph, even a two dimensional grid, the percolation probability is unknown. However, the critical probability is known for some two dimensional *undirected* models (for instance, $q_c = 1/2$ for bond-percolation on the square lattice). But computing the critical probability for the *oriented* square lattice would be probably considered a major advance in this field. Hence, any method providing approximations to $P(q)$ or even for q_c is of interest. Recently, for the oriented square lattice, some researchers have exhibited a family of polynomials $P_n(q)$ that converge pointwise to $P(q)$. Moreover, they have noticed, having obtained experimentally the first P_n's, that these polynomials also have, apparently, a *formal* limit: they converge, as formal power series, to a series $P_\infty(q)$. It is expected — but not proved — that $P_\infty(q)$ is the expansion of $P(q)$ at $q = 0$. We shall explain that the formal convergence of these polynomials is a general phenomenon that might occur on any graph.

Definition 1.2 — *A* (site) animal *A is a finite subset of V having the following two properties:*

(1) *it contains the origin O,*

(2) *for any $v \in A$, there exists a sequence $(v_0, v_1, \dots, v_n)$ of vertices lying in A, such that $v_0 = O$, $v_n = v$ and, for $1 \leq i \leq n$, $f(v_{i-1}, v_i) > 0$.*

Let A be a subset of V, and $w \in V \setminus A$. Then w is a neighbour *of A if $f(v, w) > 0$ for some v in A. The* perimeter $p(A)$ *(often called* site-perimeter*) of A is its number of neighbours. It might not be finite.*

Let us now go back to Definition 1.1. Let A be the *connected cluster*, that is, the set of connected vertices. It satisfies Properties (1) and (2) of Definition 1.2. Moreover, all its neighbours are empty (otherwise they would be connected). The system percolates if and only if A is infinite. Otherwise, A is an animal. Conversely, if there exists an animal A whose vertices are occupied and whose neighbours are empty, the system does not percolate, and A is exactly the set of connected vertices. Thus, the percolation probability is

$$P(q) = 1 - \sum_{A\in\mathcal{A}} q^{p(A)}(1-q)^{|A|-1}, \tag{1}$$

where $\mathcal{A}$ denotes the set of animals on the graph (V, f) (recall that the origin is always occupied). This set is countable, and the above summation of non-negative terms is well-defined. Note that $1 - P(q)$ is a specialization of the perimeter and area generating function of animals (the *area* of an animal being its number of vertices).

Let $(\mathcal{A}_n)_{n\geq 1}$ be an increasing sequence of finite subsets of $\mathcal{A}$, such that $\lim_{n\to\infty} \mathcal{A}_n = \mathcal{A}$. Then, for $0 \leq q \leq 1$

$$P(q) = \lim_{n\to\infty} P_n(q) \tag{2}$$

where

$$P_n(q) = 1 - \sum_{A \in \mathcal{A}_n} q^{p(A)} (1-q)^{|A|-1}. \tag{3}$$

These polynomials will often be simply denoted P_n. Equation (2) means that the sequence $P_n(q)$ converges pointwise to $P(q)$. However, in many cases, the polynomials P_n also turn out to have a *formal* limit, that is, a limit in the algebra $\mathbb{R}[[q]]$ of formal power series in the variable q. This happens in particular in the following case.

Proposition 1.3 — *Let $(\mathcal{A}_n)_{n\geq 1}$ be an increasing sequence of finite subsets of $\mathcal{A}$, such that $\lim_{n\to\infty} \mathcal{A}_n = \mathcal{A}$. Let*

$$p_n = \min\{p(A), A \in \mathcal{A}_{n+1} \setminus \mathcal{A}_n\}.$$

Then the polynomials $P_n(q)$ defined by (3) satisfy

$$P_n(q) - P_{n+1}(q) = O(q^{p_n}).$$

In particular, if

$$\lim_{n\to\infty} p_n = \infty,$$

then these polynomials have a limit $P_\infty(q)$ in $\mathbb{R}[[q]]$.

Proof. We have

$$\begin{aligned} P_n(q) - P_{n+1}(q) &= \sum_{A \in \mathcal{A}_{n+1} \setminus \mathcal{A}_n} q^{p(A)} (1-q)^{|A|-1} \\ &= O(q^{p_n}). \end{aligned} \tag{4}$$

Let $m \geq 0$, and let $N = \max\{n : \ p_n \leq m\}$. For $n > N$, the coefficient of q^m is the same in $P_n(q)$ and $P_{n+1}(q)$, since $p_n > m$. Therefore, the coefficient of q^m is the same in all the $P_n(q)$'s for $n > N$, and these polynomials converge formally.

$\blacksquare$

Remark. In most papers, the polynomials $P_n(q)$ that are used to approximate the percolation probability are *not* defined by a formula similar to (3), but rather as follows. For $v \in V$, let $d(v)$ denote the length of the shortest (oriented) path going from O to v. For $n \geq 0$, let

$$V_n = \{v \in V \ : \ d(v) = n\},$$

and let $(m_n)_{n\geq 1}$ be an increasing sequence of integers. Then the polynomial $P_n(q)$ is defined as the probability that at least one vertex of V_{m_n} is connected. This is actually a special case of our general framework. For if we let

$$\mathcal{A}_n = \{A \in \mathcal{A} \ : \ \forall v \in A \, , d(v) < m_n\}, \quad \text{for} \quad n \geq 1,$$

then the proof of Equation (1) also shows that

$$P_n(q) = 1 - \sum_{A \in \mathcal{A}_n} q^{p(A)} (1-q)^{|A|-1}.$$

Expanding Equation (4) gives a useful expression of $P_n - P_{n+1}$. For $k \geq 0$, let

$$\mathcal{A}_{n,k} = \{A \in \mathcal{A}_{n+1} \setminus \mathcal{A}_n : \ p(A) = p_n + k\}.$$

With this notation,

$$P_n - P_{n+1} = q^{p_n} \sum_{\ell \geq 0} \left(q^\ell \sum_{k=0}^{\ell} \left(\sum_{A \in \mathcal{A}_{n,k}} (-1)^{\ell-k} \binom{|A|-1}{\ell-k} \right) \right). \tag{5}$$

Let us explain how to use such a sequence (P_n). To simplify, suppose that the sequence (p_n) is increasing (as is the case in the standard examples). Then $P_n - P_\infty = O(q^{p_n})$. Usually, the first P_n's can be computed (with a computer) for small values of n. Computing these polynomials up to P_N gives the first p_N terms of the expansion of P_∞. These terms can be used to estimate the critical probability q_c and the critical exponent β ([1, 2, 4, 14]). The greater N is, the more precise the estimation. However, even when P_{N+1} cannot be computed, there is still hope of obtaining extra-terms of $P_\infty(q)$. Let us write

$$P_n - P_{n+1} = q^{p_n} \sum_{\ell \geq 0} q^\ell d_{n,\ell}.$$

The $d_{n,\ell}$'s are called *correction terms*. According to Formula (5), the coefficient $d_{n,\ell}$ only depends on animals whose perimeter is at most $p_n + \ell$. For instance, $d_{n,0}$ is simply the number of animals of $\mathcal{A}_{n+1} \setminus \mathcal{A}_n$ having p_n as perimeter. When ℓ is small, enumerating the animals of $\mathcal{A}_{n,\ell}$ is often much simpler than enumerating general animals, and a closed formula for $d_{n,\ell}$, valid for each n, can often be obtained. Suppose that such a formula is known for $0 \leq \ell \leq i-1$. We have, for $m \geq 0$:

$$\begin{aligned}
[q^m](P_N - P_\infty) &= \sum_{n \geq N} [q^m](P_n - P_{n+1}) \\
&= \sum_{n \geq N:\ p_n \leq m} d_{n,m-p_n}
\end{aligned}$$

and the expansion of P_∞ can be computed up to the coefficient of q^{p_N+i-1}. Hence, formulas for the first i correction terms give i extra coefficients of P_∞. This technique might be extremely efficient. For instance, it has been used by Baxter and Guttmann to extend the expansion of P_∞ from the 29^{th} coefficient to the 41^{th} coefficient (for *bond*-percolation on the oriented square lattice [1]). Actually, they only *conjectured* formulas for the $d_{n,\ell}$'s. We shall prove some of these conjectures in the sequel of the paper together with formulas for correction terms on other lattices.

Finally, note that the percolation probability does not depend on the exact number of edges going from a vertex v to a vertex w, but only on the existence of such edges. Multiple edges may seem useless, but they are important for *bond-percolation.*

1.2 Bond-percolation

Let us consider an infinite countable oriented graph (V, f). Let E denote the set of edges:

$$E = \{(v, w, i) : v \in V,\ w \in V,\ 1 \leq i \leq f(v, w)\}.$$

Let $(X_e)_{e \in E}$ be a (countable) family of independent equidistributed random variables:

$$\forall e \in E, \qquad P(X_e = 0) = q \quad \text{and} \quad P(X_e = 1) = 1 - q\ ,$$

with $0 \leq q \leq 1$. The edge e is said to be *occupied* if $X_e = 1$ and *empty* otherwise.

Definition 1.4 — *Let v be a vertex. Then v is* connected *if there exists a sequence of vertices $(v_0, v_1, \ldots, v_n)$, such that $v_0 = O$, $v_n = v$ and for $1 \leq i \leq n$, at least one of the edges going from v_{i-1} to v_i is occupied.*

Let $e = (v, w, i)$ be an edge of the graph. It is said to be connected if it is occupied and if the vertex v is connected. We say that the system percolates *if infinitely many vertices — or, equivalently, edges — are connected.*

We associate with any set of edges A a set $V(A)$ containing all vertices that are adjacent to at least one edge of A:

$$V(A) = \{O\} \cup \{v \in V :\ \exists\ (v, w, i) \in A\} \cup \{w \in V :\ \exists\ (v, w, i) \in A\}. \tag{6}$$

Definition 1.5 — *A* (bond) animal *A is a finite subset of E such that*

(1) *for any* $v \in V(A)$*, there exists a sequence* $(v_0, v_1, \ldots, v_n)$ *of vertices such that* $v_0 = O$*,* $v_n = v$ *and for* $1 \leq i \leq n$*, at least one edge going from* v_{i-1} *to* v_i *belongs to* A*.*

Let A *be a subset of* E*. The* neighbours *of* A *are the edges* (v, w, i)*, not belonging to* A*, such that* $v \in V(A)$*. The* perimeter $p(A)$ *of* A *is its number of neighbours. It might not be finite.*

Again, we are interested in the percolation probability $P(q)$. Let A be the set of connected edges. Then $V(A)$ is exactly the set of connected vertices. In particular, A satisfies Property (1) of Definition 1.5. Moreover, all its neighbours are empty. The system percolates if and only if A is infinite. Otherwise, A is an animal (and $V(A)$ is a site-animal). Conversely, if there exists an animal A whose edges are occupied and whose neighbours are empty, the system does not percolate. Hence, the percolation probability is

$$P(q) = 1 - \sum_{A \in \mathcal{A}} q^{p(A)} (1-q)^{|A|}$$

where $\mathcal{A}$ denotes the set of (bond) animals on the graph (V, f). This set is countable, and the above summation of non-negative terms is well-defined. Again, $1 - P(q)$ is a specialization of the perimeter and area generating function of animals (the *area* of an animal now being its number of edges).

