A method for the enumeration of various classes of column-convex polygons

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Abstract. We present a new method that allows to enumerate various classes of column-convex polygons, according to their perimeter and their area. The first step of this method leads to a functional equation which defines implicitly the generating function of the class under consideration. The second step consists in solving this equation. We apply systematically our method to all the usual classes of column-convex polygons: thus, we first refine some already known results for parallelogram polygons and for directed and convex polygons, and then obtain two new results, namely the generating function of column-convex polygons and of directed column-convex polygons.

1 Introduction

One of the most famous problems in combinatorics is (maybe) the enumeration of self-avoiding walks, or self-avoiding polygons, on a regular lattice. A closely related problem consists in enumerating polyominoes: a polyomino is a finite union of elementary cells of the lattice, whose interior is connected (Figure 1). The only lattice we will consider here is the square lattice. Note that any polygon defines a polyomino, but the converse is false since a polyomino can have holes. The interesting parameters are the perimeter (or the length for self-avoiding walks), and, for polyominoes, the area, that is the number of cells.

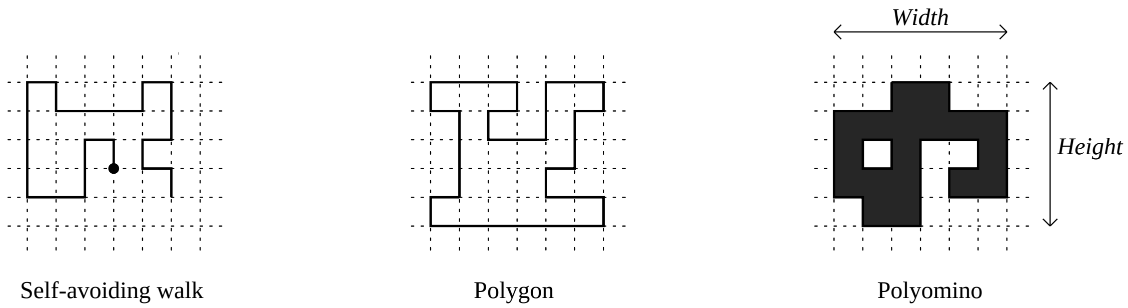

Figure 1: Self-avoiding walks, polygons and polyominoes.

These questions are not only interesting for combinatorialists, but are also of considerable importance in the study of lattice models in physics and chemistry. These models give simplified descriptions of physical phenomena, and are used to describe phase transitions. Self-avoiding walks, polygons and polyominoes arise, for example, in models of polymers, of cell-growth, of percolation... Moreover, the self-avoiding walk problem is equivalent to the resolution of the limit $n \to 0$ of the n-vector model, which generalizes the famous Ising model [8]. Finally, a quite important example is the correspondence between the enumeration of *directed* polyominoes on a regular lattice in dimension D and the resolution of a gas-model in dimension $D-1$ [12].

Despite serious efforts over the last 40 years, these problems are completely open. Though, a few asymptotic results are known. For example, let $p(n)$ denote the number of polyominoes having n cells. Klarner and Rivest proved that $(p(n))^{1/n}$ tends to a limit K, which satisfies the following inequality: $3,87 < K < 4,65$ [16]. The situation could hardly be worse, since the first digit of K is not even known...

The difficulty of these questions has led to the study of various restricted classes of polyominoes or polygons. Most of them can be defined by combining two notions: a geometric notion of *convexity*, and a notion of *directed growth*, which comes from statistical physics. A polyomino is said to be *vertically convex* (or *column-convex*) when its intersection with any vertical line is convex (Figure 2). We can define similarly a notion of *horizontal convexity*. A polyomino is *convex* if it is both vertically and horizontally convex. The problem of counting convex polyominoes was raised by D. Knuth, around 1970. Note that any column-convex polyomino is actually a polygon, so we will use indifferently the name "polyomino" or "polygon" in this case. The notion of directed structures is more recent. A polyomino P is said to be *directed* when every cell of P can be reached from a distinguished cell, called the *root*, by a path which is contained in P and only uses North and East steps (Figure 2). The problem of enumerating directed polyominoes was studied by Hakim and Nadal [14] on the one hand, Gouyou-Beauchamps and Viennot [13] on the other hand. The area generating function of directed polyominoes is an algebraic series.

We will be concerned here with families of column-convex polyominoes. Combining the two notions described above, we can already define four types of polyominoes, depending on whether they are only column-convex, or also row-convex, directed or not. Namely, here are the four main sub-classes of polygons we will deal with in this paper:

- column-convex polyominoes,
- convex polyominoes,
- directed and column-convex polyominoes,
- directed and convex polyominoes.

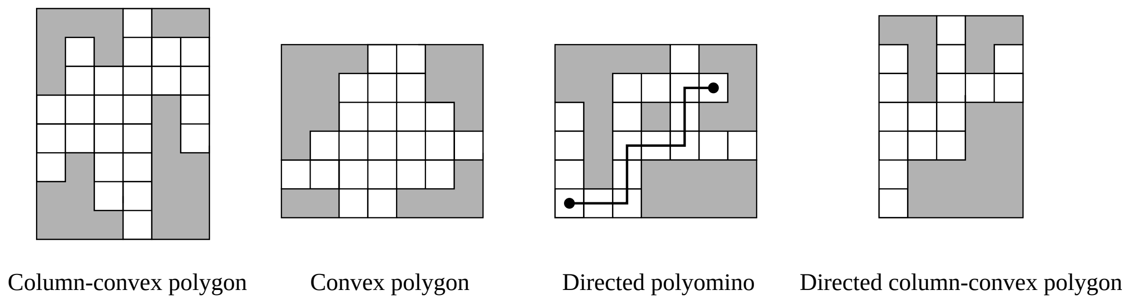

Figure 2: Convexity and directed growth.

We can also cite three very classical families of directed and convex polyominoes: the Ferrers diagrams, the stack polyominoes, and finally the parallelogram (or staircase) polyominoes. As Figure 3 shows, each of these subsets can be characterized, in the set of convex polyominoes, by the fact that two or three vertices of the minimal bounding rectangle of the polyomino must also belong to the polyomino itself. Moreover, a convex polyomino is directed if and only if it contains the south-west vertex of its minimal bounding rectangle.

