New enumerative results on two-dimensional directed animals

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Abstract

We list several open problems concerning the enumeration of directed animals on two-dimensional lattices. We show that most of these problems are special cases of two central problems: calculating the position generating function and the perimeter and area generating function for square lattice animals.

We propose a possible direction for solving these two problems: we extend Dhar's correspondence between hard particle gas models and enumeration of animals according to the area, and show that each of the main two generating functions is, essentially, the density of a one-dimensional gas model given by the stationary distribution of a probabilistic transition.

We are able to compute the density of certain stationary distributions. We thus obtain new bivariate generating functions for directed animals on the square and triangular lattices. We derive from these results the generating functions for animals on the decorated square and triangular lattices, as well as the average number of loops in directed animals.

1 Introduction

An *animal* A on a graph G is a finite *connected* set of vertices: any two vertices of A are connected through a path of G having all its vertices in A (Figure 1). The vertices of A are called *cells*, and the number of cells is the *area* of the animal. A *neighbour* of A is a vertex that does not belong to A, but is connected by an edge to a cell of A. The *(site) perimeter* of A is its number of neighbours. On a periodic infinite graph, animals are usually defined up to a translation.

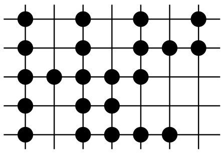

Figure 1: An animal on the square lattice (area 22, perimeter 28).

These simple combinatorial objects are also of interest in statistical physics. For instance, they are the main ingredient of cell growth models. Moreover, enumerating animals according to their perimeter and area solves the (site) percolation model on G [6].

According to Stauffer [21], *one calls these configurations "animals" since they have a certain similarity with multicellular living beings which might enter your nightmares if you counted them too long.* Indeed, enumerating animals on a periodic infinite graph seems to be a nightmarish problem. To our knowledge,

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the most precise rigorous result in this field is the following [18]: there exists a constant K such that, if a_n denotes the number of square lattice animals having n cells, then $a_n^{1/n}$ tends to K when n tends to infinity. Finding lower and upper bounds for K is difficult, and the first digit of K is not yet known:

$$3,87 < K < 4,65.$$

Some non rigorous — but much more precise — results assert that $K \approx 4.0626$ [15].

One can define more tractable animals by considering an *oriented* graph G having a distinguished vertex O, called the *origin*. If an edge goes from v to w, then w is said to be a *son* of v, and v a *father* of w. A *directed animal* A on G is a finite set of vertices containing O, such that any vertex of A can be reached from O through an oriented path of G having all its vertices in A. The origin O is the *source* of A. The vertices of A are called *cells*, and the number of cells is the *area* of the animal. The *neighbours* of A are the vertices that do not belong to A but have a father in A. The *(site) perimeter* of A is its number of neighbours. This definition generalizes the notion of (undirected) animals given above since unoriented graphs can be seen as a special kind of oriented graphs, by replacing each unoriented edge by a pair of oriented edges. From now on, we only deal with directed animals, and the word "directed" will often be omitted.

We study in this paper directed animals on the square, triangular and honeycomb lattices. Examples are given in Figure 2. The edges are oriented upwards in all lattices. The leftmost animal has area 12 and perimeter 10.

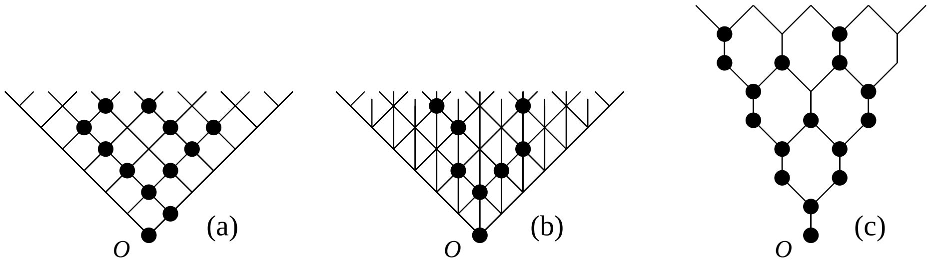


Figure 2: Directed animals on two-dimensional lattices.

Directed animals are related to directed (site) percolation models. Moreover, as shown by Dhar [12], enumerating directed animals according to the area on certain graphs is equivalent to solving a *hard particle model* on another graph. For instance, directed animals on the square lattice correspond to a simple hard particle model in one dimension, but animals on the honeycomb lattice do not correspond to any hard particle model. A combinatorial proof of Dhar's result has also been given using the notion of *heaps of pieces* [4, 22]. As illustrated in Figure 3, a hard particle model is a gas model in which two adjacent vertices cannot be simultaneously occupied by cells.

