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Polyominoes and Polygons

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Abstract. This paper is a survey on the enumeration techniques of different families of polyominoes.

1. Introduction

A very famous—and elementary—open problem in combinatorics is the enumeration of *self-avoiding-walks* on a regular lattice. These walks are defined up to a translation (Figure 1). In dimension two, the most usual lattices are the square lattice, the triangular lattice and the honeycomb lattice, but we will only be concerned here with the square lattice. A closely related problem is the enumeration of *self-avoiding-polygons* (or *polygons* for short). Finally, a third question of the same family is the enumeration of *polyominoes*. A polyomino is a finite union of elementary cells of the lattice, whose interior is connected (Figure 1). Of course, a polyomino can be associated to any polygon, but the contrary is false, since a polyomino can have holes, whereas a polygon cannot. The problem consists in finding the number of polyominoes having a given area (number of cells), or, alternatively, a given perimeter.

These three questions are interesting combinatorialists, but are also of considerable importance in the study of lattice models in physics and chemistry. These models are supposed to provide simplified versions of physical phenomena, and are used to describe phase transitions. A well-known such model is the Ising model.

In chemistry, self-avoiding walks (resp. self-avoiding polygons) are classical models of polymers (resp. ring polymers) in a dilute solution. In physics, the self-avoiding walk problem is equivalent to the resolution of the n-vector model (in the limit $n \to 0$), which generalizes the Ising model [**13**]. Polyominoes also occur, under the equivalent form of *animals*. An animal is obtained by changing every cell of a polyomino into a point (Figure 2). Animals are considered as a cell growth model, and are also closely linked to the percolation problem. A special

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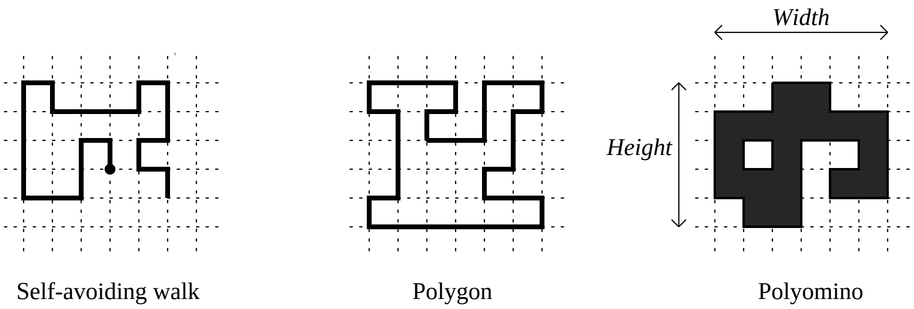

FIGURE 1. Self-avoiding walks, polygons and polyominoes.

class of animals, formed with the *directed* animals, which will be defined later on, plays an important role. These animals are naturally linked to directed percolation, but a more general result is the following: the enumeration of directed animals on any regular lattice in dimension D is equivalent to the resolution of a gas-model in dimension $D-1$ [**18**]. For example, the quite famous hard-hexagon model, which involves Rogers-Ramanujan identities and was solved in 1980 by Baxter, is equivalent to the enumeration of directed animals on the cubic lattice [**3**].

Although serious efforts have been made over the last 40 years by physicists and mathematicians, the problem of counting these objects remains completely unsolved, and their enumeration has been recognized as an extremely difficult question. Nevertheless, some asymptotic results are known. One of the main rigorous results is the following: let $p(n)$ be the number of polyominoes having n cells. Then $(p(n))^{1/n}$ tends to a limit K, which satisfies the following inequality: $3,87 < K < 4,65$. The situation could hardly be less mysterious, since the first digit of K is not even known [**22**] (Klarner has named K the Klarner Konstant, and worries about the fact it could simply be 4).

The difficulty of these problems has led to the study of various restricted—but as general as possible—classes of polyominoes and polygons. Most of the classes that have been enumerated can be defined by combining two simple notions: a notion of *convexity* and a notion of *directed growth*.

A polyomino is said to be column-convex (or vertically convex) when its intersection with any vertical line is convex. We can define similarly a notion of horizontal or diagonal convexity. A *convex* polyomino is vertically and hor-

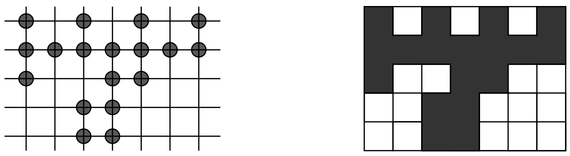

FIGURE 2. An animal on the square lattice, and the associated polyomino.