Let $(\mathcal{A}_n)_{n \geq 1}$ be an increasing sequence of finite subsets of $\mathcal{A}$ such that $\lim_{n \to \infty} \mathcal{A}_n = \mathcal{A}$. Then

$$P(q) = \lim_{n \to \infty} P_n(q)$$

where

$$P_n(q) = 1 - \sum_{A \in \mathcal{A}_n} q^{p(A)} (1-q)^{|A|}. \tag{7}$$

As in the case of site-percolation, the polynomials P_n might have a formal limit.

Proposition 1.6 — *Let* $(\mathcal{A}_n)_{n \geq 1}$ *be an increasing sequence of finite subsets of* $\mathcal{A}$ *such that* $\lim_{n \to \infty} \mathcal{A}_n = \mathcal{A}$*. Let*

$$p_n = \min\{p(A), A \in \mathcal{A}_{n+1} \setminus \mathcal{A}_n\}.$$

Then the polynomials $P_n(q)$ *defined by (7) satisfy*

$$P_n(q) - P_{n+1}(q) = O(q^{p_n}).$$

In particular, if

$$\lim_{n \to \infty} p_n = \infty,$$

then these polynomials have a limit in $\mathbb{R}[[q]]$*.*

The analogue of (5) is now

$$P_n - P_{n+1} = q^{p_n} \sum_{\ell \geq 0} \left(q^\ell \sum_{k=0}^{\ell} \left(\sum_{A \in \mathcal{A}_{n,k}} (-1)^{\ell-k} \binom{|A|}{\ell - k} \right) \right) \tag{8}$$

where for $k \geq 0$,

$$\mathcal{A}_{n,k} = \{A \in \mathcal{A}_{n+1} \setminus \mathcal{A}_n,\ p(A) = p_n + k\}. \tag{9}$$

As in the site case, researchers are interested in obtaining a good approximation to the percolation probability. They do so by first computing as many P_n's as possible and then finding formulas for the correction terms $d_{n,\ell}$ defined by:

$$P_n - P_{n+1} = q^{p_n} \sum_{\ell \geq 0} q^\ell d_{n,\ell}.$$

1.3 Example: bond-percolation on a tree

We study here bond-percolation on the infinite complete binary tree. The edges of this tree are oriented downwards as shown in Figure 1. Using some polynomials P_n that converge to the percolation probability $P(q)$, we compute both $P(q)$ and the formal limit P_∞ of the P_n's. This series coincides with $P(q)$ for $0 \leq q < q_c$. The computations are very easy in this case because the graph has no loops.

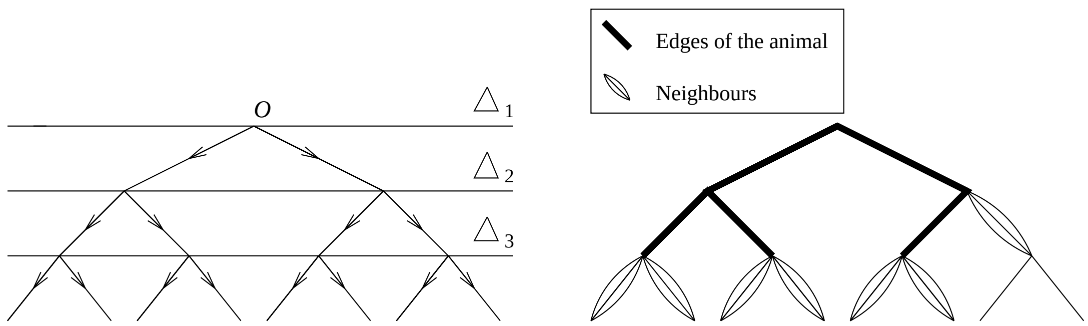

Figure 1: The infinite complete binary tree and a bond-animal on this graph.

A (bond) animal is, in this case, a binary tree. To define the *height* of an animal, it is convenient to introduce horizontal lines, denoted Δ_1, Δ_2, etc... as in Figure 1. We associate with any animal A a site-animal $V(A)$ defined by (6). The *height* of A is

$$\max\{i \geq 1 : \ V(A) \cap \Delta_i \neq \emptyset\}.$$

Hence, the empty animal has height 1. A bond-animal of height 3 is drawn in Figure 1. Note that the perimeter of an animal having p edges is $p+2$.

For $n \geq 1$, let $\mathcal{A}_n$ be the set of animals of height less than n. For instance, there is a single animal (the empty one) in $\mathcal{A}_2$. Let P_n be the probability that at least one vertex lying on Δ_n is connected:

$$P_n = 1 - \sum_{A \in \mathcal{A}_n} q^{p(A)}(1-q)^{|A|}.$$

Let $Q_n = 1 - P_n$. Then $Q_1 = 0$ and, for $n \geq 2$:

$$Q_n = (q + (1-q)Q_{n-1})^2.$$

This identity is illustrated by Figure 2, which represents the four types of animals of height less than n. Equivalently:

$$P_n = (1-q)P_{n-1}(2-(1-q)P_{n-1}). \tag{10}$$

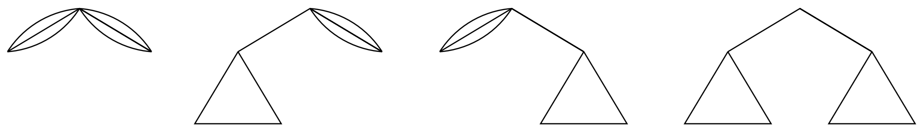

Figure 2: Binary trees.

From (10), we derive both the analytic limit $P(q)$ and the formal limit $P_\infty(q)$ of the P_n's. Each of them satisfies the following algebraic equation (in the variable x)

$$x = (1-q)x(2-(1-q)x).$$

The analytic limit is

$$P(q) = \begin{cases} \frac{1-2q}{(1-q)^2} & \text{if } \; 0 \leq q \leq \frac{1}{2} \\ 0 & \text{if } \; \frac{1}{2} \leq q \leq 1 \end{cases} .$$

Hence, the critical probability is $q_c = 1/2$ and the critical exponent is $\beta = 1$. The formal limit of the P_n's is:

$$P_\infty = \sum_{n \geq 0} (1-n)\; q^n.$$

It is the expansion at $q = 0$ of the function $\frac{1-2q}{(1-q)^2}$. Therefore, the formal limit coincides with the percolation probability in the *high-density phase*, that is when $q < q_c$.

Computing correction terms is not useful here, since we already know the exact percolation probability. Nevertheless, we illustrate the general description above by computing the first correction term. Let A be an animal of $\mathcal{A}_{n+1} \setminus \mathcal{A}_n$. Its height is n, hence, it has at least $n-1$ edges, and its perimeter is at least $n+1$. Using the notations of Proposition 1.6, this means that $p_n = n+1$, and $P_n - P_{n+1} = O(q^{n+1})$. According to Expression (8), the first correction term $d_{n,0}$ is the number of animals of $\mathcal{A}_{n+1} \setminus \mathcal{A}_n$ having $n+1$ as perimeter. This is the number of *linear trees* of height n, that is 2^{n-1}. Here are the first P_n's:

$$\begin{aligned}
&P_1 = 1, \\
&P_2 = 1 - q^2, \\
&P_3 = 1 - q^2 - 2q^3 + q^4 + 2q^5 - q^6, \\
&P_4 = 1 - q^2 - 2q^3 - 3q^4 + 4q^5 + 7q^6 + O(q^7), \\
&P_5 = 1 - q^2 - 2q^3 - 3q^4 - 4q^5 + 11q^6 + O(q^7).
\end{aligned}$$

2 Directed site-percolation on the square lattice

In this section, we study site-percolation on a square lattice whose edges are oriented downwards. Site-animals on oriented lattices are often called *directed animals*. We show such an animal in Figure 3. Its vertices are represented by black circles and its neighbours by white circles. It has 13 vertices and perimeter 11. To define its *height*, it is convenient to introduce the lines $\Delta_1, \Delta_2, \Delta_3$, etc... drawn in Figure 3. The *height* of an animal A is:

$$\max\{i \geq 1, A \cap \Delta_i \neq \emptyset\}.$$

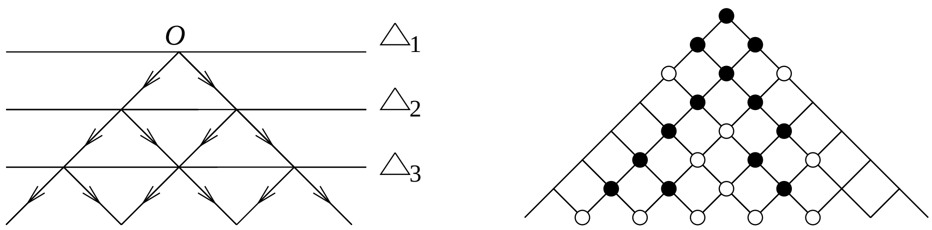

Figure 3: The oriented square lattice and a directed site-animal of height 7.

Let $\mathcal{A}_n$ be the set of animals of height less than n. Let P_n be the probability that at least one vertex of Δ_n is connected:

$$P_n = 1 - \sum_{A \in \mathcal{A}_n} q^{p(A)} (1-q)^{|A|-1}.$$

The P_n's converge pointwise to the percolation probability $P(q)$. We shall see that these polynomials have also a formal limit $P_\infty(q)$, and compute the first two correction terms. We have

$$\begin{aligned}
P_1 &= 1,\\
P_2 &= 1 - q^2,\\
P_3 &= 1 - q^2 - 3q^3 + 4q^4 - q^5,\\
P_4 &= 1 - q^2 - 3q^3 - 8q^4 + 34q^5 + O(q^6),\\
P_5 &= 1 - q^2 - 3q^3 - 8q^4 - 21q^5 + O(q^6).
\end{aligned}$$

Theorem 2.1 — *For $n \geq 1$,*

$$P_n - P_{n+1} = d_{n,0}\, q^{n+1} + d_{n,1}\, q^{n+2} + O(q^{n+3}),$$

with

$$d_{n,0} = \frac{(3n)!}{n!(2n+1)!}$$

and

$$d_{n,1} = (9n-6)\, d_{n-1,0} + \frac{139n+26}{3}\, d_{n,0} - \left(12 + \frac{22}{3}n\right) d_{n+1,0}$$

with $d_{0,0} = 1$. Equivalently, $d_{1,1} = 0$, and for $n \geq 2$,

$$d_{n,1} = -2(33n^3 + 50n^2 - 37n + 30)\frac{(3n-2)!}{(n-2)!(2n+3)!}.$$

Remark. This result has been partially conjectured by Onody and Neves [14]. More precisely, they noticed that the polynomials P_n and P_{n+1} coincide up to the coefficient of q^n, and that the first correction term is apparently $d_{n,0} = \frac{(3n)!}{n!(2n+1)!}$. They give the expansion of the (formal) percolation probability $P_\infty(q)$ up to the coefficient of q^{15}, thus extending the ten-term series of De'Bell and Essam [4]. Guttmann and Jensen have also conjectured the expression of the second correction term, as well as a very complicated recurrence for the third correction term [10].