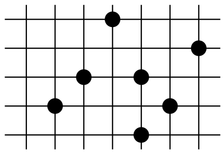
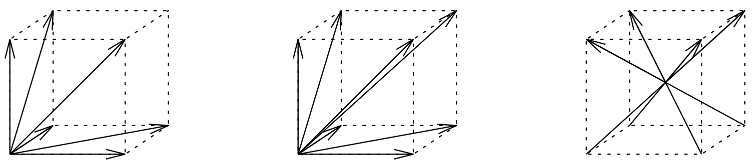


Figure 3: A hard particle configuration on the square lattice.

Actually, the correspondence between directed animals and hard particle models is not only a motivation for studying animals. It is also a very efficient way of enumerating them according to the area, for the few exact known results in this domain can be obtained by solving the corresponding gas model. The main two results are the area generating functions for directed animals on the square lattice and on the three-dimensional *next-nearest neighbour* cubic lattice drawn in Figure 4(a) [11, 12]. For square lattice animals, there exists, besides the gas model argument, a very simple and nice combinatorial proof based on the notion of heaps of pieces [3, 4, 22]. However, this combinatorial method has not (yet) been extended to animals in three dimensions, for which the very difficult solution of the corresponding gas model, called the *hard hexagon model*, remains the unique enumeration technique [2].

Proposition 1.1 [3, 4, 11, 12, 14] — *The area generating function for square lattice directed animals is*

$$S_0(t) = \frac{1}{2}\left(\left(1 - \frac{4t}{1+t}\right)^{-1/2} - 1\right).$$



(a) (b) (c)

Figure 4: Three-dimensional oriented lattices.

Looking upon animals as heaps of pieces shows that the area generating function for directed animals on the triangular lattice is obtained by replacing t by $t/(1-t)$ in the area generating function for square lattice animals. Similarly, the same substitution, performed on the generating function for animals on the lattice of Figure 4(a), gives the area generating function for animals on the lattice of Figure 4(b). Actually, what we obtain is a refinement of the area generating function. Let us describe it in the triangular lattice case (Figure 2 (b)). Let v be a cell of a directed animal A. We say that v is *only supported at the center* if the vertex placed just below in the same column is the only father of v lying in A. The number of cells of A only supported at the center is denoted $c(A)$.

Proposition 1.2 — *The generating function for directed animals on the triangular lattice, counted according to their area and number of cells only supported at the center, is*

$$\begin{aligned}\tilde{T}_0(t,w) &= \sum_A t^{|A|} w^{c(A)} \\ &= S_0\left(\frac{t}{1-tw}\right) \\ &= \frac{1}{2}\left(\left(1 - \frac{4t}{1+t-tw}\right)^{-1/2} - 1\right).\end{aligned}$$

In particular, the area generating function for directed animals on the triangular lattice is

$$T_0(t) = \frac{1}{2}\left((1-4t)^{-1/2} - 1\right). \tag{1}$$

Finally, Conway, Brak and Guttmann have conjectured simple algebraic expressions for the generating functions for directed animals on the *decorated square lattices* [9]. These conjectures have been proved by Ali [1]. More details on these lattices are given in the following sections, together with a new proof of Ali's results. We also introduce some decorated triangular lattices and give simple algebraic expressions for the generating functions of directed animals on these lattices.

The animals above are sometimes called *site-animals* to distinguish them from *bond-animals*, which are connected sets of edges. More precisely, a (directed) bond-animal A on an oriented graph G is a finite set of edges such that each edge of A belongs to an oriented path of G starting from O and having all its edges in A. The *area* of A is its number of edges. The *neighbours* of A are the edges that do not belong to A, but

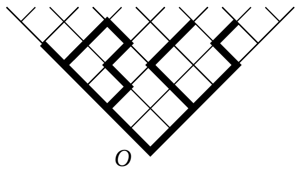


Figure 5: A directed bond-animal on the square lattice (area 21, perimeter 21).

whose starting point belongs to an edge of A. The *(bond) perimeter* of A is its number of neighbours. The enumeration of bond-animals according to their perimeter and area is related to bond-percolation models. When there is no risk of confusion, we will continue to use simply the word "animal" to denote site-animals.

As far as we know, we have mentioned *all* exact known results concerning the enumeration of animals. By elimination, one can obtain an infinite list of open problems. Here is, however, a tentative classification.