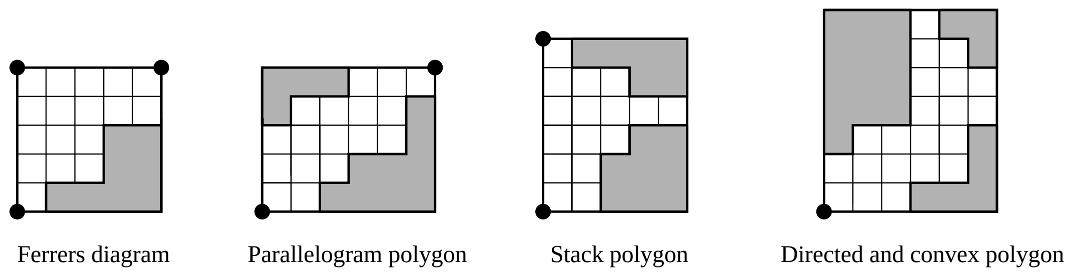

Figure 3: Directed and convex polygons.

We are interested in the enumeration of these objects according to, simultaneously, the perimeter and the area. Roughly, we can say that two kinds of generating functions occur, depending on the convexity properties of the class of polygons which is being enumerated. More precisely:

- the perimeter generating function of any usual family of convex polyominoes is an algebraic series, whereas the area generating function involves q-series; moreover, taking into account the perimeter (or the width and the height) when one already knows the area generating function is usually a rather easy task;

- the situation is different for families of column-convex polyominoes: both the perimeter generating function *and* the area generating function are algebraic, but the difficulty consists in taking into account *simultaneously* the two parameters.

The aim of this paper is to show how, by a *single* method, we can get, for any "natural" class of column-convex polyominoes, a functional equation that implicitly defines its generating function. This generating function takes into account not only the perimeter and the area, but also a few other parameters, for example the width of the polyomino. Moreover, all the equations we thus obtain are *of the same type*, and hence, they can be solved in the same way (at least theoretically). Thus, both the way of establishing an equation and the way of solving it are systematic. Practically, this "universal" method provides many enumeration results, both already known and new:

- some of them are only refinements of already known results (parallelogram polyominoes, directed and convex polyominoes),

- two others are new; we obtain the perimeter and area generating function of column-convex polygons, and of directed column-convex polygons.

Note that we did not mention above any result concerning convex polyominoes... The reason for this silence is that, for the convex case, the calculation suggested by our *theoretical* solution can not be *practically* conducted, at least for the moment. However, the problem of counting convex polygons was recently solved by a different method [4].

The following section is, in some sense, the central one. It does not contain any enumeration result, but gives all the tools that we need to enumerate families of column-convex polyominoes. Hence, the other sections can be viewed as corollaries of Section 2. Since the equation obtained for classes of *directed* column-convex polyominoes is of a particularly simple kind, we have studied separately these classes in Section 3 (parallelogram, directed and convex, directed and column-convex polyominoes). Section 4 is devoted to the two other classes: convex polygons and column-convex polygons.

2 The general technique

Our method is based on the following simple idea: a column-convex polyomino can be obtained by successively gluing columns, in a way which depends on the class of polygons one studies. This idea is sometimes known as "Temperley methodology" in statistical physics, where it has been already intensively used [19]. The first problem consists is translating this idea into an equation. The second problem is to solve this equation. A paragraph of this section is devoted to each of these two questions. Then, as an illustration, we apply our method to the set of stack polygons.

Notations and definitions. Let $\mathbb{R}[[s,t,x,y,q]]$ be the algebra of formal power series in the variables s,t,x,y,q, with real coefficients. If $X(s,t,x,y,q)$ is such a series, we will often denote it, for short, $X(s)$. Its derivative, with respect to s, will be denoted $\partial X/\partial s(s)$ or $X'(s)$. We denote $\mathcal{A}$ the sub-algebra of $\mathbb{R}[[s,t,x,y,q]]$ formed with the series S such that $S(1,t,x,y,q)$ and $S'(1,t,x,y,q)$ are convergent.

Let P be a polyomino. Its *left-height* (resp. *right-height*) is the height of its leftmost (resp. rightmost) column. We denote:

- its left-height (resp. right-height) $l(P)$ (resp. $r(P)$),
- the number of horizontal (resp. vertical) steps in its perimeter $2h(P)$ (resp. $2v(P)$) (note that $h(P)$ is also the width of the polygon P),
- its area $a(P)$.

Let $\mathcal{P}$ be a set of column-convex polyominoes. Its *generating function* is the following formal series, which lies in $\mathcal{A}$:

$$\sum_{P\in\mathcal{P}} s^{l(P)}t^{r(P)}x^{h(P)}y^{v(P)}q^{a(P)}.$$

2.1 Establishing equations

We describe here three ways of adding a new column to a polyomino, which are respectively schematized by the first polygon in Figures 4, 5 and 6. Let P be a column-convex polyomino. Let $A(P)$ (resp. $B(P)$, $C(P)$) be the set of column-convex polyominoes Q satisfying the three following properties:

- removing the first column of Q gives P;
- the top-cell of the first column of Q is higher (resp. lower, lower) than the top-cell of the first column of P;
- the bottom-cell of the first column of Q is lower (resp. lower, higher) than the bottom-cell of the first column of P.

In these definitions, a cell is said to be higher (resp. lower) than another one if its level is greater or equal (resp. less or equal). If $\mathcal{P}$ is a set of column-convex polyominoes, we define

$$A(\mathcal{P}) = \bigcup_{P\in\mathcal{P}} A(P).$$

We define $B(\mathcal{P})$ and $C(\mathcal{P})$ in a similar way.