According to Formula (5), the first correction term is the number of animals of $\mathcal{A}_{n+1} \setminus \mathcal{A}_n$ of minimal perimeter. It turns out that these animals are the so-called *diagonally convex directed animals* enumerated by several authors [5, 8, 15].

2.1 Diagonally convex directed animals on the square lattice

Let A be a directed animal on the square lattice. It is said to be *diagonally convex* if, for $i \geq 1$, the set $A \cap \Delta_i$ is either empty or formed with consecutive vertices (Figure 4).

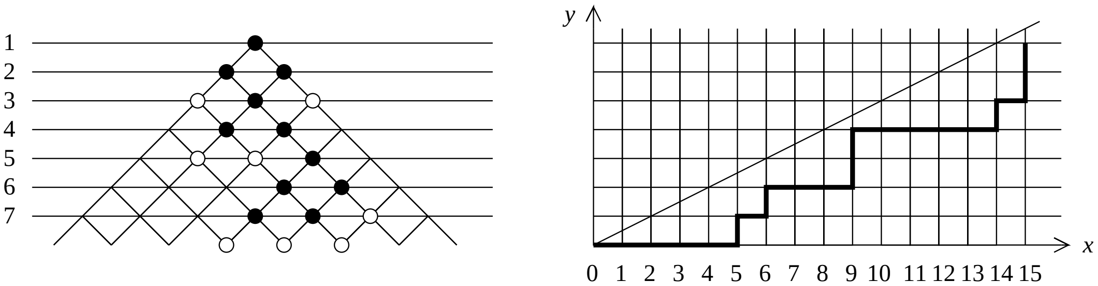

Figure 4: A diagonally convex directed animal and its associated path.

Lemma 2.2 — *Let A be a diagonally convex animal of height n. Its site-perimeter is $n+1$.*

Proof. We prove this lemma by induction on n. The result is obvious when $n = 1$ and the reader will convince himself that adding some vertices of Δ_{n+1} to a diagonally convex animal of height n increases the perimeter by 1 (Figure 5).

■

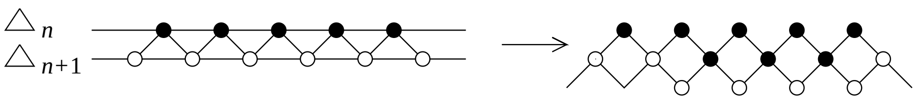

Figure 5: The perimeter of diagonally convex directed animals.

We now recall a bijection, due to Feretić and Svrtan, between diagonally convex animals and a family of paths on the lattice $\mathbb{Z} \times \mathbb{Z}$ [8]. Consider paths starting from the origin $(0,0)$ having only North and East unit steps. There is an obvious bijection Φ between these paths and words on the 2-letter alphabet $\{N, E\}$. This bijection consists of following the path from its origin to its end, replacing each North (resp. East) step by an occurrence of the letter N (resp. E). The length $|\omega|$ of a path ω is its number of steps. The length $|u|$ of a word u is its number of letters.

Let $\mathcal{P}$ be the set of paths ω such that:

- ω starts from the origin $(0,0)$ and ends at $(2n+1, n)$ for some $n \geq 0$,
- any vertex of ω, except the origin, lies strictly below the line $\{x = 2y\}$.

Figure 4 shows such a path.

Diagonally convex directed animals of height n are in one-to-one correspondence with paths of $\mathcal{P}$ of length $3n+1$. Let A be a diagonally convex animal of height n. Let $p_0 = 0$, and, for $1 \leq i \leq n$, let p_{2i-1} (resp. p_{2i}) be the number of neighbours of A lying on Δ_i, on the left-hand side (resp. right-hand side) of the animal. For instance, the sequence $(p_1, p_2, \ldots, p_{14})$ associated with the animal of Figure 4 is $(0,0,0,0,1,1,0,0,2,0,0,0,0,1)$. The site-perimeter of A is

$$n + 1 = \sum_{0 \leq i \leq 2n} p_i + |A \cap \Delta_n| + 1.$$

Therefore,

$$0 < |A \cap \Delta_n| = n - \sum_{0 \leq i \leq 2n} p_i. \tag{11}$$

For $1 \leq k < n$, the same remark can be applied to the sub-animal of A formed with the vertices of $A \cap \Delta_i$, for $i \leq k$, thus proving that:

$$0 < |A \cap \Delta_k| = k - \sum_{0 \leq i \leq 2k} p_i. \tag{12}$$

Svrtan and Feretić's bijection associates with A the path starting at $(0,0)$, ending at $(2n+1, n)$, and having p_i North steps at abscissa i, for $0 \leq i \leq 2n$. According to (11) and (12), this path belongs to $\mathcal{P}$, and the number of North steps at abscissa $2n+1$ is the number of vertices of A lying on Δ_n. Figure 4 shows an example. By convention, the path reduced to a single East step is associated with a theoretical "empty" animal of height 0.

This correspondence is one-to-one, and we are led to enumerate the paths of $\mathcal{P}$ according to their length, or, equivalently, to enumerate the words of $\Phi(\mathcal{P})$. This is very easy since $\Phi(\mathcal{P})$ is an *algebraic language* that can be generated by a *non-ambiguous grammar*. This method — encoding combinatorial objects by words of an algebraic language in order to count then — is known as Schützenberger's methodology (see [6] for an extended exposition).

Proposition 2.3 — *Let a_n be the number of diagonally convex directed animals of height n, and $f = \sum_{n \geq 0} a_n t^n$ be the corresponding generating function. Then*

$$f(t) = 1 + t f(t)^3. \tag{13}$$

Equivalently,

$$a_n = \frac{(3n)!}{n!(2n+1)!}.$$

Proof. Let $\omega \in \mathcal{P}$ be a path of length $3n+1$. If $n=0$, then ω is reduced to a single East step. Otherwise, let $\Phi(\omega)$ be the word on $\{E, N\}$ associated with ω. Then $\Phi(\omega)$ can be factorized into three words of $\Phi(\mathcal{P})$, as schematized in Figure 6. Let

$$i = \max\{k, 0 \leq k \leq n-1, (2k+1, k) \in \omega\},$$

and

$$j = \max\{k, 1 \leq k \leq n, (2k, k-1) \in \omega\}.$$

Then $0 \leq i < j \leq n$. Let us write $\Phi(\omega) = x_1 x_2 \dots x_{3n+1}$ where $x_k \in \{E, N\}$ for each k. Let $u_1 = x_1 \dots x_{3i+1}$, $u_2 = x_{3i+2} \dots x_{3j-1}$ and $u_3 = x_{3j} \dots x_{3n}$. Then $u_i \in \Phi(\mathcal{P})$ for $1 \leq i \leq 3$ and $\Phi(\omega) = u_1 u_2 u_3 N$. The sequence (u_1, u_2, u_3) is called the *canonical factorization* of $\Phi(\omega)$. Conversely, let $(u_1, u_2, u_3) \in \Phi(\mathcal{P})^3$. Let $\omega = u_1 u_2 u_3 N$. Then $\omega \in \Phi(\mathcal{P})$ and its canonical factorization is (u_1, u_2, u_3).

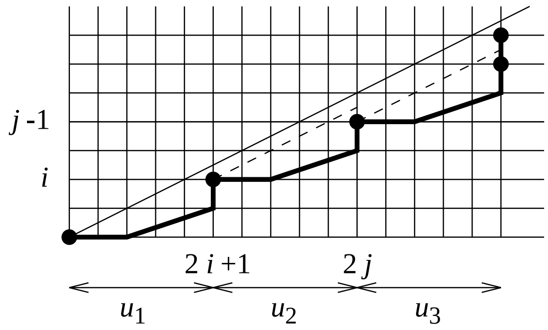

Figure 6: Factorization of a path of $\mathcal{P}$.

In other words, if G denotes the non-commutative generating function of $\Phi(\mathcal{P})$ defined by

$$G = \sum_{u \in \Phi(\mathcal{P})} u,$$

then we have the following non-commutative algebraic equation

$$G = E + G^3 N. \tag{14}$$

Equation (14) implies that the commutative generating function of $\Phi(\mathcal{P})$ defined by

$$g(t) = \sum_{u \in \Phi(\mathcal{P})} t^{|u|} = \sum_{\omega \in \mathcal{P}} t^{|\omega|} \tag{15}$$

satisfies

$$g(t) = t + t g(t)^3. \tag{16}$$

Recall that the number of diagonally convex directed animals of height n is the number of words of $\Phi(\mathcal{P})$ of length $3n+1$. Hence, $g(t) = t f(t^3)$ which gives Equation (13). The expression of a_n is then obtained from the Lagrange inversion formula.

∎

In order to compute the second correction term, we need to solve two more enumerative problems. The first is a refinement of the previous result. Let A be a diagonally convex directed animal and $v \in A \cap \Delta_k$. We say that v is a *cut-vertex* of A if $A \cap \Delta_k = \{v\}$.

Proposition 2.4 — *Let $a_{n,m}$ be the number of diagonally convex directed animals of height n having m cut-vertices and $f_0(t,x) = \sum_{n,m} a_{n,m}\, t^n x^m$ be the corresponding generating function. Then*

$$f_0(t,x) = \frac{1 - t x f(t)^2}{1 - t x f(t)\,[1 + f(t)]}$$

where $f(t)$ is the height generating function of diagonally convex animals defined by (13).

Proof. Again, use the encoding of diagonally convex animals by paths of $\mathcal{P}$ described above. According to Formulas (11) and (12), the animal A has a cut-vertex on Δ_k if and only if the vertex $(2k, k-1)$ belongs to the path associated with A. Let ω be a path of $\mathcal{P}$ of length $3n+1$. Define two parameters $c_0(\omega)$ and $c_1(\omega)$ as follows:

$$c_0(\omega) = |\{1 \leq k \leq n,\ (2k, k-1) \in \omega\}|,$$

and

$$c_1(\omega) = |\{0 \leq k \leq n,\ (2k+1, k) \in \omega\}|.$$

Let

$$g_0(t,x) = \sum_{\omega \in \mathcal{P}} t^{|\omega|} x^{c_0(\omega)},$$

and

$$g_1(t,x) = \sum_{\omega \in \mathcal{P}} t^{|\omega|} x^{c_1(\omega)}.$$

Then $g_0(t,1) = g_1(t,1) = g(t)$ where $g(t)$ is defined by (15), and $g_0(t,x) = tf_0(t^3,x)$. Using the decomposition of paths of $\mathcal{P}$ described in the proof of Proposition 2.3, we find:

$$g_0(t,x) = t + tg_0(t,x)g_1(t,x)g(t)$$

$$g_1(t,x) = tx + txg_1(t,x)g(t)^2.$$

This gives:

$$g_0(t,x) = t\frac{1 - txg(t)^2}{1 - txg(t)\,[t + g(t)]},$$

from which we derive the expression of $f_0(t,x)$.