- The area is the only parameter for which exact enumerations are available. One could try to take into account other parameters, such as the perimeter, given its crucial role in percolation models. It is known that the perimeter generating function for square lattice animals is not algebraic [7, 8].
- Nothing is known about bond-animals.
- The honeycomb lattice raises serious difficulties, for animals on this lattice do not behave like animals on the square or triangular lattice. In particular, their area generating function does not seem to be algebraic [16], and the asymptotic number of animals having n cells, which is respectively $3^n/\sqrt{n}$ and $4^n/\sqrt{n}$ for the square and triangular lattices, is $(2.0251...)^n/\sqrt{n}$ for the honeycomb lattice [13].
- Other lattices could be studied in two, three (and more...) dimensions. In particular, the enumeration of directed animals on the lattice of Figure 4(c) corresponds to the famous unsolved *hard square model.*

We focus in this paper on two-dimensional directed animals. We list in the following section several open problems concerning their enumeration. We show that all these problems (including the enumeration of bond-animals on the square lattice, the enumeration of site-animals on the honeycomb lattice, etc.) are special cases of two central problems: calculating the *position generating function* and the *perimeter and area generating function* for square lattice animals.

How can we compute these generating functions? A natural idea is to extend one of the two methods for enumerating square lattice animals according to the area, i.e., the link with hard particle models on the one hand, and the idea of heaps on the other hand. It turns out that the idea of using heaps is not, at least at first sight, easily generalized. Its main drawback is that it turns the arrangement of rows of an animal upside down whereas both the position of cells and the perimeter are closely linked to this arrangement. Therefore, we have concentrated on Dhar's idea, and have extended it to take into account additional parameters.

Our central result is that the position generating function and the area and perimeter generating function for square lattice animals — and consequently all generating functions mentioned in Section 2 — are, essentially, the density of a certain one-dimensional gas model. The cell distribution of this gas is the *stationary distribution* of a simple *probabilistic transition.*

These transitions are described in Section 4. They are characterized by four parameters p_1, p_2, p_3 and p_4. When

$$p_1p_4(1-p_2)(1-p_3) = p_2p_3(1-p_1)(1-p_4), \tag{2}$$

the stationary distribution and its density have simple expressions. Alas, the position generating function is related to the transition $(p_1, p_2, p_3, p_2p_3/p_1)$ while the area and perimeter generating function is related to the transition (p_1, p_2, p_2, p_2), and neither of these transitions satisfies (2)... However, when $p_3 = 0$, the first transition satisfies (2) and is combinatorially significant. We thus obtain a new bivariate generating function for directed animals on the square lattice. We derive from this result the generating functions for animals on decorated square lattices, as well as the average number of *loops* in animals of given area as conjectured by Conway [7, 8].

Similarly, we obtain for animals on the triangular lattice a new bivariate generating function, from which we derive the generating functions for animals on decorated triangular lattices as well as the average number of loops in animals of given area as conjectured by Conway. These results do not follow from the corresponding square lattice results. Surprisingly, animals on decorated triangular lattices are equinumerous with triangular lattice animals.

To finish, here is the history of this paper. I had conjectured the values of the two bivariate generating functions mentioned above (Propositions 6.1 and 6.4). Since I was not able to prove them "combinatorially", I tried to extend Dhar's method, and discovered that the main open problems on two-dimensional lattices were equivalent to the solution of a gas model. Although I could only solve this model in some special cases — thus proving the two conjectures — I believe that the *general* correspondence between directed animals and gas models is worth being presented, and could maybe lead in the future to the solution of certain other open questions.

2 A survey of open problems

2.1 The position of cells in a square lattice animal

Let us consider a directed animal A on the square lattice. Let $v \neq O$ be a cell of A. Three cases occur, illustrated by Figure 6. In the first case, we say that the cell v is *only supported on the right*, in the second case, that it is *only supported on the left*, and in the third case, that v is a *loop* (in Figure 6, the vertices belonging to the animal are denoted by black circles, and the others by crosses). We denote $r(A)$ (resp.

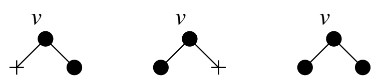


Figure 6: The three cases for the square lattice.

$\ell(A)$) the number of cells of A only supported on the right (resp. left). The number of loops of A is $b(A) = |A| - r(A) - \ell(A) - 1$.

Definition 2.1 — *The* position generating function *for directed animals on the square lattice is*

$$S_1(t,u,v) = \sum_A t^{|A|} u^{r(A)} v^{\ell(A)}. \tag{3}$$

The loop generating function *for directed animals on the square lattice is*

$$\begin{aligned} S_\ell(t,w) &= \sum_A t^{|A|} w^{b(A)} \\ &= w^{-1} S_1(tw, w^{-1}, w^{-1}). \end{aligned} \tag{4}$$

The sums are over all the square lattice directed animals.

Of course, we have $S_1(t,1,1) = S_0(t)$ and $S_1(t,u,v) = S_1(t,v,u)$. The series $S_1(t,u,v)$ has two interesting specializations, namely $S_1(t,u,u)$ and $S_1(t,u,1)$:

- the perimeter and area generating function for bond-animals on the square lattice can be computed from $S_1(t,u,u)$ (equivalently, from $S_\ell(t,w)$),
- the area generating function for animals on the decorated square lattices, studied by Conway, Brak and Guttmann [9], can be computed from $S_1(t,u,1)$,
- the average number of loops in animals of given area can also be derived from $S_1(t,u,1)$.