Proposition 2.1 *Let $\mathcal{P}$ be a set of column-convex polygons, and $X(s,t,x,y,q)$ be its generating function.*
1. The generating function of $A(\mathcal{P})$ is:

$$\frac{x}{(1-syq)^2}X(sq).$$

2. The generating function of $B(\mathcal{P})$ is:

$$\frac{xsq}{(1-sq)(1-syq)}\left(X(1)-X(sq)\right).$$

3. The generating function of $C(\mathcal{P})$ is:

$$\frac{xsq}{1-sq}\frac{\partial X}{\partial s}(1)-\frac{xs^2q^2}{(1-sq)^2}(X(1)-X(sq)).$$

Proof. First, note that we just have to prove this proposition in the case where the set $\mathcal{P}$ is reduced to a single polygon P.
1. The polygons lying in $A(P)$ can be obtained in two steps (Figure 4).

- We first duplicate the leftmost column of P; in terms of generating functions, this corresponds to multiplying $X(sq)$ by x.
- Then, we glue two pieces of columns, the first one at the top of the new column, the other one at its bottom. This corresponds to a multiplication by $1/(1-syq)^2$.

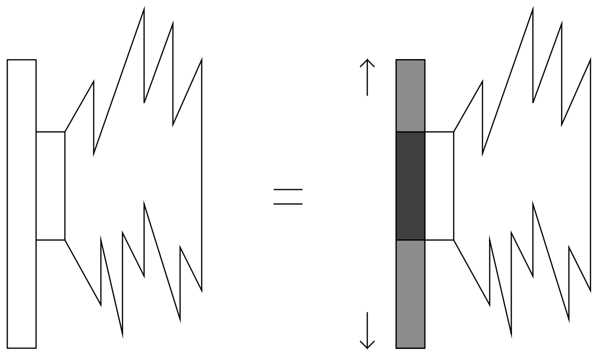

Figure 4: Construction A.

2. The polygons lying in $B(P)$ can be obtained in two steps (Figure 5).

- We first add a new cell c at the left of the lowest cell of the first column of P. Note that the left-height of the new polyomino is now 1. The associated generating function is hence $xsqX(1)$.
- Then, we glue two pieces of columns, the first one at the top of c, the other one at the bottom of c. This operation corresponds to multiplying by $1/((1-sq)(1-syq))$.

But, by doing this, we built some polyominoes which do NOT lie in $B(P)$: such polyominoes occur when the column added at the top of c is "too" high. These "undesired" polyominoes can be independently constructed in a way similar to construction A, but by adding a non-empty piece of column at the top of the first column. Their generating function is hence

$$\frac{xsq}{(1-sq)(1-syq)}X(sq).$$

3. The polygons lying in $C(P)$ can be obtained in two steps (Figure 6).

- We first add a new cell c anywhere on the left of the first column of P. Note that the left-height of the new polyomino is now 1. The associated generating function is $xsq\partial X/\partial s(1)$.
- Then, we glue one piece of column at the top of c. This corresponds to a multiplication by $1/(1-sq)$.

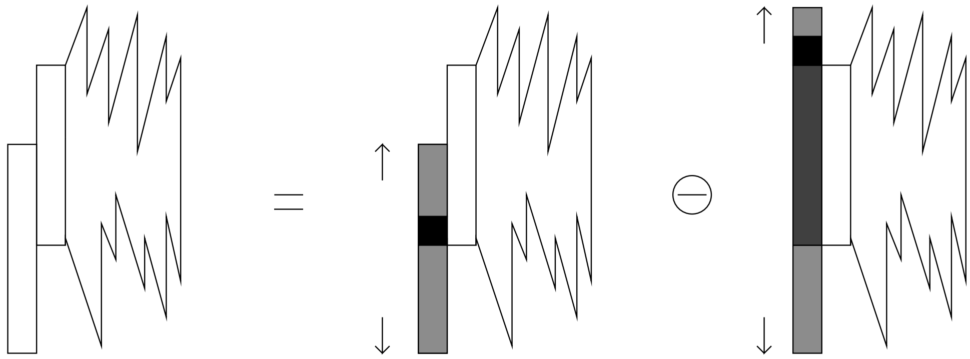

Figure 5: Construction B.

But, by doing this, we built some polyominoes which do NOT lie in $C(P)$: such polyominoes occur when the column added at the bottom of c is "too" high. These "undesired" polyominoes can be independently constructed in a way similar to construction B, but by adding a non-empty piece of column at the top of the first column. Their generating function is hence

$$\frac{xs^2q^2}{(1-sq)^2}\left(X(1)-X(sq)\right).$$

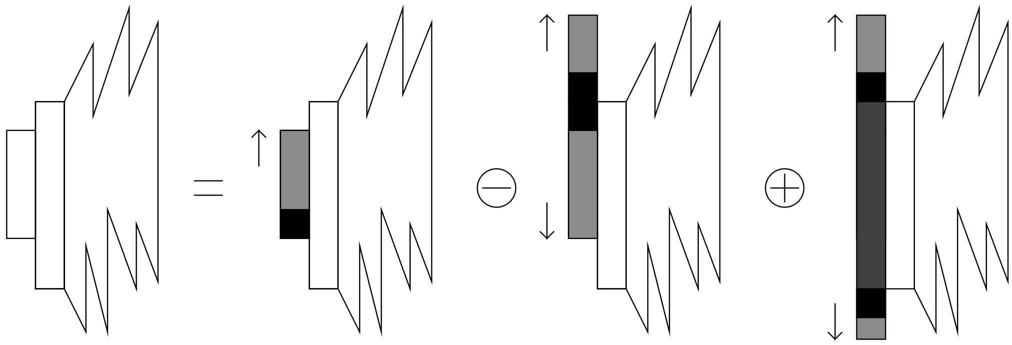

Figure 6: Construction C.