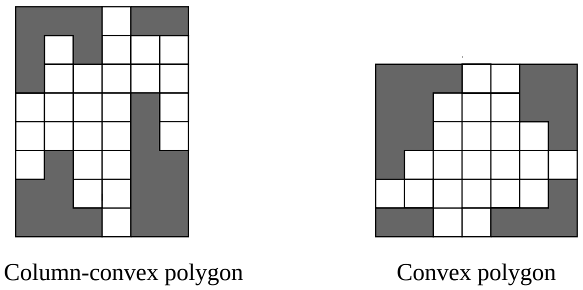

FIGURE 3. Convexity.

izontally convex (Figure 3). Note that a column-convex polyomino defines a polygon, hence we will use just as well the name "polyomino" or "polygon" in this case. The problem of counting convex polyominoes was apparently raised by D. Knuth, around 1970 (see [**23**]).

The notion of directed structures is more recent, and comes from statistical physics. An animal P is said to be directed when every point of P can be reached from one distinguished point, called the *root*, by a path using only North and East steps, whose vertices are contained in P. We can also define directed animals with several roots (Figure 4). Note that, in this case, the interior of the corresponding "polyomino" might not be connected. The roots of a *compact-rooted* directed animal are consecutive points of the NW-SE diagonal.

To conclude this part, let us present a few asymptotic results. They give an idea of the generality of the different classes of polyominoes under consideration.

- The number of directed animals having n cells is asymptotically equivalent, up to a multiplicative constant, to $\frac{3^n}{\sqrt{n}}$ [**21**].
- The number of compact-rooted directed animals having n cells is *exactly* 3^{n-1} [**21**].
- The number of vertically convex polyominoes having n cells is equivalent to $(3,205...)^n$ [**23**].
- The number of convex polyominoes having n cells is equivalent, up to a multiplicative constant, to $(2,309...)^n$ [**4**] [**23**].

Thus, the class of vertically convex polygons is the largest one. However, it is exponentially small in the class of all polyominoes.

These asymptotic results, but the last one, were actually derived from exact

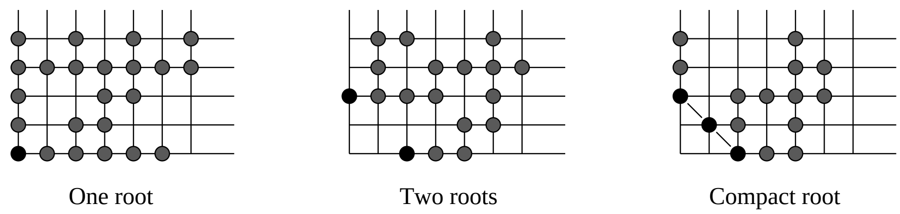

FIGURE 4. Directed animals.

results. The aim of this paper is to give an idea of the methods which have been used to enumerate these families of polyominoes. We will describe in particular the case of directed animals (Section 3), and the case of convex polygons (Section 4).

2. How to enumerate polyominoes?

Let $\mathcal{P}$ be a family of polyominoes. In order to enumerate the elements of $\mathcal{P}$, it is often very convenient to consider the corresponding generating function. For example, the *area-generating function* of $\mathcal{P}$ is the formal power series

$$\sum_{n\geq 1} a_n t^n,$$

where a_n denotes the number of elements of $\mathcal{P}$ whose area is n. Actually, we will often be interested in more precise, multivariate generating functions. For example, we will enumerate convex polygons according to three parameters, their *width*, their *height* and their *area* (Figure 1). The associated generating function is

$$\sum_{n\geq 1, m\geq 1, a\geq 1} c_{n,m,a} x^n y^m q^a, \tag{2.1}$$

where $c_{n,m,a}$ is the number of convex polygons having width n, height m and area a.

Almost all the classical methods used to enumerate classes of polyominoes can be basically described in two steps as follows.

(i) The first step consists in finding a "good decomposition" for these polyominoes. A "good decomposition" is a kind of recursive definition that can be translated into an equation—or a system of equations—which defines implicitly the generating function of the studied class. A formal study of these decompositions is currently being developped by Dutour and Fédou [**20**].

(ii) The second step consists in solving these equations, which can be, unfortunately, of any type.

Example. This principle can be immediately illustrated by the simple example of Ferrers diagrams (Figure 13.1). Ferrers diagrams are particular convex polyominoes, and are well-known as the usual representation for partitions of integers [**1**]. Let $F(x, y, q)$ be the generating function of these diagrams, counted as in (2.1) according to their width, height and area. Let us consider a Ferrers diagram. It might be reduced to a single cell. Otherwise, let us consider its columns of maximal height. If there is only one such column, one obtains a new Ferrers diagram by removing its lowest cell. If there are several such columns, one obtains a new Ferrers diagram by removing one of them.