■

We now come to the second enumerative problem we need to solve in order to compute the second correction term. Let A be a diagonally convex directed animal, and let v be a vertex of $A \cap \Delta_k$. Then v is said to be a *border vertex* of A if it is either the rightmost or the leftmost vertex of $A \cap \Delta_k$. It is said to be *useless* if $A \setminus \{v\}$ is still an animal. Let v and w be two distinct useless vertices of $A \cap \Delta_k$ such that v lies to the left of w. Then (A, v, w) is said to be a *doubly-marked diagonally convex animal* if

- neither v nor w is a border vertex,
- any vertex of $A \cap \Delta_k$ lying between v and w is also useless.

This means that, if $A \cap \Delta_{k+1}$ is not empty, then v and w are either both on the right-hand side of $A \cap \Delta_{k+1}$, or both on the left-hand side of $A \cap \Delta_{k+1}$ (as in Figure 7).

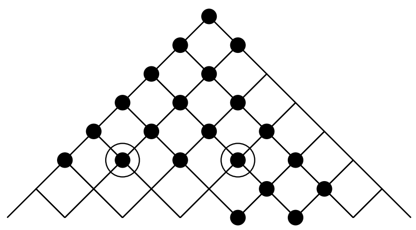

Figure 7: A doubly-marked diagonally convex animal.

Proposition 2.5 — *Let b_n be the number of doubly-marked diagonally convex animals of height n, and $f_1(t) = \sum_{n\geq 1} b_n t^n$ be the corresponding generating function. Then*

$$f_1(t) = \frac{1}{3t}\left[13t + 19 + (9t^2 + 32t - 19)f(t) + t(27t^2 + 172t - 26)f'(t)\right] \tag{17}$$

where $f(t)$ is the height generating function of diagonally convex directed animals defined by (13), *and $f'(t)$ is its derivative.*

Proof. We use the notations introduced at the beginning of this section to describe Svrtan and Feretić's bijection between diagonally convex directed animals and paths of $\mathcal{P}$. Let A be a diagonally convex directed animal of height n. Any vertex of $A \cap \Delta_n$ is useless. Moreover, if $1 \leq k < n$, then

- the number of useless vertices of $A \cap \Delta_k$ lying on the left-hand side of $A \cap \Delta_{k+1}$ is $p_{2k+1} - 1$ if both $p_{2k+2} = 0$ and $|A \cap \Delta_{k+1}| = 1$, and it is p_{2k+1} otherwise;
- the number of useless vertices of $A \cap \Delta_k$ lying on the right-hand side of $A \cap \Delta_{k+1}$ is $p_{2k+2} - 1$ if both $p_{2k+1} = 0$ and $|A \cap \Delta_{k+1}| = 1$, and it is p_{2k+2} otherwise.

The bijection between diagonally convex directed animals and paths of $\mathcal{P}$ induces another bijection between doubly-marked diagonally convex animals and some coloured paths. Let $\mathcal{P}_0$ be the set of paths of $\mathcal{P}$ where two North steps, lying at the same abscissa, have been coloured. Let $\mathcal{P}_1$ be the subset of $\mathcal{P}_0$ formed with the paths ω such that, if $|\omega| = 3n + 1$, the following properties are verified for $1 \leq i \leq 2n + 1$:

(C_0) the highest North step at abscissa i is not coloured

(C_1) if $i = 2j + 1$ and $(2j + 1, j) \in \omega$, then the lowest North step at abscissa i is not coloured

(C_2) if $i = 2j$, $p_{2j-1} = 0$ and $(2j, j - 1) \in \omega$, then the lowest North step at abscissa i is not coloured.

Then the number of doubly-marked diagonally convex animals of height n is the number of paths of $\mathcal{P}_1$ of length $3n + 1$. Hence, we must enumerate these paths according to their length. The canonical factorization defined for paths of $\mathcal{P}$ can naturally be extended to coloured paths. We use seven different families of coloured paths to factorize the paths of $\mathcal{P}_1$.

First, let g_1 be the (length) generating function of the paths of $\mathcal{P}_1$. Then $g_1(t) = tf_1(t^3)$, where $f_1(t)$ is the (height) generating function of doubly-marked diagonally convex animals. For $2 \leq i \leq 7$, let g_i be the generating function of $\mathcal{P}_i$, where the $\mathcal{P}_i$'s are defined as follows:

- $\mathcal{P}_2$ is the set of paths of $\mathcal{P}_0$ satisfying C_0 for $1 \leq i \leq 2n + 1$,
- $\mathcal{P}_3$ is the set of paths of $\mathcal{P}_2$ satisfying: for $1 \leq i \leq 2n + 1$,

(C_3) if $i = 2j + 1$, $(2j + 1, j) \in \omega$ and $p_{2j} = 0$, the lowest North step at abscissa i is not coloured,

- $\mathcal{P}_4$ is the set of paths of $\mathcal{P}_0$ satisfying C_0 for $1 \leq i \leq 2n$,
- $\mathcal{P}_5$ is the set of paths of $\mathcal{P}_4$ satisfying C_3 for $i = 2n + 1$,
- $\mathcal{P}_6$ is the set of paths of $\mathcal{P}$ where *one* North step lying at abscissa $2n + 1$ has been coloured,
- $\mathcal{P}_7$ is the set of paths of $\mathcal{P}_6$ satisfying C_3 for $i = 2n + 1$.

Note that in all these definitions, the paths are always supposed to have length $3n+1$. By factorizing all these families of paths, we obtain

$$\begin{aligned}
g_1 &= tg^2(g_1+g_2+g_3),\\
g_2 &= tg^2(2g_2+g_4),\\
g_3 &= tg^2(g_2+g_3+g_5),\\
g_4 &= g_2+tg^2g_6,\\
g_5 &= tg^2(2g_2+g_5+g_7),\\
g_6 &= tg^2(g+g_6),\\
g_7 &= tg^2(g+g_7)-t^3g,
\end{aligned}$$

where g is the generating function of (uncoloured) paths of $\mathcal{P}$. Using the equation satisfied by g (see (16)), we have

$$g_1 = -t^3\frac{4t+3t^4-g(4+10t^3)+11t^2g^2}{1-5t^3+t^2g(3t^3-1)+tg^2(4t^3-1)}.$$

But $g_1(t)=tf_1(t^3)$, thus

$$f_1 = -t\frac{4+3t-f(4+10t)+11tf^2}{1-5t+tf(3t-1)+tf^2(4t-1)}.$$

Using $f=1+tf^3$, it is easy to check that this is equivalent to Equation (17).

■

2.2 The first correction term

This subsection and the following are devoted to the proof of Theorem 2.1. According to Proposition 1.3, we first need to know the minimal perimeter of an animal of height n.

Definition 2.6 — *Let A be an animal. A* diagonal hole *of A is a non-empty maximal finite sequence of consecutive vertices lying on the same diagonal and not belonging to A.*

Proposition 2.7 — *Let A be a (directed) animal of height n, having h diagonal holes. Then its (site) perimeter is $n+h+1$.*

We can check that the animal drawn in Figure 3 satisfies this proposition: its height is 7, it has 3 diagonal holes, and its perimeter is 7+3+1=11.

Proof. We prove this result by induction on h. It could also be proved by induction on n.

If $h=0$, then A has no hole. In other words, it is a diagonally convex directed animal, and according to Lemma 2.2, its perimeter is $n+1$.

Now, suppose $h>0$. Let Δ_i be the first (highest) diagonal containing a hole $H=(v_1,\dots,v_k)$. Let $A'=A\cup\{v_1,\dots,v_k\}$. The vertices $v_1,\dots,v_k$ are neighbours of A, and hence, A' is still an animal; its site-perimeter is

$$p(A')=p(A)-k+(k-1).$$

More precisely, k is the number of neighbours of A that are not neighbours of A' (these are the vertices $v_1,\dots,v_k$), and $k-1$ is the number of neighbours of A' that are not neighbours of A (see Figure 8). Hence, $p(A)=p(A')+1$. But A' has $h-1$ holes, and its height is n. By induction on h, its perimeter is $n+h$, and finally $p(A)=n+h+1$.

■

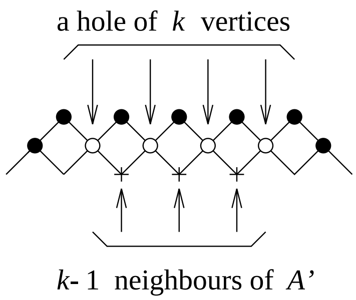

Figure 8: The perimeter of a directed animal.

We now apply Proposition 1.3. According to Proposition 2.7, the minimal perimeter of animals of height n is $p_n = n+1$. Hence, the polynomials P_n defined at the beginning of this section satisfy

$$P_n - P_{n+1} = O(q^{n+1}).$$

The first correction term $d_{n,0}$ is the number of animals of height n having $n+1$ as perimeter. These are precisely the diagonally convex directed animals of height n. By Proposition 2.3, we have thus obtained

$$P_n - P_{n+1} = d_{n,0}\, q^{n+1} + O(q^{n+2}),$$

where

$$d_{n,0} = \frac{(3n)!}{n!(2n+1)!}.$$

2.3 The second correction term

We now turn to the computation of the second correction term. According to Equation (5):

$$d_{n,1} = |\mathcal{A}_{n,1}| - \sum_{A \in \mathcal{A}_{n,0}} (|A|-1).$$

According to Proposition 2.7, $|\mathcal{A}_{n,1}|$ is the number of animals of height n having exactly one diagonal hole. Moreover, $\sum_{A \in \mathcal{A}_{n,0}} (|A|-1)$ is the number of *marked diagonally convex animals*, that is the number of pairs (A, v), where $A \in \mathcal{A}_{n,0}$ and $v \in A \setminus \{O\}$. We compute $d_{n,1}$ in three steps.

Step 1: cancellation of terms. Let A be an animal of $\mathcal{A}_{n,1}$. Two cases occur:
a) either the diagonal hole of A consists of a single vertex, or
b) the diagonal hole of A has at least two vertices.
Now, let (A, v) be a marked diagonally convex animal. Suppose that the distinguished vertex v lies on Δ_i. Again, two cases occur:
α) either v is not a border-vertex of A, or
β) v is a border-vertex of A.
The reader will easily convince himself that animals in case a are in one-to-one correspondence with marked animals in case α. Hence, we only have to enumerate animals in cases b and β, but that has essentially been done in Subsection 2.1.