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Remark. The above proposition gives the theme, here are a few useful variations. Again, a polygon Q is obtained by adding a column C to a polygon P in such a way that:

- **Construction $\mathbf{A_1}$:** the top-cell (resp. the bottom-cell) of C is higher (resp. strictly lower) than the top-cell (resp. the bottom-cell) of the first column of P; the associated generating function is $xsyqX(sq)/(1-syq)^2$.
- **Construction $\mathbf{A_2}$:** the top-cell (resp. the bottom-cell) of C is strictly higher (resp. lower) than the top-cell (resp. the bottom-cell) of the first column of P; the associated generating function is again $xsyqX(sq)/(1-syq)^2$.
- **Construction $\mathbf{A_3}$:** the top-cell (resp. the bottom-cell) of C is strictly higher (resp. strictly lower) than the top-cell (resp. the bottom-cell) of the first column of P; the associated generating function is $xs^2y^2q^2X(sq)/(1-syq)^2$.
- **Construction $\mathbf{B'}$:** the top-cell (resp. the bottom-cell) of C is higher (resp. higher) than the top-cell (resp. the bottom-cell) of the first column of P; the associated generating function is of course the same as for construction B, that is $xsq(X(1)-X(sq))/((1-sq)(1-syq))$.

- **Construction $\mathbf{B_1}$:** the top-cell (resp. the bottom-cell) of C is lower (resp. strictly lower) than the top-cell (resp. the bottom-cell) of the first column of P; the associated generating function is $xs^2yq^2(X(1)-X(sq))/((1-sq)(1-syq))$.
- **Construction $\mathbf{B_1'}$:** the top-cell (resp. the bottom-cell) of C is stricly higher (resp. higher) than the top-cell (resp. the bottom-cell) of the first column of P; the associated generating function is of course the same as for construction B_1, that is $xs^2yq^2(X(1)-X(sq))/((1-sq)(1-syq))$.

2.2 Solving equations

It becomes clear from the previous paragraph that we will have to deal with affine equations expressing $X(s)$ in terms of $X(sq)$, $X(1)$ and $\partial X/\partial s(1)$. The following lemma gives the solution of such equations.

Lemma 2.2 *Let $X(s,t,x,y,q)$ be a formal power series lying in $\mathcal{A}$. Suppose that:*

$$X(s) = xe(s) + xf(s)X(1) + xg(s)X(sq) + xh(s)\frac{\partial X}{\partial s}(1), \tag{1}$$

where $e(s)$, $f(s)$, $g(s)$ and $h(s)$ are some given power series in $\mathcal{A}$. Then

$$X(1) = \frac{E(1)+H(1)E'(1)-E(1)H'(1)}{1-F(1)-H'(1)-H(1)F'(1)+F(1)H'(1)}, \tag{2}$$

where

$$E(s) = \sum_{n\geq 0} x^{n+1}g(s)g(sq)\dots g(sq^{n-1})e(sq^n),$$

$$F(s) = \sum_{n\geq 0} x^{n+1}g(s)g(sq)\dots g(sq^{n-1})f(sq^n),$$

and

$$H(s) = \sum_{n\geq 0} x^{n+1}g(s)g(sq)\dots g(sq^{n-1})h(sq^n).$$

Moreover:

$$X(s) = E(s) + \frac{F(s)\Big[E(1)+H(1)E'(1)-E(1)H'(1)\Big] + H(s)\Big[E'(1)+E(1)F'(1)-F(1)E'(1)\Big]}{1-F(1)-H'(1)-H(1)F'(1)+F(1)H'(1)}.$$

Proof. Let us substitute s by sq in Equation (1). We get an expression of $X(sq)$ in terms of $X(1)$, $\partial X/\partial s(1)$ and $X(sq^2)$. Now, we substitute $X(sq)$ by this expression in Equation (1): we obtain an expression of $X(s)$ in terms of $X(1)$, $\partial X/\partial s(1)$ and $X(sq^2)$. Iterating this procedure shows that, for $k \geq 1$:

$$\begin{aligned} X(s) &= \sum_{n=0}^{k-1} x^{n+1}g(s)g(sq)\dots g(sq^{n-1})e(sq^n) + X(1)\sum_{n=0}^{k-1} x^{n+1}g(s)g(sq)\dots g(sq^{n-1})f(sq^n) \\ &+ x^k g(s)g(sq)\dots g(sq^{k-1})X(sq^k) + \frac{\partial X}{\partial s}(1)\sum_{n=0}^{k-1} x^{n+1}g(s)g(sq)\dots g(sq^{n-1})h(sq^n). \end{aligned}$$

But, when $k \to \infty$, the sequence x^k, as a formal power series, goes to zero. Therefore, taking the limit when $k \to \infty$ of this identity gives, with the notations defined in the lemma:

$$X(s) = E(s) + X(1)F(s) + X'(1)H(s). \tag{3}$$

Let us differentiate this identity with respect to s:

$$X'(s) = E'(s) + X(1)F'(s) + X'(1)H'(s).$$

Substituting s by 1 leads to a system of two linear equations in $X(1)$ and $X'(1)$, with a non-zero determinant. Hence, we first compute $X(1)$ and $X'(1)$, and then we deduce $X(s)$ from these results and from Equation (3).

■

The resolution of Equation (1) is greatly simplified when no partial derivative arises, that is when the series $h(s)$ equals 0. According to Proposition 2.1, this partial derivative only occurs when we use the construction C. Looking at Figure 6 suggests that we will not use this construction as soon as the class of polygons under consideration is formed with *directed* polygons. So, in all these cases, we will apply the following simplified version of Lemma 2.2.

Lemma 2.3 *Let $X(s,t,x,y,q)$ be a formal power series in $\mathcal{A}$. Suppose that:*

$$X(s) = xe(s) + xf(s)X(1) + xg(s)X(sq),$$

where $e(s)$, $f(s)$ and $g(s)$ are some given power series in $\mathcal{A}$. Then

$$X(1) = \frac{E(1)}{1 - F(1)},$$

where

$$E(s) = \sum_{n \geq 0} x^{n+1} g(s)g(sq)\ldots g(sq^{n-1})e(sq^n),$$

and

$$F(s) = \sum_{n \geq 0} x^{n+1} g(s)g(sq)\ldots g(sq^{n-1})f(sq^n).$$

Moreover:

$$X(s) = \frac{E(s) + E(1)F(s) - E(s)F(1)}{1 - F(1)}.$$

Remark. In this study, we take into account the most usual three parameters: the area, the vertical perimeter and the horizontal perimeter. The left-height and the right-height of the polygon also occur: the former is essential to use our method, and the latter has been choosen for symmetry reasons. However, this method allows to consider many other parameters that might be of interest, especially in statistical physics: for example, the site-perimeter and the directed perimeter.