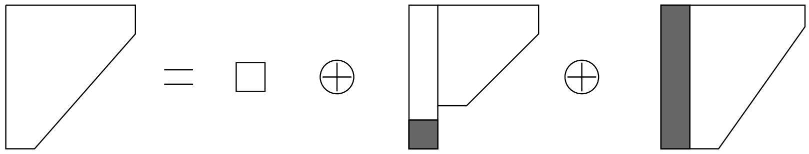

FIGURE 5. The decomposition of Ferrers diagrams.

This decomposition is schematized in Figure 5, and leads to the following equation defining F:

$$F(x,y,q) = xyq + yqF(x,y,q) + xF(x,yq,q),$$

or

$$F(x,y,q) = \frac{xyq}{1-yq} + \frac{x}{1-yq}F(x,yq,q).$$

In order to solve this equation, we just have to iterate it infinitely many times. We get:

$$F(x,y,q) = y\sum_{n\geq 1}\frac{x^n q^n}{(yq)_n},$$

where we use the classical notation $(a)_n = (1-a)(1-aq)\ldots(1-aq^{n-1})$.

Each of the steps (i) and (ii) described above can raise serious difficulties. When there is no decomposition, the problem remains completely open (as for general polyominoes for example). In many cases, one can obtain an equation, but does not know how to solve it.

We are going to illustrate these difficulties by describing two important cases: the case of directed animals and the case of convex polygons. For directed animals, "the" decomposition happened to be hard to guess, but the associated equation was easy to solve. On the contrary, several rather natural decompositions exist for convex polygons, but the resolution of the associated equations has just been achieved recently [**8**] [**9**].

3. Directed animals

Let us start with a very simple fact.

PROPOSITION 3.1. — *The number of directed animal having a compact root and n cells is 3^{n-1}.*

This assertion can be checked by hand for small values of n (Figure 6). One can hardly imagine a simpler result. It suggests that one can consider all the animals of size n, add a new cell to each of them in three different ways, and thus get all the animals of size $n+1$. But ... how to prove this?

The first proof of this result was given by Gouyou-Beauchamps and Viennot [**21**]. They precisely described the three ways of adding a cell. Their construction is very impressive, but also very complicated, and does not really explain

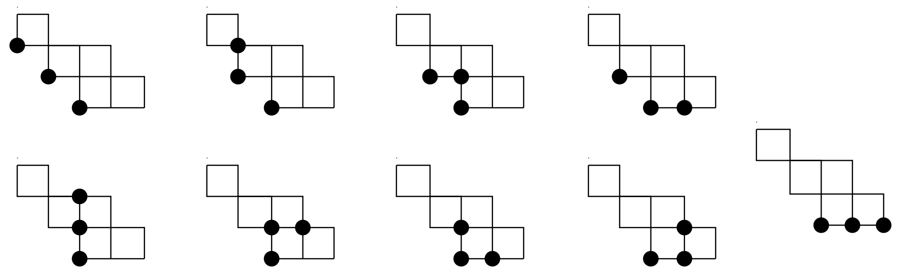

FIGURE 6. There are 9 compact-rooted directed animals with 3 cells.

the structure of directed animals. "The" good decomposition was found a few years later, when Viennot developped the theory of *heaps*, a geometric version of partially commutative monoids, which were introduced in 1969 by Cartier and Foata [**12**] [**29**]. The purpose of this paper is not to give the details of this theory, and, actually, we do not really need them here. Let us just give an intuitive definition: a heap is what one gets by dropping vertically some solid pieces of anything, one after the other. Thus, a piece lies either on the "floor" (then it is said to be *minimal*), or on another piece—at least partially. Figure 7 shows a heap of segments, having two minimal pieces, and a heap of rectangles.

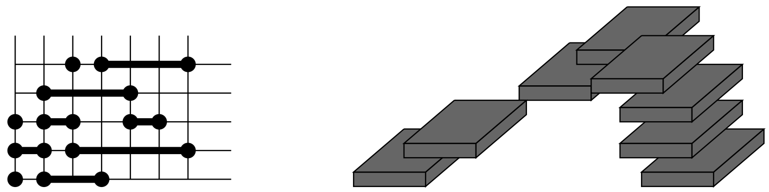

FIGURE 7. Heaps of pieces.

Directed animals *are*, basically, heaps. To make it clear, replace each point of an animal by a small segment, named a *domino* (Figure 8). Note that, because of the structure of the square lattice, two dominoes of the heap thus obtained cannot be perfectly superposed. Moreover, if the animal has a unique root, the associated heap has a unique minimal piece. Such heaps are named *pyramids*.