Step 2: Case b. Let A be an animal in this class. Let $v_1, \ldots, v_k$ be, from left to right, the vertices of the hole of A, and $A' = A \cup \{v_1, \ldots, v_k\}$. Then (A', v_1, v_k) is a doubly-marked diagonally convex animal, as defined in Subsection 2.1. Actually, animals in this class are in one-to-one correspondence with doubly-marked diagonally convex animals. Hence, the series $f_1(t)$ given in Proposition 2.5 is also the height generating function of animals having a single hole containing several vertices.

Step 3: Case β. Let A be a diagonally convex animal of height n. The sum of the number of border-vertices and the number of cut-vertices of A is $2n$. Hence, the height generating function of marked diagonally convex animals whose distinguished vertex is a border-vertex different from the origin is:

$$f_2(t) = \frac{\partial}{\partial x}\left(\frac{f_0\left(tx^2, \frac{1}{x}\right) - 1}{x}\right)\Bigg|_{x=1}$$

where the series $f_0(t, x)$ is given in Proposition 2.4. Using $f(t) = 1 + tf(t)^3$, we obtain

$$f_2(t) = \frac{1}{3t}\left[3t + 5 + (6t - 5)f(t) + t(33t - 4)f'(t)\right].$$

We finally combine the results of the second and third step to obtain the generating function of the second correction terms:

$$\begin{aligned} \textstyle\sum_{n\geq 1} d_{n,1}\, t^n &= f_1(t) - f_2(t) \\ &= \tfrac{1}{3t}\left[10t + 14 + (9t^2 + 26t - 14)f(t) + t(27t^2 + 139t - 22)f'(t)\right]. \end{aligned}$$

This concludes the proof of Theorem 2.1.

■

3 Directed bond-percolation on the square lattice

The square lattice is oriented as in the previous section, but we now deal with bond-percolation. Let A be a bond-animal on the square lattice. Its *width* (resp. *height*) is the width (resp. height) of the smallest rectangle containing it. Its *size* is the sum of its height and its width. We associate with A a site-animal $V(A)$ defined by (6); it contains the origin O and the vertices adjacent to at least one edge of A. The *directed height* of A is

$$\max\{i \geq 1, V(A) \cap \Delta_i \neq \emptyset\}.$$

Note that the empty animal has directed height 1. Figure 9 shows a bond-animal of height 5, width 5 and directed height 8. Its perimeter is 20.

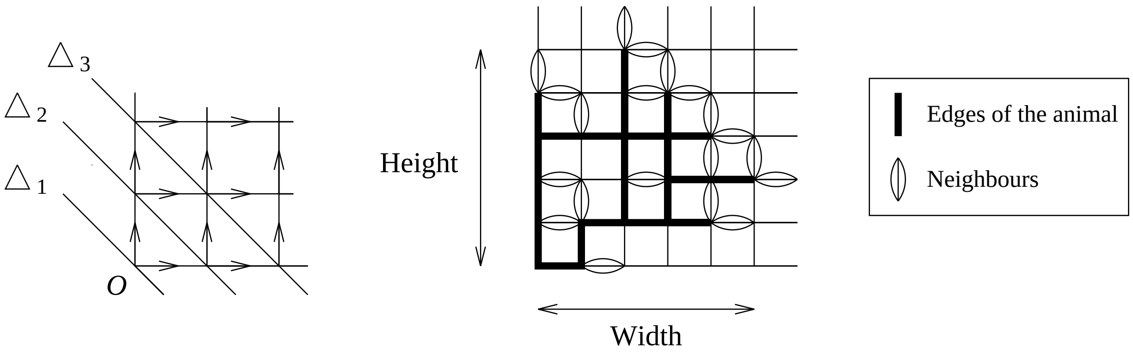

Figure 9: A bond-animal on the oriented square lattice.

Let $\mathcal{A}_n$ be the set of animals of directed height less than n, and let P_n be the probability that at least one vertex of Δ_n is connected:

$$P_n = 1 - \sum_{A\in\mathcal{A}_n} q^{p(A)}(1-q)^{|A|}.$$

The P_n's converge pointwise to the percolation probability $P(q)$. We shall see that they also converge formally to a series $P_\infty(q)$ and compute the first two correction terms. We have

$$\begin{aligned}
P_1 &= 1,\\
P_2 &= 1 - q^2,\\
P_3 &= 1 - q^2 - 2q^3 + q^4 + 2q^5 - q^6,\\
P_4 &= 1 - q^2 - 2q^3 - 4q^4 + 6q^5 + 11q^6 + O(q^7),\\
P_5 &= 1 - q^2 - 2q^3 - 4q^4 - 8q^5 + 25q^6 + O(q^7),\\
P_6 &= 1 - q^2 - 2q^3 - 4q^4 - 8q^5 - 17q^6 + O(q^7).
\end{aligned}$$

Theorem 3.1 — *For $n \geq 1$,*

$$P_n - P_{n+1} = d_{n,0}\, q^{n+1} + d_{n,1}\, q^{n+2} + O(q^{n+3}),$$

where $d_{n,0}$ is the n^{th} Catalan number

$$d_{n,0} = C_n = \frac{1}{n+1}\binom{2n}{n},$$

and

$$d_{n,1} = 2C_n - C_{n+1}.$$

Note that the second correction term is much simpler than in the previous model. Actually, Baxter and Guttmann, who studied this model, were able to give conjectures up to the 12^{th} correction term. This allowed them to extrapolate their expansion of the (formal) percolation probability $P_\infty(q)$ from the 29^{th} to the 41^{st} term [1]. The longest previously published series was the eight-term series of Blease [2].

According to Formula (8), we first need to characterize animals of directed height n having a minimal perimeter.

Definition 3.2 — *Let A be an animal. Let $(i,j,k) \in \mathbb{N}^2 \times \mathbb{N}^*$, and for $0 \leq \ell \leq k$, let v_ℓ be the vertex of coordinates $(i, j+\ell)$. For $1 \leq \ell \leq k$, let e_ℓ be the vertical edge going from $v_{\ell-1}$ to v_ℓ. The sequence $(e_1, e_2, \ldots, e_k)$ is said to be a* vertical hole *of A if:*

- *for $1 \leq \ell \leq k$, $e_\ell \notin A$,*
- *for $1 \leq \ell \leq k-1$, $v_\ell \notin V(A)$,*
- *$\{v_0, v_k\} \subset V(A)$.*

We define *horizontal holes* in a similar manner. A *hole* is a vertical or horizontal hole.

Proposition 3.3 — *Let A be a (bond) animal of size s having h holes. Then its perimeter is:*

$$s + h + 2.$$

For example, the animal in Figure 9 has size 10 (more precisely, width 5 and height 5), 8 holes, and its perimeter is 10+8+2=20.

Proof of Proposition 3.3. Suppose that A has width p and height q (with $p+q=s$). For $0 \leq i \leq p$, the number of vertical neighbours of A lying at abscissa i is the number of vertical holes of A at abscissa i, plus one. We conclude the proof by counting horizontal neighbours in a similar manner.

■

Let A be an animal of directed height n. As $V(A)$ has at least one vertex on Δ_n, the size of A is at least $n-1$, and if it is exactly $n-1$, then $|V(A) \cap \Delta_n| = 1$. Consequently, the perimeter of A is at least $n+1$. Conversely, any holeless animal of directed height n that fits into a rectangle of size $n-1$ has $n+1$ as perimeter. As in Proposition 1.6, we define $p_n = \min\{p(A), A \in \mathcal{A}_{n+1} \setminus \mathcal{A}_n\}$. Then $p_n = n+1$, and therefore, $P_n - P_{n+1} = O(q^{n+1})$.

3.1 The first correction term

Let us call *staircase animals* the animals of $\mathcal{A}_{n+1} \setminus \mathcal{A}_n$ having $n+1$ as perimeter. The following simple procedure provides all of them. First, choose two North-East paths ω_1 and ω_2 on $\mathbb{Z} \times \mathbb{Z}$, having $n-1$ steps, starting from the origin O and having the same endpoint, such that ω_1 lies below ω_2 (Figure 10). The set of steps of ω_1 and ω_2 is an animal of directed height n, whose size is $n-1$. Then, fill the holes between ω_1 and ω_2. The animal thus obtained has directed height n and perimeter $n+1$.

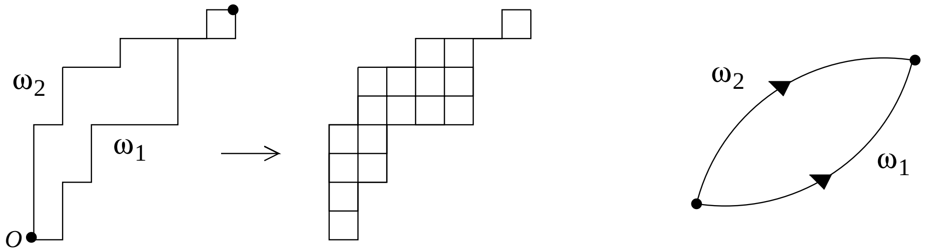

Figure 10: Staircase animals.

Conversely, a staircase animal is totally characterized by its *bounding paths* ω_1 and ω_2 (Figure 10 shows the schematized form of a generic staircase animal). These animals are closely related to *staircase* (or *parallelogram*) *polyominoes* which have been extensively studied [6, 12]. Hence, it is not at all surprising that Catalan numbers appear in the correction terms.

According to Equation (8), the first correction term $d_{n,0}$ is the number of staircase animals of directed height n.

Lemma 3.4 — *For $n \geq 1$, let a_n be the number of staircase animals of directed height n, and let $f = \sum_{n \geq 1} a_n t^n$ be the corresponding generating function. Then*

$$f = t(1+f)^2.$$

Equivalently,

$$a_n = \frac{1}{n+1}\binom{2n}{n}.$$

Proof. We give a standard proof of this result. Let A be a staircase animal of directed height n, with $n \geq 2$. Let $k = \max\{i, 1 \leq i < n,\ |V(A) \cap \Delta_i| = 1\}$.

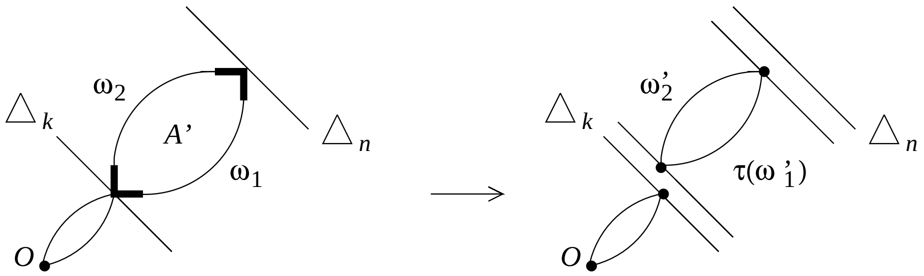

Figure 11: The decomposition of staircase animals.