2.3 Example: stack polygons

Lemma 2.4 *Let $S(s,t,x,y,q)$ be the generating function of stack polygons. Then:*

$$S(s) = \frac{xstyq}{1 - styq} + \frac{x}{(1 - syq)^2} S(sq). \tag{4}$$

Proof. Applying construction A to the set of stack polygons provides all the stack polygons of width > 1. Moreover, the generating function of stack polygons of width 1 is $xstyq/(1 - styq)$.

■

Notation. Let a be an indeterminate. Then

$$(a)_n = \prod_{i=0}^{n-1} (1 - aq^i).$$

Lemma 2.5 *The generating function $S(s,t,x,y,q)$ of stack polyominoes is given by:*

$$S(s,t,x,y,q) = sty \sum_{n \geq 1} \frac{x^n q^n}{(syq)_{n-1}^2 (1 - styq^n)}.$$

Proof. Apply Lemma 2.3.

■

3 Families of directed and column-convex polygons

We study here three classes of directed column-convex polyominoes: parallelogram polyominoes, directed and convex polyominoes, directed and column-convex polyominoes. For each of these classes, we first give the q-equation which defines its generating function, and then solve this equation by a blind application of Lemma 2.3.

3.1 Parallelogram polygons

Lemma 3.1 *Let $P(s,t,x,y,q)$ be the generating function of parallelogram polyominoes. Then:*

$$P(s) = \frac{xstyq}{1-styq} + \frac{xsq}{(1-sq)(1-syq)}(P(1) - P(sq)).$$

Proof. Applying construction B to the set of parallelogram polyominoes provides all the parallelogram polyominoes of width > 1.

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Theorem 3.2 *The generating function $P(s,t,x,y,q)$ of parallelogram polyominoes satisfies:*

$$P(1,t,x,y,q) = ty\frac{J_1(1)}{J_0(1)},$$

with

$$J_0(s) = \sum_{n\geq 0} \frac{(-1)^n x^n s^n q^{\binom{n+1}{2}}}{(sq)_n (syq)_n}$$

and

$$J_1(s) = \sum_{n\geq 1} \frac{(-1)^{n-1} x^n s^n q^{\binom{n+1}{2}}}{(sq)_{n-1}(syq)_{n-1}(1-styq^n)}.$$

Moreover:

$$P(s,t,x,y,q) = ty\frac{J_1(s)J_0(1) - J_1(1)J_0(s) + J_1(1)}{J_0(1)}.$$

Proof. Apply Lemma 2.3.

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Remarks.

1. The parameters "left-height" and "right-height" play obviously the same role in the generating function of parallelogram polyominoes. Hence, $P(s,t,x,y,q)$ is symmetric in the variables s and t. Since its denominator $J_0(1)$ depends neither on s nor on t, this means that its numerator $J_1(s)J_0(1) - J_1(1)J_0(s) + J_1(1)$ is symmetric. One could try to write it in a way where this symmetry becomes apparent.
2. Special important cases of this theorem were proved by Klarner and Rivest [17] (area generating function), Delest and Fédou [11] (width and area generating function), Bousquet-Mélou and Viennot [5] (width, height and area generating function). Brak and Guttmann [6] give the perimeter and area generating function under a form which seems a bit more complicated but is finally equivalent.
3. Though it is not clear on the above expression, the width and height generating function is algebraic of degree 2 (see for example [3]).

3.2 Directed and convex polygons

Lemma 3.3 *Let $DC(s,t,x,y,q)$ be the generating function of directed and convex polyominoes. Then:*

$$DC(s) = xstyq\frac{1-syq}{1-styq} + syqS(s) + \frac{xsq}{(1-sq)(1-syq)}(DC(1) - DC(sq)),$$

where $S(s)$ is the generating function of stack polyominoes.

Proof. Let P be a directed and convex polyomino. Three cases can occur.

1. **First case**: P is reduced to a single column. The generating function of such polyominoes is $xstyq/(1-styq)$.

2. **Second case**: P has at least two columns, and the top-cell of its first column is strictly higher than the top-cell of its second column. (P is actually a stack polyomino). Removing the highest cell of its first column proves that the generating function of such polyominoes is

$$syq\left(S(s) - \frac{xstyq}{1-styq}\right).$$

3. **Third case**: P has at least two columns, and the top-cell of its first column is lower than the top-cell of its second column. Such polygons can be obtained by applying construction B to the set of directed and convex polyominoes.

Combining these three cases, we get the lemma.

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Theorem 3.4 *The generating function $DC(s,t,x,y,q)$ of directed and convex polyominoes satisfies:*

$$DC(1,t,x,y,q) = ty\frac{M_1(1)}{J_0(1)},$$

with

$$J_0(s) = \sum_{n\geq 0} \frac{(-1)^n x^n s^n q^{\binom{n+1}{2}}}{(sq)_n (syq)_n}$$

and

$$M_1(s) = \sum_{n\geq 1} \frac{x^n q^n}{(syq)_{n-1}(1-styq^n)}\left(\frac{s}{(syq)_{n-1}} + \sum_{m=1}^{n-1} \frac{(-1)^m s^m q^{\binom{m}{2}}}{(sq)_m (syq^{m+1})_{n-m-1}}\right). \qquad (5)$$

Moreover:

$$DC(s,t,x,y,q) = ty\frac{M_1(s)J_0(1) - M_1(1)J_0(s) + M_1(1)}{J_0(1)}.$$

Proof. Using the notations of Lemma 2.3, we obtain

$$E(s) = ty\sum_{n\geq 1} \frac{x^n q^n E_n}{(syq)_{n-1}(1-styq^n)},$$

with

$$E_n = \frac{(-1)^{n-1} s^n q^{\binom{n}{2}}(1-syq^n)}{(sq)_{n-1}} + s^2 y \sum_{m=0}^{n-1} \frac{(-1)^m s^m q^{\binom{m+2}{2}}}{(sq)_m (syq^{m+1})_{n-m-1}}.$$

To get the expression of M_1 given in (5), just note that the sequence E_n can be defined by $E_1 = s$ and, for $n \geq 1$, $E_{n+1} = (-1)^n s^n q^{\binom{n}{2}}/(sq)_n + E_n/(1-syq^n)$.