The reason why using the notion of heap is interesting here is that there exists

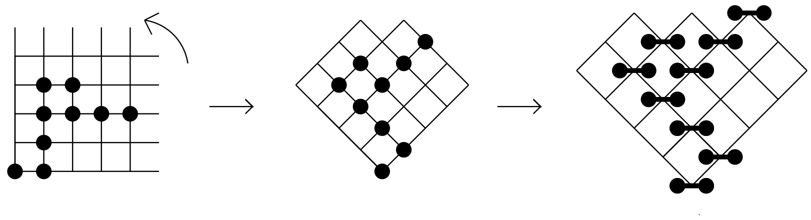

FIGURE 8. A directed animal and the associated pyramid.

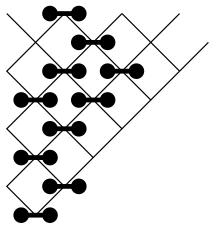

FIGURE 9. A half-pyramid of dominoes.

a *monoid structure* on the set of heaps (the product of two heaps is obtained by putting one heap above the other and dropping its pieces). Hence there are also factorizations (obtained by pushing upwards one or several pieces of the heap). But a factorization is nothing but a decomposition.

Let us begin with the decomposition of pyramids. A pyramid is either a *half-pyramid* (Figure 9), or the product of a half-pyramid and a pyramid (Figure 10). Let $P(t)$ denote the generating function of pyramids, counted according to the number of pieces, and let $H(t)$ denote the generating function of half-pyramids. Then $P(t) = H(t)(1 + P(t))$.

Now, a half-pyramid having several pieces is the product of a single piece and of one or two half-pyramids (Figure 11), which implies $H(t) = t + tH(t) + tH^2(t)$. A trivial computation finally provides the following result.

PROPOSITION 3.2. — *The area-generating function of one-rooted directed animals is:*

$$P(t) = \frac{1}{2}\left(\sqrt{\frac{1+t}{1-3t}} - 1\right).$$

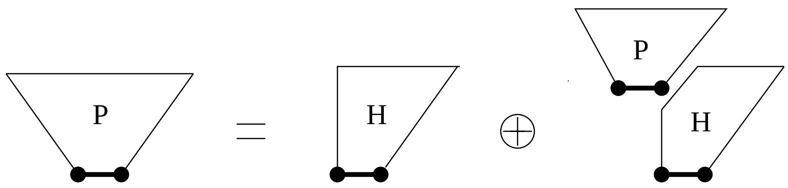

FIGURE 10. The decomposition of pyramids.

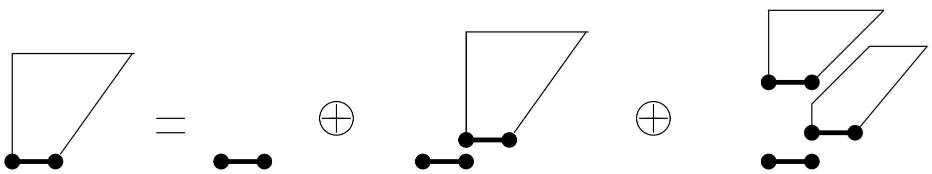

FIGURE 11. The decomposition of half-pyramids.

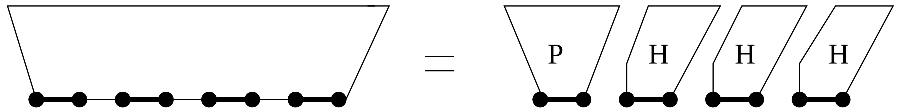

FIGURE 12. The decomposition of heaps having a compact basis.

The enumeration of compact-rooted directed animals is equivalent to the enumeration of heaps having a *compact basis* (that is "consecutive" minimal pieces, see Figure 12). The generating function of heaps having a compact basis formed with k pieces is $P(t)H(t)^{k-1}$, which implies that the generating function of compact-rooted directed animals is

$$\frac{P(t)}{1-H(t)},$$

that is

$$\frac{t}{1-3t}.$$

Hence, this decomposition of heaps seems quite natural—once it has been found—and proves without any difficulty Proposition 3.1 [**5**] [**6**].

4. Convex polygons

The problem raised by Knuth—counting convex polygons—was actually solved step by step, by successively enumerating more and more general sub-classes of convex polygons. We already defined Ferrers diagrams. The other important sub-classes can also be defined by requiring that one or several vertices of the minimal bounding rectangle of the polygon also belongs to the polygon itself. Stack polyominoes and parallelogram polyominoes share (at least) two vertices with their minimal bounding rectangle. Moreover, note that the polyominoes which have one vertex in common with their bounding rectangle are exactly the directed and convex polyominoes (Figure 13).