We compute $S_1(t,u,1)$ in Section 6.

2.1.1 Directed bond-animals on the square lattice

Let B be a directed bond-animal on the square lattice. Let $V(B)$ be the set of vertices belonging to an edge of B. By convention, $V(B) = \{O\}$ if B is the empty animal. Then $V(B)$ is a site-animal. Since two edges start from each cell of $V(B)$, the bond-perimeter of B is

$$p(B) = 2|V(B)| - |B|. \tag{5}$$

Conversely, let A be a site-animal. We wish to obtain all bond-animals B such that $V(B) = A$. Let B be such an animal. For any vertex v belonging to $A \setminus \{O\}$, B must contain an edge ending at v and having its starting point in A. If v is not a loop of A, there is a unique way of choosing such an edge. Otherwise, one can either choose one of the two edges ending at v or both of them.

Hence, the area generating function for bond-animals B such that $V(B) = A$ is

$$t^{r(A)+\ell(A)} \left(2t + t^2\right)^{b(A)}.$$

More precisely, according to (5), their area and perimeter generating function is

$$x^{2|A|} \left(\frac{t}{x}\right)^{r(A)+\ell(A)} \left(2\frac{t}{x} + \frac{t^2}{x^2}\right)^{b(A)}.$$

This gives the following result.

Proposition 2.2 — *Let $S_b(t,x)$ be the area and perimeter generating function for bond-animals on the square lattice:*

$$S_b(t,x) = \sum_A t^{|A|} x^{p(A)}.$$

We have

$$S_b(t,x) = \frac{x}{t} S_\ell \left(tx, 2 + \frac{t}{x}\right)$$

where $S_\ell(t,w)$ is the loop generating function for site-animals, defined by (4).

2.1.2 Directed animals on decorated square lattices

In their attempt to understand why the area generating functions for animals on the square and honeycomb lattices do not have similar behaviours, Conway, Brak and Guttmann have introduced new lattices, called *strange lattices* [9] or *decorated square lattices* [1]. They conjectured algebraic expressions for the area generating functions of directed animals on these lattices. We show here that these generating functions are specializations of $S_1(t,u,1)$.

The $n-$*decorated square lattice* is obtained from the usual square lattice by adding n vertices on each South-East/North-West edge. The edges of the new lattice are still oriented upwards (Figure 7).

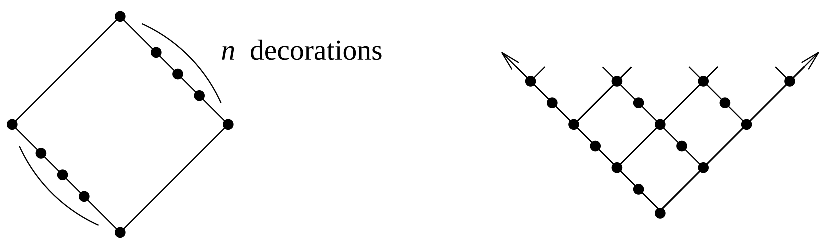


Figure 7: Square lattices with n decorations and 1 decoration.

Let B be a directed animal on the $n-$decorated (square) lattice. The set $S(B)$ formed with the cells of B lying on the original square lattice is a square lattice animal. Conversely, let A be a square lattice animal. We wish to obtain all animals B on the $n-$decorated lattice such that $S(B) = A$. Such an animal B contains the cells of A. Moreover, if a cell v of A is only supported on the right, then B contains the n vertices lying on the SE/NW edge ending at v. On the remaining SE/NW edges starting from a cell of A (there are $|A| - r(A)$ such edges), B may have $0, 1, 2, \ldots$, or n cells (Figure 8).

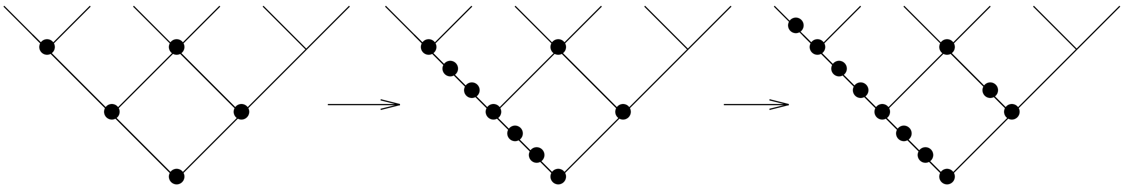


Figure 8: From an animal on the square lattice to an animal on the 2-decorated lattice.