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Remarks.

1. The width, height and area generating function of directed and convex polygons was already given by Bousquet-Mélou and Viennot [5].

2. The width and height generating function is algebraic of degree 2 [18].

3.3 Directed and column-convex polygons

Lemma 3.5 *Let $DV(s,t,x,y,q)$ be the generating function of directed and column-convex polyominoes. Then:*

$$DV(s) = \frac{xstyq}{1-styq} + \frac{xsq}{(1-sq)(1-syq)}DV(1) + \frac{xsq(y-1)}{(1-sq)(1-syq)^2}DV(sq).$$

Proof. Applying the constructions A_2 and B to the set of directed and column-convex polyominoes provides all the directed and column-convex polyominoes of width > 1.

∎

Theorem 3.6 *The generating function $DV(s,t,x,y,q)$ of directed and column-convex polyominoes satisfies:*

$$DV(1,t,x,y,q) = ty\frac{L_1(1)}{L_0(1)}, \tag{6}$$

with

$$L_0(s) = 1 - \sum_{n\geq 1} \frac{x^n s^n (y-1)^{n-1} q^{\binom{n+1}{2}}}{(sq)_n (syq)_{n-1} (syq)_n}$$

and

$$L_1(s) = \sum_{n\geq 1} \frac{x^n s^n (y-1)^{n-1} q^{\binom{n+1}{2}}}{(sq)_{n-1} (syq)_{n-1}^2 (1 - styq^n)}.$$

Moreover:

$$DV(s,t,x,y,q) = ty\frac{L_1(s)L_0(1) - L_1(1)L_0(s) + L_1(1)}{L_0(1)}.$$

Proof. Apply Lemma 2.3.

■

Remarks.

1. From this theorem - which is new -, we deduce easily that the generating function of directed column-convex polygons according to their left-height, right-height, width and area is a *rational function*. In particular, the area generating function is:

$$DV(1,1,1,1,q) = \frac{q(1-q)}{1-3q+q^2}.$$

   This result was proved by Delest and Dulucq [10]. See also [1] and [2].

2. Delest and Dulucq also proved that the width and perimeter generating function is algebraic of degree 3, which is not clear from the expression given above.

4 Families of non-directed column-convex polygons

4.1 Convex polygons

Lemma 4.1 *Let $C(s,t,x,y,q)$ be the generating function of convex polyominoes. Then:*

$$C(s) = xstyq\frac{(1-syq)^2}{1-styq} + 2syqDC(s) - s^2y^2q^2S(s) + \frac{xsq}{1-sq}\frac{\partial C}{\partial s}(1) - \frac{xs^2q^2}{(1-sq)^2}(C(1) - C(sq)),$$

where $S(s)$ (resp. $DC(s)$) is the generating function of stack polyominoes (resp. directed and convex polyominoes).

Proof. Let us consider a convex polygon P.

1. **First case:** P is reduced to a single column. The generating function of such polygons is $xstyq/(1-styq)$.

2. **Second case:** P has at least two columns, the top-cell of its first column is strictly higher than the top-cell of its second column, and the bottom-cell of its first column is strictly lower than the bottom-cell of its second column. Hence, P is a stack polygon, and the generating function of this class is

$$s^2y^2q^2\left(S(s) - \frac{xstyq}{1-styq}\right).$$

3. **Third case:** P has at least two columns, the top-cell of its first column is lower than the top-cell of its second column, and the bottom-cell of its first column is strictly lower than the bottom-cell of its second column. Hence, P is a directed and convex polygon, and the generating function of this class is

$$syq\left(DC(s) - \frac{xstyq}{1-styq}\right) - s^2y^2q^2\left(S(s) - \frac{xstyq}{1-styq}\right).$$

4. **Fourth case:** P has at least two columns, the top-cell of its first column is strictly higher than the top-cell of its second column, and the bottom-cell of its first column is higher than the bottom-cell of its second column. This class of polygons and the previous one are symmetric, and hence have the same generating function.

5. **Fifth case:** P has at least two columns, the top-cell of its first column is lower than the top-cell of its second column, and the bottom-cell of its first column is higher than the bottom-cell of its second column. Such polygons can be obtained by applying construction C to the set of convex polygons.

We prove the lemma by combining these five cases.

■

Remark. There will not be any "Theorem" following this "Lemma", since, in this case, we were not able to compute explicitly the series E, F and H occurring in Lemma 2.2. However, the width, height and area generating function of convex polyominoes has been derived by a different method [4], and the final form of the result has some similarities with the formal expression (2).

4.2 Column-convex polygons

Lemma 4.2 *Let $V(s,t,x,y,q)$ be the generating function of column-convex polyominoes. Then:*

$$V(s) = \frac{xstyq}{1-styq} + \frac{xsq}{1-sq}\frac{\partial V}{\partial s}(1) + \frac{xs^2q^2(2y-syq-1)}{(1-sq)^2(1-syq)}V(1) + \frac{xs^2q^2(1-y)^2}{(1-sq)^2(1-syq)^2}V(sq).$$

Proof. Applying the constructions A_3, B_1, B_1' and C to the set of column-convex polygons provides all the column-convex polygons of width > 1. (This sequence of constructions is choosen in such a way that every column-convex polygon is obtained once).