The set of edges of A lying between Δ_1 and Δ_k is a staircase animal of directed height k (Figure 11). The set of edges of A lying between Δ_k and Δ_n is a staircase animal A' of directed height $n-k+1$, having the following additional property

$$\forall i \in \{k+1, \ldots, n-1\},\ |V(A') \cap \Delta_i| \geq 2.$$

If $k = n-1$, then A' is reduced to a single edge. If $k \leq n-2$, let ω_1 and ω_2 be the bounding paths of A'. Let ω'_1 (resp. ω'_2) be the path obtained by removing from ω_1 (resp. ω_2) its first

step and its last step. Let τ denote the translation of vector $(-1,1)$. Then the paths $\tau(\omega_1')$ and ω_2' define a staircase animal of directed height $n-k-1$. Hence, we have split A into two smaller animals.
Combining the cases $k=n-1$ and $k<n-1$ provides the following recurrence, for $n \geq 2$:

$$a_n = 2a_{n-1} + \sum_{k=1}^{n-2} a_k a_{n-k-1}.$$

Since $a_1 = 1$, this gives

$$f = t + 2tf + tf^2,$$

and we compute a_n from the Lagrange inversion formula.

3.2 The second correction term

According to Equation (8), the second correction term is

$$d_{n,1} = |\mathcal{A}_{n,1}| - \sum_{A \in \mathcal{A}_{n,0}} |A|$$

where for $k \in \{0,1\}$, $\mathcal{A}_{n,k}$ is the set of animals of directed height n having perimeter $n+k+1$.
Let A be an animal of $\mathcal{A}_{n,1}$. Three cases occur (see Proposition 3.3):

a) either A has one hole, containing one edge, and its size is $n-1$, or

b) A has one hole, containing several edges, and its size is $n-1$, or

c) A has no hole, but its size is n.

We have enumerated in Figure 12 the animals of $\mathcal{A}_{n,1}$ $(n \leq 4)$.

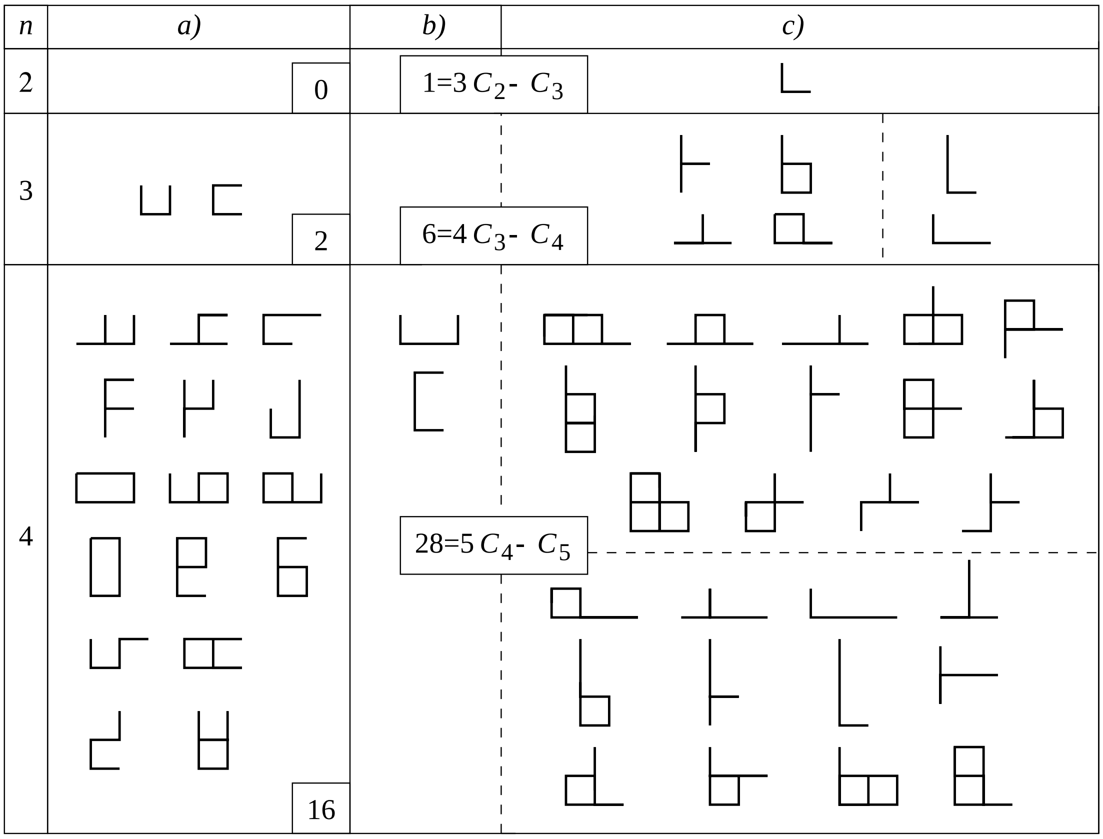

Figure 12: Animals of $\mathcal{A}_{n,1}$ for $n \leq 4$.

Now, note that $\sum_{A \in \mathcal{A}_{n,0}} |A|$ is the number of *marked staircase animals* of directed height n, that is the number of pairs (A, e) where $A \in \mathcal{A}_{n,0}$ and $e \in A$. Again, three cases occur:
α) either $A \setminus \{e\}$ is an animal of directed height n, or
β) $A \setminus \{e\}$ is an animal of directed height $n-1$, or
γ) $A \setminus \{e\}$ is not an animal.
We have enumerated in Figure 13 the marked staircase animals of directed height at most 4.

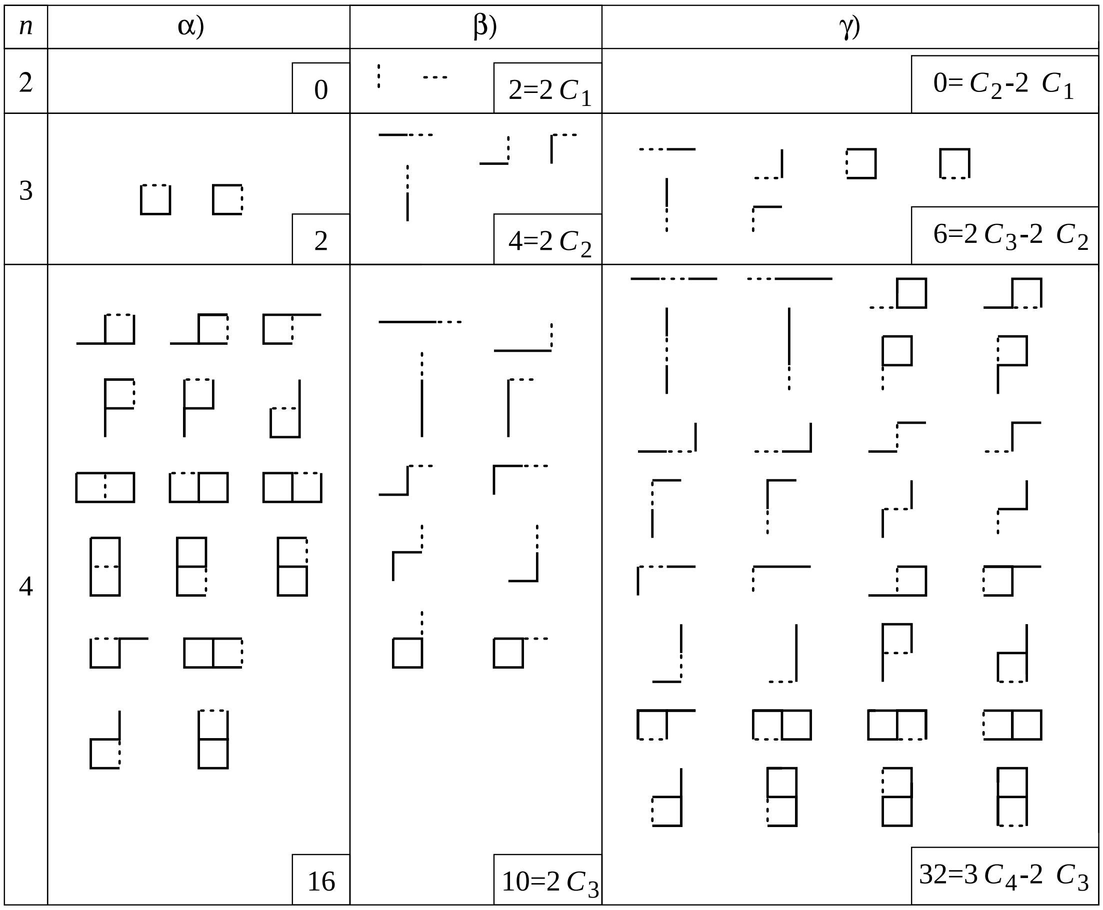

Figure 13: Marked staircase animals of directed height at most 4.

In Case α, $A \setminus \{e\}$ has one hole reduced to the edge e, and animals in case α are in one-to-one correspondence with animals in case a.

We now count separately animals in cases β, γ, b and c. The simplest cases are β and γ.

Case β happens when A has a single edge e — which is the distinguished edge — between Δ_{n-1} and Δ_n. Then $A \setminus \{e\}$ is a staircase animal of directed height $n-1$. Hence, the number of animals in this class is $2C_{n-1}$.

Case γ happens when e is a horizontal (resp. vertical) edge of the lower (resp. upper) bounding path of A, different from the last step of this path. Since the sum of the number of horizontal edges lying on ω_1 and the number of vertical edges lying on ω_2 is $n-1$, the number of animals in this class is $(n-1)C_n - 2C_{n-1}$.

Case b. Assume that the hole $(e_1, \ldots, e_k)$, with $k \geq 2$, is a vertical hole, and that e_i goes from v_{i-1} to v_i for $1 \leq i \leq k$. Since v_k belongs to $V(A)$, the horizontal edge ending at v_k belongs to A. Using the fact that A has a single hole and size $n-1$, we state the following properties:

- for $0 \leq i < k$, $V(A)$ has no vertex on the right-hand side of v_i (on the same row), and

- the horizontal edge ending at v_0 belongs to A.

An example is given in Figure 14 together with the schematized form of these animals.

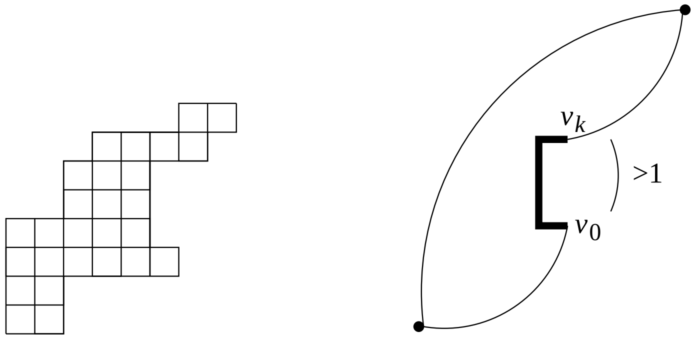

Figure 14: Case b.