Let us give to B the weight

$$x^{|S(B)|} y^{|B|-|S(B)|}.$$

The generating function for animals B on the $n-$decorated lattice such that $S(B) = A$ is

$$x^{|A|} \left(y^n\right)^{r(A)} \left(1 + y + y^2 + \cdots + y^n\right)^{|A|-r(A)}.$$

This gives the following result.

Proposition 2.3 — *Let $S_{d,n}(x, y)$ be the area generating function for animals on the $n-$decorated square lattice:*

$$S_{d,n}(x, y) = \sum_A x^{|S(A)|} y^{|A|-|S(A)|}$$

where $|S(A)|$ is the number of cells of A on the underlying square lattice. We have

$$S_{d,n}(x, y) = S_1\left(x\frac{1-y^{n+1}}{1-y}, y^n \frac{1-y}{1-y^{n+1}}, 1\right)$$

where $S_1(t, u, v)$ is defined by (3).

2.1.3 The mean number of loops

For $n \geq 1$, let

$$b_n = \sum_{|A|=n} b(A)$$

be the *total* number of loops in animals of area n. The mean number of loops in animals of area n is b_n divided by the number of animals of area n. Let $S_m(t)$ be the generating function for the b_n's:

$$S_m(t) = \sum_{n\geq 1} b_n t^n = \sum_A b(A) t^{|A|}.$$

Of course, we have

$$S_m(t) = \frac{\partial S_\ell}{\partial w}(t, 1)$$

where $S_\ell(t, w)$ is the loop generating function, defined by (4). However, $S_m(t)$ can also be derived from $S_1(t, u, 1)$.

Proposition 2.4 — *The generating function $S_m(t)$ is given by*

$$S_m(t) = t\frac{dS_0}{dt}(t) - S_0(t) - 2\frac{\partial S_1}{\partial u}(t, 1, 1)$$

where the series $S_0(t)$ and $S_1(t, u, v)$ are defined in Proposition 1.1 *and Definition* 2.1 *respectively.*

Proof. Use $b(A) = |A| - r(A) - \ell(A) - 1$ and $S_1(t, u, 1) = S_1(t, 1, u)$.

∎

To finish this section, we show that the number of cells only supported on the right is indirectly related to the site perimeter. Let A be a square lattice animal. Let us call *right* (resp. *left*) *neighbour* of A any neighbour of A lying to the north-east (resp. north-west) of a cell of A (Figure 9). A neighbour of A can be simultaneously a right and a left neighbour.

Figure 9: Right and left neighbours.

Lemma 2.5 — *Let A be an animal on the square lattice. The number of right (resp. left) neighbours of A is $1+r(A)$ (resp. $1+\ell(A)$) where $r(A)$ (resp. $\ell(A)$) denotes the number of cells of A only supported on the right (resp. left).*

Proof. Consider the set of vertices that lie to the north-east of a cell of A. It has cardinality $|A|$, and it is the disjoint union of the set of cells of A supported on the left and the set of right neighbours of A. One concludes the proof by noticing that the number of cells of A supported on the left is $|A| - r(A) - 1$.

■

2.2 The site perimeter of directed animals

Let $S_2(t, x)$ be the area and perimeter generating function for animals on the square lattice:

$$S_2(t, x) = \sum_A t^{|A|} x^{p(A)}.$$

Similarly, let $H_2(t, x)$ be the area and perimeter generating function for animals on the honeycomb lattice. No formula for these series is known. Moreover, no formula is known for $H_2(t, 1)$ whereas $S_2(t, 1) = S_0(t)$ is given in Proposition 1.1. Both series are related to *directed site-percolation models*, on the square and honeycomb lattices respectively [5, 6]. The percolation probability on the square lattice is

$$P_S(p) = 1 - \frac{1}{p} S_2(p, 1-p) \tag{6}$$

where p denotes the probability for a site to be occupied. A similar identity holds for the percolation probability on the honeycomb lattice:

$$P_H(p) = 1 - \frac{1}{p} H_2(p, 1-p). \tag{7}$$

It turns out that computing $S_2(t, x)$ is equivalent to computing $H_2(t, x)$ [13].

Proposition 2.6 — *The area and perimeter generating functions for animals on the square and honeycomb lattices are related as follows:*

$$H_2(t, x) = tx + S_2\left(t^2, x(1+t)\right).$$

Proof. There is a one-to-one correspondence Φ between animals of area ≥ 2 on the honeycomb lattice and pairs (A, A') such that A is a square lattice animal and A' is a set of neighbours of A. This correspondence has the following properties, which imply the proposition: if $\Phi(B) = (A, A')$, then

$$|B| = 2|A| + |A'| \quad \text{and} \quad p(B) = p(A).$$

To describe Φ, we first draw on the honeycomb lattice $\mathcal{H}$ a square lattice $\mathcal{S}$ whose vertices are the vertices of $\mathcal{H}$ having two sons (Figure 10). Let B be an animal of area ≥ 2 on the honeycomb lattice. Let A be the set of vertices of B lying on $\mathcal{S}$. Then A is an animal on $\mathcal{S}$. Let A' be the set of vertices of $\mathcal{S}$ that do not lie in B but whose father (in $\mathcal{H}$) belongs to B. Let $\Phi(B) = (A, A')$. One checks easily that Φ is one-to-one and satisfies the announced properties.