■

Proposition 4.3 *The generating function $V(s,t,x,y,q)$ of column-convex polyominoes satisfies:*

$$V(1,t,x,y,q) = \frac{E(1)+H(1)E'(1)-E(1)H'(1)}{1-F(1)-H'(1)-H(1)F'(1)+F(1)H'(1)}, \tag{7}$$

with

$$E(s) = ty\sum_{n\geq 1}\frac{x^n s^{2n-1}(1-y)^{2n-2}q^{n^2}}{(sq)_{n-1}^2(syq)_{n-1}^2(1-styq^n)},$$

$$F(s) = \sum_{n\geq 1}\frac{x^n s^{2n}(1-y)^{2n-2}q^{n(n+1)}(2y-syq^n-1)}{(sq)_n^2(syq)_{n-1}(syq)_n},$$

and

$$H(s) = \sum_{n\geq 1}\frac{x^n s^{2n-1}(1-y)^{2n-2}q^{n^2}}{(sq)_{n-1}(sq)_n(syq)_{n-1}^2}.$$

Moreover:

$$V(s) = E(s) + \frac{F(s)\Big[E(1)+H(1)E'(1)-E(1)H'(1)\Big]+H(s)\Big[E'(1)+E(1)F'(1)-F(1)E'(1)\Big]}{1-F(1)-H'(1)-H(1)F'(1)+F(1)H'(1)}.$$

Proof. Apply Lemma 2.2.

■

The derivatives $E'(1)$, $F'(1)$ and $H'(1)$ can be easily explicited from the values of $E(s)$, $F(s)$ and $H(s)$. But the generating function of column-convex polygons, given by Expression (7), remains obviously more complicated than, for example, the generating function of directed column-convex polygons (6). The reason for this complexity is that we do not deal any longer with directed polyominoes.

However, in the case where the variable t is equal to 1, it turns out that both the numerator and the denominator of Expression (7) have a more pleasant expression, actually, a simple expansion in the variable x. In what follows, we compute these two expansions, and finally obtain a nice expression for the generating function $V(1,1,x,y,q)$. The impatient reader might skip the technicalities of Lemma 4.4 to contemplate the final formula in Theorem 4.5.

Notations. From now on, the variable t is set to 1, thus all the power series we consider lie in $\mathbb{R}[[s,x,y,q]]$. Moreover, the q-equations we are going to handle now do not involve series of the type $S(sq^i)$, for $i \geq 0$, but of the type $S(s,xq^i,y,q)$. Hence, we choose the following conventions.

Let $S(s,x,y,q)$ be a formal power series lying in $\mathbb{R}[[s,x,y,q]]$. It will be convenient to denote it $S(x)$, or even S, if there is no ambiguity. We define $\bar{S}$ to be the series $S(1,x,y,q)$ (when defined). The derivatives are still taken with respect to s. For example, $\bar{S}'$ stands for $\frac{\partial S}{\partial s}(1,x,y,q)$.

Let S and T be two power series lying in $\mathbb{R}[[s,x,y,q]]$. We denote $[S,T]$ the following element of $\mathbb{R}[[x,y,q]]$ (when defined):

$$[S,T] = \bar{S}'\bar{T} - \bar{S}\bar{T}'.$$

Note that $[T,S] = -[S,T]$ and $[sS,sT] = [S,T]$.

Lemma 4.4 1. *Let I be the following power series:*

$$I = \sum_{n\geq 1} \frac{x^n s^{2n-1}(1-y)^{2n-2}q^{n^2}}{(sq)_{n-1}(sq)_n(syq)_{n-1}(syq)_n}.$$

Then I satisfies the q-equation:

$$q^2 I(x) - sq\alpha I(xq) + s^2\beta I(xq^2) - s^3 y\alpha I(xq^3) + s^4y^2 I(xq^4) = xsq^3(1-s)(1-sy), \tag{8}$$

with $\alpha = (1+y)(1+q)$ and $\beta = y+q+2yq+y^2q+yq^2-xq^3(1-y)^2$.

2. *Let X, Y and Z be the elements of $\mathbb{R}[[x,y,q]]$ defined by:*

$$\begin{aligned} X &= [I, sI(xq)] + \frac{\bar{I}-\bar{I}(xq)}{1-y}, \\ Y &= [I, s^2I(xq^2)] + \frac{\bar{I}-\bar{I}(xq^2)}{1-y}, \\ Z &= [I, s^3I(xq^3)] + \frac{\bar{I}-\bar{I}(xq^3)}{1-y}. \end{aligned} \tag{9}$$

Then, X, Y and Z are totally determined by the following system:

$$\left\{ \begin{array}{llllll} q^2X(x) & & -\ \beta X(xq) & +\ y\alpha Y(xq) & -\ y^2Z(xq) & = 0 \\ q^2Y(x) & -\ q\alpha X(xq) & & +\ y\alpha X(xq^2) & -\ y^2Y(xq^2) & = 0 \\ q^2Z(x) & -\ q\alpha Y(xq) & +\ \beta X(xq^2) & & -\ y^2X(xq^3) & = 0 \end{array} \right. \tag{10}$$

and the initial condition

$$\frac{\partial X}{\partial x}(0) = \frac{q}{(1-y)(1-yq)}.$$

The expansions of X, Y and Z can be deduced from this system:

$$X = \frac{xq}{(1-y)(1-yq)} + \sum_{n\geq 2} \frac{(-1)^{n+1}x^n(1-y)^{2n-4}q^{\binom{n+1}{2}}(y^2q)_{2n-2}}{(q)_{n-1}(yq)_{n-2}(yq)_{n-1}^2(yq)_n(y^2q)_{n-1}}, \tag{11}$$

and

$$Y = \frac{xq(1+q)}{(1-y)(1-yq)} + (1+y)(1+q)\sum_{n\geq 2} \frac{(-1)^{n+1}x^n(1-y)^{2n-4}q^{\binom{n+2}{2}-2}(y^2q)_{2n-3}}{(q)_{n-1}(yq)_{n-2}^2(yq)_{n-1}(yq)_n(y^2q)_{n-1}}. \tag{12}$$

(The expansion of Z is a bit more complicated and will not be used).