Since we are going to count simultaneously animals in cases b and c, we now describe case c.

Case c. Note that $|V(A) \cap \Delta_n| \in \{1, 2\}$. If $V(A)$ has two vertices on Δ_n, then these vertices are consecutive. The number of such animals is clearly $C_{n+1} - 2C_n$.

Now, suppose that $V(A)$ has a single vertex on Δ_n. Let R_1 be the smallest rectangle containing this vertex and the origin (Figure 15). Then R_1 has size $n - 1$. The set of edges of A contained in R_1 is a staircase animal of directed height n. The smallest rectangle R_2 containing A is larger than R_1: it has either an extra column or an extra row. Assume that it has an extra column. Then A can be schematized as in Figure 15. An example of such an animal is also drawn in this figure.

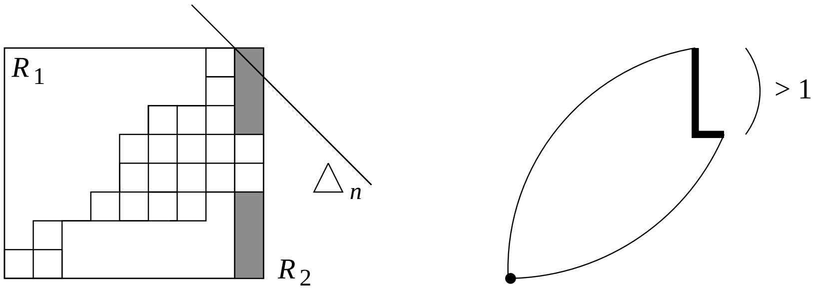

Figure 15: Case c.

Finally, we can give a general schema (Figure 16) including simultaneously

- animals of case b having a vertical hole,
- animals of case c having a single vertex on Δ_n, and such that R_2 is wider than R_1.

We could call these animals *bumpy* staircase animals. A simple transformation allows to count them. Let A be a "bumpy" staircase animal, and suppose that the vertex v_0 lies on Δ_i (Figure 17). Let $k = \max\{j, 1 \leq j < i,\ |V(A) \cap \Delta_j| = 1\}$. The set of edges of A lying between Δ_1 and Δ_k is a staircase animal of height k. Let A' be the animal formed by the edges of A lying between Δ_k and Δ_n. Let us define the paths $\omega_{1,1}$, $\omega_{1,2}$ and ω_2 as in Figure 17:

- $\omega_{1,1}$ starts on Δ_k and ends at v_0,
- $\omega_{1,2}$ starts on Δ_{i+1} and ends on Δ_n, and
- ω_2 starts on Δ_k and ends on Δ_n.

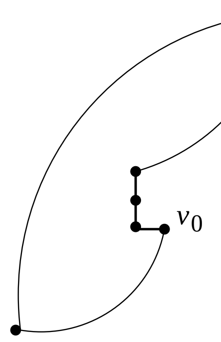

Figure 16: Cases b and c.

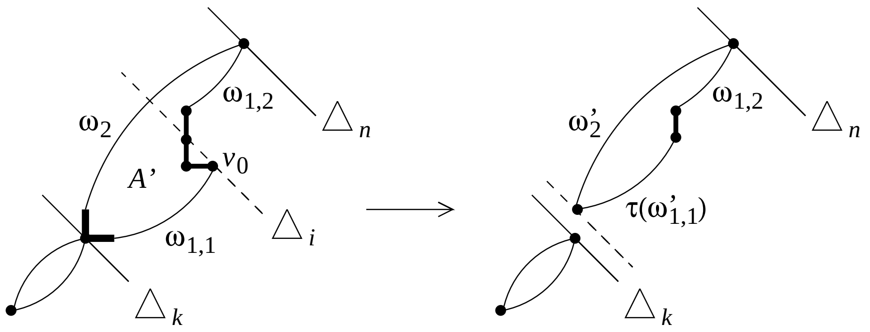

Figure 17: The decomposition for cases b and c.

Let $\omega'_{1,1}$ (resp. ω'_2) be obtained by removing from $\omega_{1,1}$ (resp. ω_2) its first step. We apply to $\omega'_{1,1}$ a translation of vector $(-1,1)$. The animal A' is thus replaced by a staircase animal of height $n-k$ having a distinguished vertical step on its lower bounding path.

Not forgetting the symmetric case, the number of animals in cases b and c is

$$C_{n+1} - 2C_n + \sum_{k=1}^{n-2} C_k C_{n-k}(n-k-1).$$

Combining all the cases leads to

$$d_{n,1} = C_{n+1} - 2C_n + \sum_{k=1}^{n-2} C_k C_{n-k}(n-k-1) - (n-1)C_n.$$

Therefore, the generating function of the $d_{n,1}$'s is

$$\sum_{n\geq 1} d_{n,1}t^n = tf'(f-1) + \frac{1}{t}(f - t - tf - tf^2),$$

where $f = \sum_{n\geq 1} C_n t^n$. Using $f = t(1+f)^2$, we obtain

$$\sum_{n\geq 1} d_{n,1}t^n = \left(2 - \frac{1}{t}\right) f + 1,$$

so for $n \geq 1$, $d_{n,1} = 2C_n - C_{n+1}$.

■

4 Directed bond-percolation on the honeycomb lattice

We could study several other lattices (e.g. the triangular lattice), and compute for each of them the first few corrections terms, but the two previous examples (directed site- and bond-percolation on the square lattice) probably suffice to illustrate our techniques, the limits and the difficulties of this approach. Nevertheless, we compute in this section the first correction term for a third model: bond-percolation on the honeycomb lattice. This model has been studied by Onody, but he could not even conjecture the value of the first correction term [13].

The honeycomb lattice is oriented as shown in Figure 18. This figure also shows a bond-animal on this lattice. To define the *height* of an animal, we introduce again some horizontal lines, denoted $\Delta_1, \Delta_2, \Delta_3$, etc... We associate with each (bond) animal A a site-animal $V(A)$ defined by (6): it contains the origin and the vertices adjacent to at least one edge of A. The *height* of an animal A is:

$$\max\{i \geq 1, V(A) \cap \Delta_i \neq \emptyset\}.$$

Note that the empty animal has height 1.

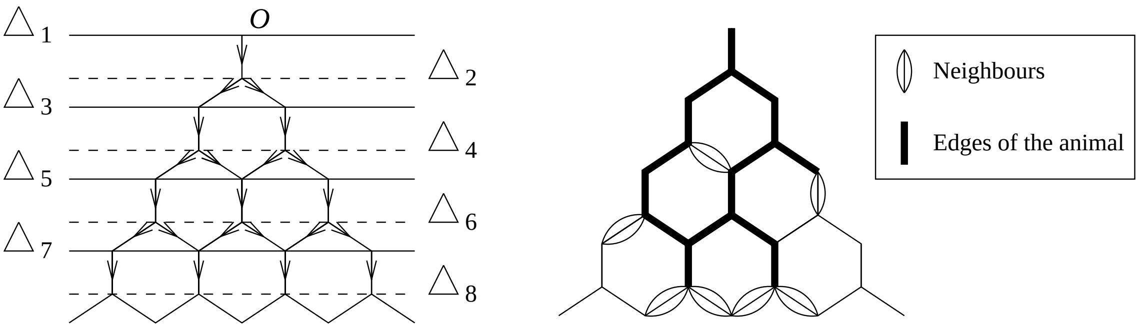

Figure 18: The oriented honeycomb lattice and an animal of height 8.

Let $\mathcal{A}_n$ be the set of animals of height less than $2n-1$. Let P_n be the probability that at least one vertex of Δ_{2n-1} is connected:

$$P_n = 1 - \sum_{A \in \mathcal{A}_n} q^{p(A)}(1-q)^{|A|}.$$

The P_n's converge pointwise to the percolation probability $P(q)$. We shall see that these polynomials have also a formal limit $P_\infty(q)$ and compute the first correction term. We have

$$\begin{aligned}
P_1 &= 1, \\
P_2 &= 1 - q - q^2 + q^3, \\
P_3 &= 1 - q - 4q^2 + 4q^3 + 8q^4 + O(q^5), \\
P_4 &= 1 - q - 4q^2 - 12q^3 + 63q^4 + O(q^5), \\
P_5 &= 1 - q - 4q^2 - 12q^3 - 45q^4 + O(q^5).
\end{aligned}$$

Theorem 4.1 — *For $n \geq 1$,*

$$P_n - P_{n+1} = d_{n,0}\, q^n + O(q^{n+1})$$

where the generating function f of the $d_{n,0}$ defined by

$$f = \sum_{n \geq 1} d_{n,0}\, t^{n-1},$$

is characterized by the algebraic equation

$$f = 1 + tf + \frac{t}{4}\left((7+t)f^2 + f^3\right).$$

According to Proposition 1.6, we must first characterize the animals of $\mathcal{A}_{n+1} \setminus \mathcal{A}_n$ having minimal perimeter, and then count them: the number of such animals is exactly the first correction term $d_{n,0}$. Let A be an animal on the honeycomb lattice. It has two kind of edges: vertical edges lying in a strip bounded by Δ_{2i-1} and Δ_{2i} for some i, and oblique edges lying in a strip bounded by Δ_{2i} and Δ_{2i+1} for some i.

Definition 4.2 — *Let A be an animal. A* diagonal hole *is a non-empty maximal finite sequence of consecutive oblique edges lying in the same horizontal strip and not belonging to A. An animal having no diagonal hole is called a* diagonally convex *animal.*

Let (V, f) be an infinite graph, and let A be a bond-animal on this graph. Let $e = (v, w, i)$ be an edge of A. Then e is said to have a *child* (resp. a *father*) if A contains an edge of type (w, x, j) (resp. (x, v, j)) for some (x, j).

Proposition 4.3 — *Let A be an animal of height $2n-1$ or $2n$ having h diagonal holes, and such that k of its vertical edges have no child. Its perimeter is $n + h + k$.*

We can check that the animal of Figure 18 satisfies this proposition: its height is 8, it has one diagonal hole, and two of its vertical edges have no child. Its perimeter is 4+1+2=7.

Proof of Proposition 4.3. Let A be an animal of height $2n-1$ or $2n$, having h diagonal holes, and such that k of its vertical edges have no child.

1. Suppose $h = 0$ and $k = 0$. Then the height of A is $2n-1$, otherwise k would be positive. We prove the proposition by induction on n. The result is obvious if $n = 1$, and the reader will easily convince himself that adding some edges lying between Δ_{2n-1} and Δ_{2n+1} to an animal of height $2n-1$ such that $k = h = 0$ increases the perimeter by 1 (see Figure 19).