■

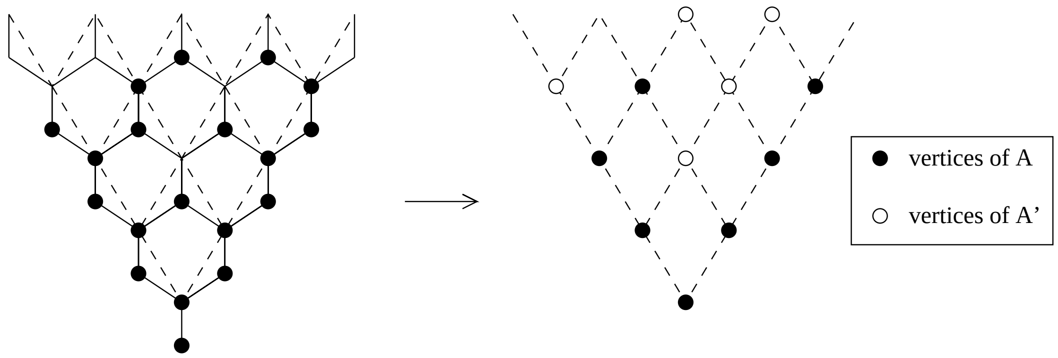


Figure 10: From the honeycomb lattice to the square lattice.

Remark. Proposition 2.6 implies that the directed site-percolation probabilities on the square and honeycomb lattices (see Equations (6) and (7)) satisfy $P_H(p) = pP_S(p^2)$. In particular, the *critical probabilities* satisfy $p_c^S = (p_c^H)^2$.

The correspondence Φ provides another identity between two generating functions. Let $\mathcal{H}_1$ be the set of vertices of $\mathcal{H}$ having a unique son. Let $\mathcal{H}_2 = \mathcal{H} \setminus \mathcal{H}_1$. Let A be an animal on the honeycomb lattice. We denote $|A|_1 = |A \cap \mathcal{H}_1|$ and $|A|_2 = |A \cap \mathcal{H}_2|$.

Proposition 2.7 — *Let $\tilde{H}_2(u, v)$ be the following generating function for animals on the honeycomb lattice:*

$$\tilde{H}_2(u, v) = \sum_A u^{|A|_1} v^{|A|_2}.$$

We have

$$\tilde{H}_2(u, v) = u + S_2(uv, 1 + u).$$

2.3 The position of cells in a triangular lattice animal

The results presented here are very similar to those of Section 2.1 and the proofs are omitted.

Let A be an animal on the triangular lattice. Let $v \neq O$ be a cell of A. Seven cases occur now, defined in Figure 11. We denote $r(A)$ (resp. $\ell(A)$, $c(A)$) the number of cells of A only supported on the right (resp.

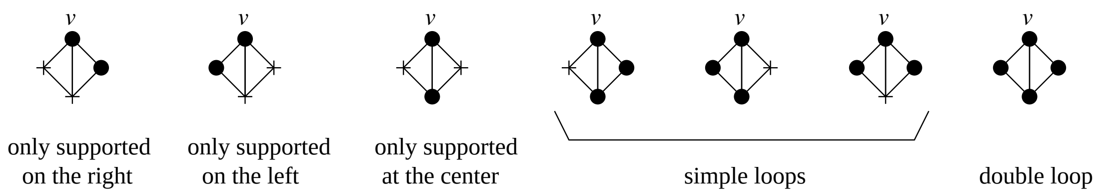


Figure 11: The seven cases for the triangular lattice.

on the left, at the center). The number of simple loops is denoted $sb(A)$. The number of double loops is $db(A) = |A| - r(A) - \ell(A) - c(A) - sb(A) - 1$.

Definition 2.8 — *The* position generating function *for directed animals on the triangular lattice is*

$$T_1(t, u, v, w, x) = \sum_A t^{|A|} u^{r(A)} v^{\ell(A)} w^{c(A)} x^{sb(A)}. \tag{8}$$

The loop generating function *for directed animals on the triangular lattice is*

$$T_\ell(t,x,y) = \sum_A t^{|A|} x^{sb(A)} y^{db(A)} \qquad (9)$$
$$= y^{-1} T_1(ty, y^{-1}, y^{-1}, y^{-1}, xy^{-1}).$$

The sums are over all directed animals of the triangular lattice.