We are going to describe two decompositions of parallelogram polyominoes. Each of them can be extended to directed and convex polyominoes, and then to convex polyominoes. We will not give the details of these extensions, but

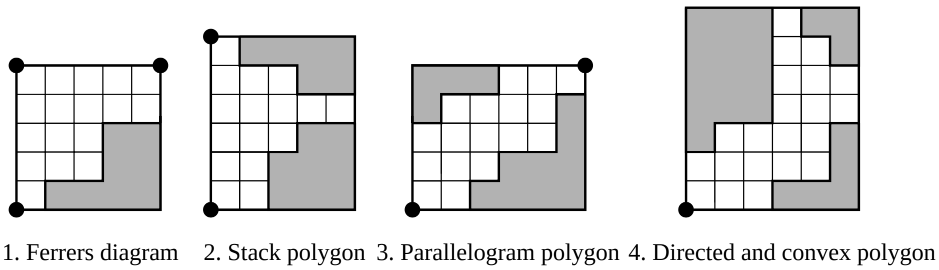

FIGURE 13. Families of directed and convex polygons.

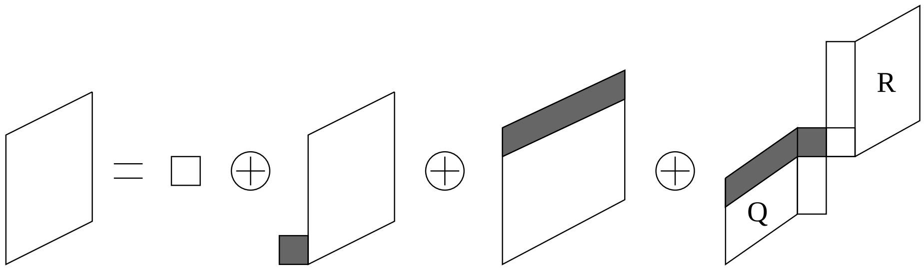

FIGURE 14. The first decomposition of parallelogram polygons.

only the corresponding equations (or systems of equations) defining the associated generating function. Roughly speaking, the first decomposition consists in splitting a polygon into two smaller polygons, and the second one consists in removing the first column of the polygon.

4.1. Divide and conquer. The following decomposition, basically, was used by Delest and Fédou to compute the width and area generating function of parallelogram poyominoes [**16**]. Let us consider a parallelogram polyomino P. Three cases can occur.

1. The first column is reduced to a single cell. By removing this cell, one gets either a new parallelogram polyomino or...nothing.
2. By removing a cell from the top of each column of P, one gets a new parallelogram polyomino, having the same width as P.
3. By removing a cell from the top of each column of P, one does not get a new parallelogram polyomino having the same width as P. This means that two consecutive columns of P are "glued" by a unique cell, and P can be split into two smaller polygons Q and R.

These cases are schematized in Figure 14.

Let $X(x, y, q)$ (sometimes denoted $X(x)$, or even X) be the width, height and area generating function of parallelogram polyominoes. The previous remarks lead to

$$X(x, y, q) = xyq + xqX(x, y, q) + yX(xq, y, q) + X(x, y, q)X(xq, y, q),$$

that is:

$$X(x, y, q) = \frac{xyq}{1-xq} + \frac{y+X}{1-xq}X(xq, y, q).$$

Unfortunately, there exists no general technique to solve q-algebraic equations. How could we *guess* the solution?

We suggest here a method which is likely to "give an idea" of the solution and can serve as a guide for solving the equation. This method consists in solving, first, the *associated ordinary differential equation*. Let us set y to 1, and denote

$$\hat{X}(x) = \frac{X(x(1-q)^2, 1, q)}{1-q}.$$

Then:

$$\frac{\hat{X}(x)-\hat{X}(xq)}{x(1-q)} = q + q(1-q)\hat{X}(x) + \frac{\hat{X}(x)\hat{X}(xq)}{x}.$$

This is a *q-differential equation.* It implies that the series $\hat{X}$ has a limit $\tilde{X}$ when $q \to 1$, which satisfies the *ordinary differential equation*:

$$\tilde{X}' = 1 + \frac{\tilde{X}^2}{x}.$$

We solve this new equation by searching for a solution of the form $-xf'/f$. We get:

$$\tilde{X} = \frac{\tilde{J}_1}{\tilde{J}_0},$$

with

$$\tilde{J}_1 = \sum_{n\geq 1} \frac{(-1)^{n-1}x^n}{(n-1)!n!} \qquad \text{and} \qquad \tilde{J}_0 = \sum_{n\geq 0} \frac{(-1)^n x^n}{n!^2}. \tag{4.1}$$

The problem reduces then in finding the "right" q-analogue of $\tilde{J}_1$ and $\tilde{J}_0$. This is not very complicated in this case, and the solution is finally given by the following proposition [**10**] [**11**] [**16**].