Proof. 1. For $n \geq 1$, let I_n be the coefficient of x^n in the expansion of I.
Then, for $n \geq 2$:

$$(1-sq^{n-1})(1-sq^n)(1-syq^{n-1})(1-syq^n)I_n = s^2(1-y)^2q^{2n-1}I_{n-1}.$$

Hence:
$$\sum_{n\geq 2}(1-sq^{n-1})(1-sq^n)(1-syq^{n-1})(1-syq^n)I_n x^n = xs^2q(1-y)^2\sum_{n\geq 1}q^{2n}I_n x^n,$$
and this identity, combined with the initial condition $I_1 = sq/\big((1-sq)(1-syq)\big)$, is nothing but the announced equation (8).
2. Let $1\leq i\leq 3$. One gets the i^{th} equation of System (10) by applying the following algorithm:

- divide each side of Equation (8) by $s^i I(xq^i)$,
- differentiate with respect to s the identity thus obtained,
- set s to 1.

Now, let X_n (resp. Y_n, Z_n) be the coefficient of x^n in X (resp. Y, Z). After a few reductions, one finds that System (10) implies that, for any n:
$$\begin{cases}(1-q^{n-1})(1-yq^{n-2})(1-yq^{n-1})(1-yq^n)(1-y^2q^{n-1})X_n = -q^n(1-y)^2(1+yq^{n-1})(1-y^2q^{2n-3})X_{n-1},\\ (1+yq^{n-1})Y_n = q^{n-1}(1+y)(1+q)X_n,\\ (1-q^{n-2}(y+q+2yq+y^2q+yq^2))X_n + yq^{n-2}(1+y)(1+q)Y_n - y^2q^{n-2}Z_n = -q^n(1-y)^2X_{n-1}.\end{cases}$$
These three equations do not suffice to determine uniquely X, Y and Z. The initial condition $X_1 = q/\big((1-y)(1-yq)\big)$ is needed, and allows to compute by induction X_n, Y_n and Z_n.

∎

Theorem 4.5 *The generating function $V(s,t,x,y,q)$ of column-convex polyominoes satisfies:*
$$V(1,1,x,y,q) = y\frac{(1-y)X}{1+W+yX},$$
with
$$X = \frac{xq}{(1-y)(1-yq)} + \sum_{n\geq 2}\frac{(-1)^{n+1}x^n(1-y)^{2n-4}q^{\binom{n+1}{2}}(y^2q)_{2n-2}}{(q)_{n-1}(yq)_{n-2}(yq)_{n-1}^2(yq)_n(y^2q)_{n-1}},$$
and
$$W = \sum_{n\geq 1}\frac{(-1)^n x^n(1-y)^{2n-3}q^{\binom{n+1}{2}}(y^2q)_{2n-1}}{(q)_n(yq)_{n-1}^3(yq)_n(y^2q)_{n-1}}. \tag{13}$$

Proof. Let N and D be the numerator and the denominator of the expression of $V(1,1,x,y,q)$ given by (7):
$$\begin{aligned} N &= \bar{E}+[E,H],\\ D &= 1-\bar{F}-\bar{H}'+[H,F].\end{aligned}$$
In order to compute N and D, let us express the series E, F and H in terms of I. First, note that:
$$E = yI(x) - syI(xq), \tag{14}$$
$$H = I(x) - syI(xq). \tag{15}$$
Moreover, one can check that:
$$\begin{aligned} xq^2(1-y)F(s) &= -qI(x) + s(y+q+yq-xq^2(1-y)(1-2y))I(xq)\\ &\quad -s^2y(1+y+q)I(xq^2) + s^3y^2I(xq^3) + xsq^2(1-sy).\end{aligned} \tag{16}$$
Using (14) and (15), we can now compute N:
$$\begin{aligned} N &= \bar{E}+[E,H]\\ &= y\Big(\bar{I}(x)-\bar{I}(xq) + [I(x)-sI(xq), I(x)-syI(xq)]\Big)\\ &= y\Big(\bar{I}(x)-\bar{I}(xq) + (1-y)[I(x), sI(xq)]\Big)\\ &= y(1-y)X,\end{aligned}$$
the series X being defined by (9).
Using (15) and (16), a similar calculation proves that:

$$xq^2(1-y)D = (y+q-xq^2(1-y)(1-2y))X(x)+y^2(1+q+y)X(xq)-y(1+q+y)Y(x)-y^3Y(xq)+y^2Z(x),$$

where X, Y and Z are defined by (9).
We now use the first equation of System (10) to get rid of the series Z. We get:

$$xq^3(1-y)D(xq) = q^2X(x)-yq(2+y+q-xq^2(1-y))X(xq)+y^2(1+y+q)X(xq^2)+y^2qY(xq)-y^3Y(xq^2).$$

Substituting $Y(xq^2)$ by its value derived from the third equation of System (10) gives:

$$xq^2(1-y)\big(D(xq)-yX(xq)\big) = qX(x)+y(yq-1)X(xq)-y^3X(xq^2)-yqY(x)+y^2Y(xq).$$

Finally, the expansions of X and Y given by (11) and (12) lead to the announced expression for D:

$$D = 1+W+yX,$$

the series W being defined by (13).

■

Remarks.

1. Again, the parameters "left-height" and "right-height" play the same role in the generating function of column-convex polyominoes. Hence, $V(s,t,x,y,q)$ is symmetric in the variables s and t. One could try to write V in a way where this symmetry becomes apparent.

2. From this theorem - which is new -, we deduce easily that the generating function of column-convex polygons according to their left-height, right-height, width and area is a *rational function.* In particular, the area generating function is:

$$V(1,1,1,1,q) = \frac{q(1-q)^3}{1-5q+7q^2-4q^3}.$$

   This result can be found at several places [9] [15].

3. Delest [9], and then Brak, Enting and Guttmann [7], have proved that the perimeter generating function is an algebraic series of degree 4.

4. Brak and Guttmann [6] found another expression for the perimeter and area generating function of column-convex polyominoes. But it is much more complicated than the formula given in Theorem 4.5.

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