Of course, we have $T_1(t,1,1,w,1) = \tilde{T}_0(t,w)$ (see Proposition 1.2) and $T_1(t,u,v,w,x) = T_1(t,v,u,w,x)$. A more amazing (and non-trivial) result is proved in Section 6:

$$T_1(t,u,1,1,1) = T_1(t,1,1,u,1). \qquad (10)$$

In other words,

$$\sum_A t^{|A|} u^{r(A)} = \sum_A t^{|A|} u^{\ell(A)} = \sum_A t^{|A|} u^{c(A)}.$$

The series $T_1(t,u,v,w,x)$ has again two interesting specializations, namely $T_\ell(t,x,y)$ and $T_1(t,u,1,1,1)$.

2.3.1 Directed bond-animals on the triangular lattice

The perimeter and area generating function for directed bond-animals on the triangular lattice can be derived from the loop generating function.

Proposition 2.9 — *Let $T_b(t,x)$ be the area and perimeter generating function for bond-animals on the triangular lattice:*

$$T_b(t,x) = \sum_A t^{|A|} x^{p(A)}.$$

We have

$$T_b(t,x) = \frac{x}{t} T_\ell \left(tx^2, 2 + \frac{t}{x}, 3 + 3\frac{t}{x} + \frac{t^2}{x^2} \right)$$

where $T_\ell(t,x,y)$ is the loop generating function for site-animals, defined by (9)*.*

2.3.2 Directed animals on decorated triangular lattices

Although they have not been studied in literature, we can define decorated triangular lattices, by adding vertices on the SE/NW edges of a triangular lattice (Figure 12).

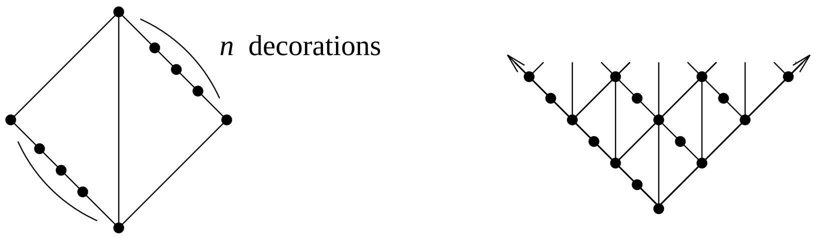


Figure 12: Triangular lattices with n decorations and 1 decoration.

Proposition 2.10 — *Let $T_{d,n}(x,y)$ be the area generating function for animals on the $n-$decorated triangular lattice:*

$$T_{d,n}(x,y) = \sum_A x^{|T(A)|} y^{|A|-|T(A)|}$$

where $|T(A)|$ is the number of cells of A on the underlying triangular lattice. We have

$$T_{d,n}(x,y) = T_1 \left(x\frac{1-y^{n+1}}{1-y}, y^n \frac{1-y}{1-y^{n+1}}, 1, 1, 1 \right)$$

where $T_1(t,u,v,w,x)$ is defined by (8)*.*

2.3.3 The mean number of loops

For $n \geq 1$, let

$$b_n = \sum_{|A|=n} b(A)$$

where $b(A) = sb(A) + db(A)$, be the total number of loops in animals of area n.

Proposition 2.11 — *Let $T_m(t)$ be the generating function of the b_n's:*

$$T_m(t) = \sum_A b(A) t^{|A|}.$$

We have

$$T_m(t) = t\frac{dT_0}{dt}(t) - T_0(t) - 2\frac{\partial T_1}{\partial u}(t,1,1,1,1) - \frac{\partial T_1}{\partial w}(t,1,1,1,1),$$

the series $T_0(t)$ and $T_1(t,u,v,w,x)$ being defined by (1) and (8) respectively.

According to (10), this identity can also be written as

$$T_m(t) = t\frac{dT_0}{dt}(t) - T_0(t) - 3\frac{\partial T_1}{\partial u}(t,1,1,1,1).$$

3 Animals of bounded width

To state our results rigorously, we need to introduce animals of bounded width, already considered in [19].

3.1 Animals on the square lattice

Consider a cyclic oriented square lattice having N cells in each row (Figure 13). Edges are oriented away from the center, and the vertices of the first two rows are labelled with $1, 2, \ldots, N$. It is convenient to consider that the labels belong to the group $\mathbb{Z}/N\mathbb{Z}$, denoted $[N]$. We deal in this section with animals that may have a source formed of several vertices.