Proposition 4.1. — *The width, height and area generating function of parallelogram polyominoes is:*

$$X(x,y,q) = y\frac{J_1}{J_0}, \tag{4.2}$$

with

$$J_1 = \sum_{n\geq 1} \frac{(-1)^{n-1}x^n q^{\binom{n+1}{2}}}{(q)_{n-1}(yq)_n} \qquad \textit{and} \qquad J_0 = \sum_{n\geq 0} \frac{(-1)^n x^n q^{\binom{n+1}{2}}}{(q)_n(yq)_n}.$$

This decomposition has been extended to directed and convex polyominoes, and then to convex polyominoes [**7**]. Let $Y(x,y,q)$ (resp. $Z(x,y,q)$) be the generating function of directed and convex polygons (resp. convex polygons). Then, Y and Z are entirely characterized by the following system of q-equations.

$$\begin{pmatrix} Y \\ Y_1 \end{pmatrix} = \frac{xyq}{1-xq}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{y+X}{1-xq}\begin{pmatrix} 1 & xq \\ 1 & 1 \end{pmatrix}\begin{pmatrix} Y \\ Y_1 \end{pmatrix}(xq)$$

and

$$\begin{pmatrix} Z \\ Z_1 \\ Z_2 \end{pmatrix} = \frac{xyq}{1-xq}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{2}{y+X}\begin{pmatrix} V^2 \\ VW \\ W^2 \end{pmatrix} + \frac{y}{(1-xq)^2}\begin{pmatrix} 1 & 2xq & x^2q^2 \\ 1 & 1+xq & xq \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} Z \\ Z_1 \\ Z_2 \end{pmatrix}(xq),$$

where

$$V = Y - \frac{xyq}{1-xq} \qquad \text{and} \qquad W = Y_1 - \frac{xyq}{1-xq}.$$

Again, there is no general method to solve these equations. However, this system has been solved, by applying the method suggested above [**9**]. We completely skip the details of the computation, which are quite long and heavy. The most difficult part is of course the system defining $Z(x,y,q)$. By solving the associated differential system, we first get:

$$\lim_{q\to 1} \frac{Z(x(1-q)^2,1,q)}{1-q} = \frac{\tilde{J}_1 + \tilde{\tilde{J}}_1}{2\tilde{J}_0} + \frac{\tilde{\tilde{J}}_1}{2\tilde{J}_0}(1 + \tilde{J}_0\tilde{\tilde{K}}_1 - \tilde{K}_0\tilde{\tilde{J}}_1),$$

where $\tilde{J}_0$ et $\tilde{J}_1$ are given in (4.1),

$$\tilde{\tilde{J}}_1 = -\tilde{J}_1(-x),$$

$$\tilde{K}_0 = 2\sum_{n\geq 1}\left(\frac{(-x)^n}{n!^2}\sum_{k=1}^{n}\frac{1}{k}\right)$$

and

$$\tilde{\tilde{K}}_1 = 1 + \sum_{n\geq 1}\left(\frac{x^n}{(n-1)!n!}\left(2\sum_{k=1}^{n}\frac{1}{k} - \frac{1}{n}\right)\right).$$

The final result gives the generating function of convex polygons under the following form:

$$Z(x,y,q) = 2y^2\frac{M_1(\bar{J}_1\beta - \bar{K}_1\alpha)}{J_0} - y\bar{J}_1 b_1 + y\bar{K}_1 a_1, \tag{4.3}$$

where J_0, $\bar{J}_1$, $\bar{K}_1$, M_1, α, β, a_1 and b_1 are some q-series *expanded* in the variable x. This means that each of them is written as $\sum_{n\geq 0} x^n R_n(y,q)$, where $R_n(y,q)$ is an explicit rational function in y and q.

Expression (4.3) is, in some sense, almost as simple as (4.2): it is an algebraic expression involving a finite number of q-series which are expanded in the variable x. There might be a simpler expression, but it remains to be found... However, we will see in the next section that the other decomposition of convex polygons, which is really different from this one, leads to the same expression for the generating function $Z(x,y,q)$. This might mean that Expression (4.3) is The Right Expression... Another formula, much more complicated, is given in [**24**].

Remark. We shall see that this decomposition is not the most convenient method to compute the width, height and area generating function of convex polyominoes. However, it is probably the simplest explanation for the algebraicity of the width and height generating function, obtained by setting q to 1. Actually, the width and height generating function of any classical family of column-convex polyominoes is an algebraic series [**7**] [**14**] [**15**] [**17**] [**25**].