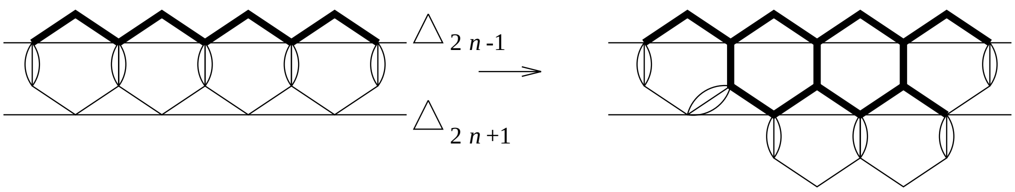

Figure 19: The perimeter of animals such that $h = k = 0$.

2. Suppose $h > 0$ and $k = 0$. Again, the height of A is $2n-1$. Let us consider the first (highest) horizontal strip containing a hole $(e_1, e_2, \ldots, e_m)$. Let A' be obtained by adding to A the edges $(e_1, \ldots, e_m)$ and their fathers that do not already belong to A. Then A' is still an animal of height $2n-1$. It has $h-1$ holes, and all its vertical edges have a child. Moreover, its perimeter is $p(A)-1$ (see Figure 20). Hence, by induction on h, we prove the proposition for animals such that $k = 0$.

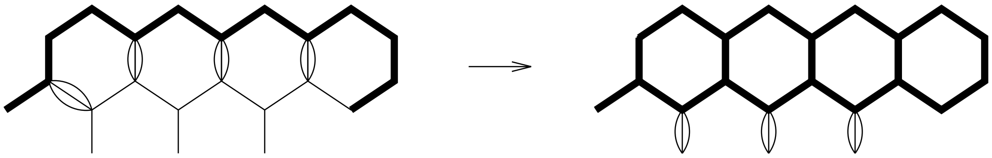

Figure 20: Filling a hole of an animal such that $k = 0$ decreases the perimeter by 1.

3. Suppose $k > 0$. Let e be a vertical edge of A having no child, and $A' = A \setminus \{e\}$. Then A' is still an animal of height $2n-1$ or $2n$. It has as many diagonal holes as A, and $k-1$ of its vertical edges have no child. Moreover, its perimeter is $p(A)-1$ (see Figure 21). We conclude the proof by induction on k.

■

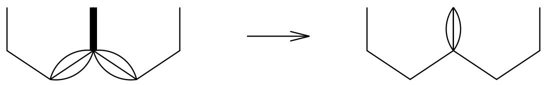

Figure 21: Removing a vertical edge having no child decreases the perimeter by 1.

Corollary 4.4 — *Let $\mathcal{A}_n$ be the set of animals of height less than $2n-1$. Then $p_n = n$, and an animal A of $\mathcal{A}_{n+1} \setminus \mathcal{A}_n$ has perimeter p_n if and only if*

- *A is diagonally convex, and*
- *each of its vertical edges has one or two childs.*

As in Equation (9), we denote $\mathcal{A}_{n,0}$ the set of such animals. Then the first correction term is $d_{n,0} = |\mathcal{A}_{n,0}|$. We are now going to enumerate the animals of $\mathcal{A}^{(0)} = \bigcup_{n\geq 1} \mathcal{A}_{n,0}$ according to their height, and this will conclude the proof of Theorem 4.1.

Proposition 4.5 — *Let $f(t) = \sum_{n\geq 1} d_{n,0} t^{n-1}$ be the generating function associated with the $d_{n,0}$'s. Then f is characterized by the algebraic equation*

$$f = 1 + tf + \frac{t}{4}\left((7+t)f^2 + f^3\right).$$

Proof. We proceed as for the enumeration of diagonally convex directed site-animals (Section 2.1): we first describe a one-to-one correspondence Ω between $\mathcal{A}^{(0)}$ and some paths of $\mathcal{P}$ having black and red steps. Then, we use the canonical decomposition of these paths illustrated in Figure 6 to enumerate them.

Let A be an animal of $\mathcal{A}_{n,0}$. Its height is $2n-1$, and its perimeter is n. For $1 \leq i \leq n-1$,

- let $p^{(1)}_{2i+1}$ be the number of vertical neighbours of A lying between Δ_{2i-1} and Δ_{2i} on the left-hand-side of A,
- let $p^{(1)}_{2i+2}$ be the number of vertical neighbours of A lying between Δ_{2i-1} and Δ_{2i} on the right-hand-side of A,
- let $p^{(2)}_{2i+1}$ be the number of oblique neighbours of A lying between Δ_{2i} and Δ_{2i+1} on the left-hand-side of A, and
- let $p^{(2)}_{2i+2}$ be the number of oblique neighbours of A lying between Δ_{2i} and Δ_{2i+1} on the right-hand-side of A.

Since any vertical edge of A has a child, $p^{(2)}_i$ equals 0 or 1 for any i. For $0 \leq i \leq 2$, we set $p^{(1)}_i = p^{(2)}_i = 0$. The perimeter of A is

$$n = \sum_{0\leq i\leq 2n} \left(p^{(1)}_i + p^{(2)}_i\right) + |V(A) \cap \Delta_{2n-1}|.$$

Therefore,

$$\sum_{0\leq i\leq 2n} \left(p^{(1)}_i + p^{(2)}_i\right) < n. \tag{18}$$

For $1 \leq k < n$, the same remark can be applied to the sub-animal of A formed by the edges lying above Δ_{2k-1}, thus proving that

$$\sum_{0\leq i\leq 2k} \left(p^{(1)}_i + p^{(2)}_i\right) < k. \tag{19}$$

For $2 \leq k \leq n$, a similar argument can be applied to the sub-animal of A formed by the edges lying above Δ_{2k-2}. We obtain

$$\sum_{0 \leq i \leq 2k-2} \left(p_i^{(1)} + p_i^{(2)}\right) + p_{2k-1}^{(1)} + p_{2k}^{(1)} < k-1.$$

In particular, for $k \geq 2$, if $\sum_{0 \leq i \leq 2k} \left(p_i^{(1)} + p_i^{(2)}\right) = k-1$, then $p_{2k-1}^{(2)} + p_{2k}^{(2)} > 0$.
We associate with A the path starting from $(0,0)$, ending at $(2n+1, n)$,

- having, at abscissa i, $p_i^{(1)}$ black North steps followed by $p_i^{(2)}$ red North steps (for $0 \leq i \leq 2n$),
- having only black steps at abscissa $2n+1$.

According to (18) and (19), all vertices of this path other than the origin, lie strictly below the line $\{x = 2y\}$.

Example. For the animal drawn in Figure 22, $(p_3^{(1)}, p_4^{(1)}, \ldots, p_{12}^{(1)}) = (0,0,0,0,1,0,0,0,0,1)$ and $(p_3^{(2)}, p_4^{(2)}, \ldots, p_{12}^{(2)}) = (0,0,1,0,1,0,0,0,1,0)$. The associated path is shown in Figure 22 as well.

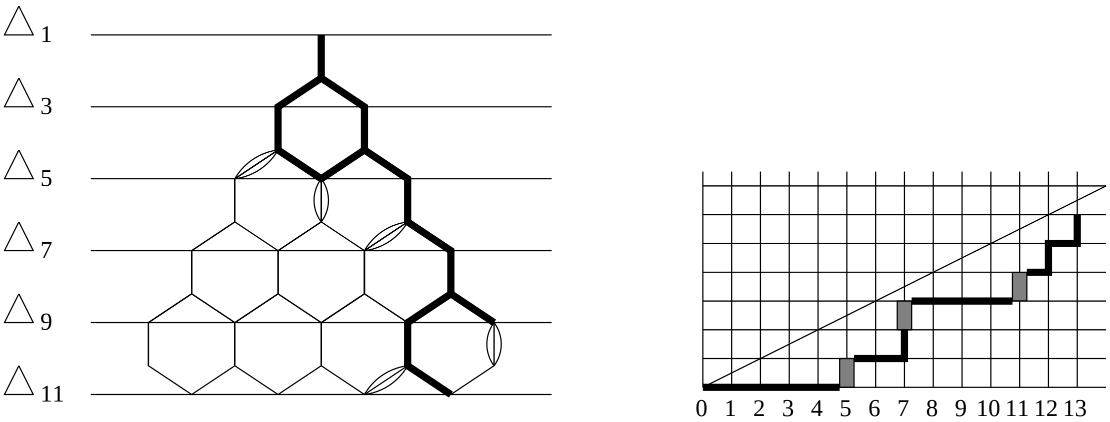

Figure 22: The coding of animals of $\mathcal{A}^{(0)}$ by coloured paths.

Let $\mathcal{P}_0$ be the set of black and red paths going from (0,0) to $(2n+1, n)$ for some n, that have all their vertices other than the origin strictly below the line $\{x = 2y\}$ and that, at each abscissa, the only North step that might be red is the highest. Let $\mathcal{P}_1$ be the subset of $\mathcal{P}_0$ containing the paths ω:

- having no red step at abscissa $2n+1$, $\qquad (C_0)$

and such that, for $2 \leq k \leq n$:

- if $(2k, k-1) \in \omega$, there is a red step at abscissa $2k-1$ or $2k$. $\qquad (C_1)$

Note that $\mathcal{P}_1$ contains the path reduced to a single East step. For $n \geq 1$, the correspondence described above induces a bijection between $\mathcal{A}_{n,0}$ and the paths of $\mathcal{P}_1$ of length $3n+1$. Hence, we are led to enumerate the paths of $\mathcal{P}_1$ according to their length. The factorization of the paths of $\mathcal{P}$ described in Section 2 can be extended to black and red paths. Again, different families of paths are involved in successive factorizations. First of all, let g_1 be the (length) generating function of the paths of $\mathcal{P}_1$. Then $g_1(t) = t + t^4 f(t^3)$, where f is the generating function defined in Proposition 4.5. For $2 \leq i \leq 4$, let g_i be the generating function of $\mathcal{P}_i$, where the $\mathcal{P}_i$'s are defined as follows:

- $\mathcal{P}_2$ is the subset of $\mathcal{P}_0$ containing the paths ω, such that, if $|\omega| = 3n+1$:
  for $1 \leq k \leq n$, if $(2k, k-1)$ belongs to ω, there is a red step at abscissa $2k+1$ or $2k$.
- $\mathcal{P}_3$ is the set of paths of $\mathcal{P}_0$ satisfying C_0,
- $\mathcal{P}_4$ is the set of paths of $\mathcal{P}_3$ having a red step at abscissa $2n$.

Applying the canonical factorization to these families of paths gives the following system:

$$\begin{aligned}
g_1 &= t + t g_1 g_2 g_3,\\
g_2 &= t + t g_2 (2g_3 - t)(g_3 + g_4) + t^2 g_2 (g_3 - t),\\
g_3 &= t + t(2g_3 - t)^2 g_3,\\
g_4 &= t(2g_3 - t)^2 g_4 + t^2 (2g_3 - t)(g_3 - t).
\end{aligned}$$

By elimination, we obtain the algebraic equation satisfied by g_1 from which we deduce the equation satisfied by f.

■

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