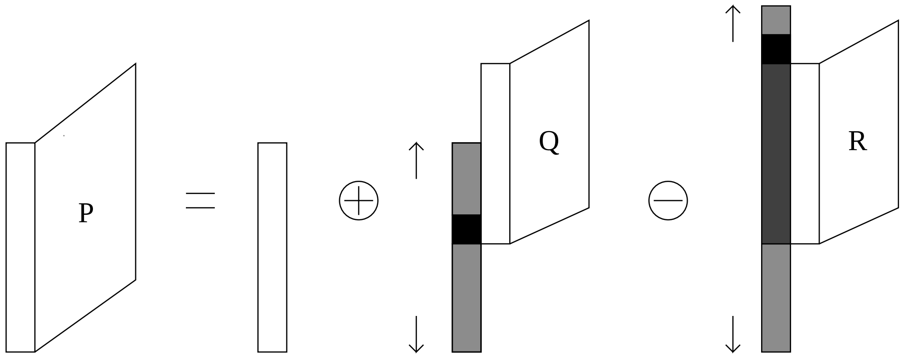

FIGURE 15. The second decomposition of parallelogram polygons.

4.2. Removing a column. A very natural idea is to build a column-convex polyomino by gluing some columns together. This is the starting point of what physicists sometimes call "Temperley's methodology". We show how this idea can be translated into q-equations that can be solved systematically ([**8**], see also [**28**]).

Let us consider a parallelogram polyomino P. If it is not reduced to a unique column, removing its first column provides a new parallelogram polyomino Q. Conversely, P can be obtained from Q as follows.

- First, add a new column C, of height one, to the polyomino Q.
- Then, append to C two "pieces of column": the first one at the bottom of C, and the other one at its top.

However, note that, if the second piece of column is too high, the polyomino thus obtained is *not* a parallelogram polyomino. Hence, this procedure gives too many polyominoes, and we have to remove some of them. These can be obtained as follows.

- Choose a parallelogram polyomino R, and duplicate its first column. Let C denote the first column of the new polygon.
- Append to C two "pieces of column": the first one at the bottom of C, and the other one—that cannot be empty—at its top.

This construction is represented by Figure 15.

Let $X(s,x,y,q)$ (often denoted $X(s)$) be the generating function of parallelogram polyominoes, counted according to their width, their height, their area and the height of their first column by the variables x, y, q and s respectively. The translation of the previous decomposition in terms of X is the following (see [**8**] for detailed explanations):

$$X(s) = \frac{xsyq}{1-syq} + \frac{xsq}{(1-sq)(1-syq)}(X(1) - X(sq)).$$

Iterating this equation infinitely many times leads to the identity

$$X(s,x,y,q) = y\sum_{n\geq 1}\frac{(-1)^{n-1}x^n s^n q^{\binom{n+1}{2}}}{(sq)_{n-1}(syq)_n} + X(1,x,y,q)\sum_{n\geq 1}\frac{(-1)^{n-1}x^n s^n q^{\binom{n+1}{2}}}{(sq)_n(syq)_n},$$

from which we deduce the width, height and area generating function of parallelogram polyominoes:

$$X(1,x,y,q) = y\frac{J_1}{J_0},$$

with

(4.4)

$$J_1 = \sum_{n\geq 1}\frac{(-1)^{n-1}x^n q^{\binom{n+1}{2}}}{(q)_{n-1}(yq)_n} \qquad \text{and} \qquad J_0 = \sum_{n\geq 0}\frac{(-1)^n x^n q^{\binom{n+1}{2}}}{(q)_n(yq)_n}.$$

Hence we have obtained a new and simpler proof of Proposition 4.1. The same technique works for directed and convex polyominoes. Let $Y(s,x,y,q)$ be their generating function according to the same parameters as above. We get:

$$Y(s) = xsyq + syqS(s) + \frac{xsq}{(1-sq)(1-syq)}(Y(1)-Y(sq)), \tag{4.5}$$

where

$$S(s) = sy\sum_{m\geq 1}\frac{x^m q^m}{(syq)_{m-1}(syq)_m} \tag{4.6}$$

is the generating function of stack polyominoes (Figure 13.2). Iterating this equation provides easily the following proposition [**8**] [**10**]:

Proposition 4.2. — *The generating function of directed and convex polyominoes is:*

$$Y(1,x,y,q) = y\frac{M_1}{J_0},$$

where J_0 is given by (4.4) and

$$M_1 = \sum_{n\geq 1}\frac{x^n q^n}{(yq)_{n-1}}\sum_{k=0}^{n-1}\frac{(-1)^k q^{\binom{k}{2}}}{(q)_k(yq^{k+1})_{n-k}}. \tag{4.7}$$

The variables x, y and q account respectively for the width, the height and the area of polygons.

This kind of decomposition can be applied to any classical class of column-convex polygons. However, when this class is not composed of *directed* polyominoes, a new term appears in the equation defining the associated generating function, denoted $A(s)$. The equation has now the following type:

$$A(s) = xe(s) + xf(s)A(1) + xg(s)A(sq) + xh(s)\frac{\partial A}{\partial s}(1). \tag{4.8}$$

In particular, the generating function $Z = Z(s,x,y,q)$ of convex polygons satisfies:

$$Z = xsyq(1-syq)+2syqY - s^2y^2q^2S + \frac{xsq}{1-sq}\frac{\partial Z}{\partial s}(1) - \frac{xs^2q^2}{(1-sq)^2}(Z(1)-Z(sq)),$$

where $Y = Y(s)$ and $S = S(s)$ are defined by (4.5) and (4.6) respectively. Solving this more general kind of equation is a bit more delicate, but it can be done systematically [**8**], and the general solution of (4.8) explains the form of Expression (4.3).

PROPOSITION 4.3. — *The width, height and area generating function for convex polygons is:*

$$Z(1,x,y,q) = \bar{J_1}\frac{\partial E}{\partial s}(1) - \bar{K_1}E(1),$$

where

$$E(s) = 2y^2\frac{M_1}{J_0}\alpha(s) - ya(s),$$

$$\bar{J_1} = \sum_{n\geq 1}\frac{x^nq^n}{(q)_{n-1}(q)_n},$$

$$\bar{K_1} = -1 + \sum_{n\geq 1}\left(\frac{x^nq^n}{(q)_{n-1}(q)_n}\left(\sum_{k=1}^{n-1}\frac{2q^k}{1-q^k} + \frac{q^n}{1-q^n}\right)\right),$$

$$\alpha(s) = \sum_{m\geq 1}\left(\frac{(-1)^mx^ms^mq^{\binom{m+1}{2}}}{(sq)_m}\sum_{n=1}^{m}\frac{(-1)^ns^nq^{\binom{n+1}{2}}}{(sq)_{n-1}(syq^n)_{m-n+1}}\right),$$

and

$$a(s) = \sum_{m\geq 1}\left(\frac{x^mq^m}{1-syq^m}\left(-\frac{s^{2m-1}q^{m(m-1)}}{(sq)_{m-1}^2} + y\sum_{n=1}^{m-1}\frac{s^{2n}q^{n^2}(syq^n-2)}{(sq)_{n-1}^2(syq^n)_{m-n}^2}\right.\right.$$

$$\left.\left. +2y\sum_{1\leq n\leq k<m}\frac{(-1)^{n+k}s^{n+k}q^{\binom{n+1}{2}}q^{\binom{k}{2}}}{(sq)_{n-1}(syq^n)_{m-n}(sq)_k(syq^{k+1})_{m-k-1}}\right)\right),$$

the series J_0 and M_1 being given by (4.4) *and* (4.7) *respectively.*

NOTE. This method, as we said, can be systematically applied to any class of column-convex polygons, thus unifying many enumeration results. In particular, it also provides the generating function of directed column-convex polygons and the generating function of directed column-convex polygons [**8**].

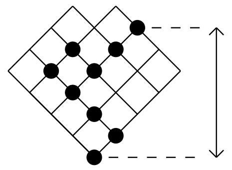

FIGURE 16. The directed height of an animal.

5. Open problems

The most general questions, that is the enumeration of self-avoiding walks, self-avoiding polygons and polyominoes are, as we said, completely open.
Let us mention a few more precise questions. They can be classified in two classes, depending on whether they deal with other parameters or other models.

- **Other parameters**
  1. What is the average *directed height* of directed animals having n cells (Figure 16)? This question has some importance in physics, and an asymptotic value of $n^{9/11}$ has been conjectured [**26**]. But, apparently, none of the decompositions which are known for the moment allows to take the directed height into account. However, this question has been solved for column-convex directed animals [**2**].
  2. Physicists are also interested in the number of *interactions* (pairs of nearest-neighbour vertices) in self-avoiding walks or polygons [**27**]. For some classes of polygons, the method described in Section 4 allows to write equations defining the generating function, but these equations are not yet solved.
- **Other models**
  1. In dimension two, other lattices can be considered. In particular, counting directed animals on a triangular lattice leads to an algebraic generating function (as for the square lattice), but the question of counting directed animals on the *honeycomb* lattice is still open [**19**].
  2. Finally, a wide family of new problems is obtained by trying to count animals or polygons in higher dimensions. This idea is motivated by the link between directed animals and gas-models and by Baxter's very nice solution of the hard-hexagon model [**3**] [**18**].

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