Definition 3.1 — *Let $C \subset [N]$ be a subset of vertices of the first row. A directed animal A* of source *C is a finite set of vertices containing C such that any vertex of A can be reached from a vertex of C through an oriented path having all its vertices in A.*

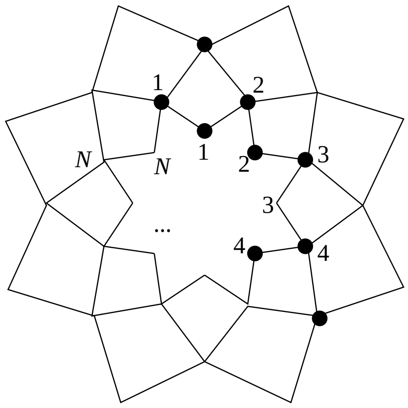


Figure 13: The cyclic square lattice and an animal of source $\{1, 2, 4\}$.

All definitions given above can be extended to animals of bounded width (area, perimeter, loops, cells only supported on the right, etc). Since the lattice has now a finite width, we can write a *finite* system of equations defining the generating functions for animals having a given source. All of them are rational functions. Let $C \subset [N]$ be a subset of vertices of the first row. We denote $\mathcal{N}(C)$ the set of neighbours of C and $\mathcal{N}_\ell(C)$ (resp. $\mathcal{N}_r(C)$) the set of vertices that are only left (resp. right) neighbours of C. More precisely,

$$\mathcal{N}(C) = C \cup \{i : i-1 \in C\}, \quad \mathcal{N}_\ell(C) = \{i \in C : i-1 \notin C\} \quad \text{and} \quad \mathcal{N}_r(C) = \{i \notin C : i-1 \in C\}. \tag{11}$$

Lemma 3.2 — *For $C \subset [N]$, let*

$$S_1^{(N)}(C) = \sum_A t^{|A|} u^{r(A)} v^{\ell(A)}$$

be the position generating function for animals of source C, and let

$$S_2^{(N)}(C) = \sum_A t^{|A|} x^{p(A)}$$

be their area and perimeter generating function. By convention, $S_1^{(N)}(\emptyset) = S_2^{(N)}(\emptyset) = 1$. We have

$$S_1^{(N)}(C) = t^{|C|} \sum_{D \subset \mathcal{N}(C)} S_1^{(N)}(D) u^{|D \cap \mathcal{N}_\ell(C)|} v^{|D \cap \mathcal{N}_r(C)|}. \tag{12}$$

Similarly:

$$S_2^{(N)}(C) = t^{|C|} \sum_{D \subset \mathcal{N}(C)} S_2^{(N)}(D) x^{|\mathcal{N}(C) \setminus D|}. \tag{13}$$

Proof. If A is an animal of source C, then $A' = A \setminus C$ is an animal whose source D is a subset of $\mathcal{N}(C)$. ■

The series $S_1^{(N)}(C)$ and $S_2^{(N)}(C)$ are related to the series $S_1(t,u,v)$ and $S_2(t,x)$ defined in Section 2 by

$$\lim_{N \to \infty} S_1^{(N)}(\{1\}) = S_1(t,u,v) \quad \text{and} \quad \lim_{N \to \infty} S_2^{(N)}(\{1\}) = S_2(t,x).$$

We give in Section 5 a probabilistic interpretation of the recurrence relation (12).

3.2 Animals on the honeycomb lattice

Consider the cyclic oriented honeycomb lattice of Figure 14. For $C \subset [N]$, animals of source C are defined as in Definition 3.1. However, we distinguish now between animals *of type* 1, that have their source in the first row of the lattice, and animals *of type* 2, that have their source in the second row of the lattice.

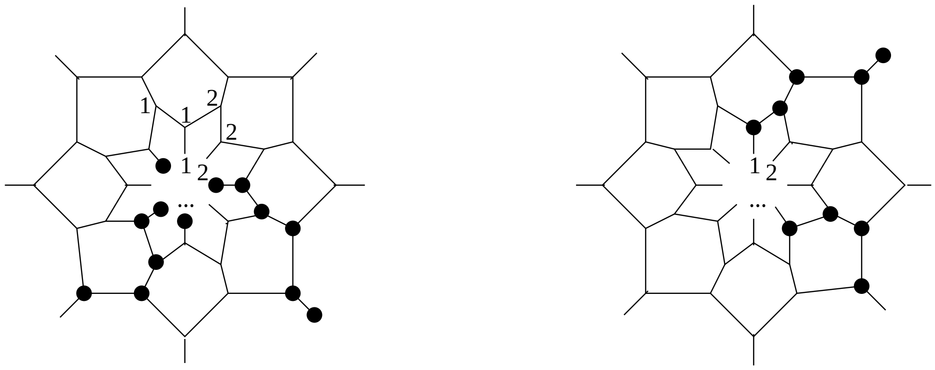


Figure 14: Animals of type 1 and 2 on the cyclic honeycomb lattice.

Let $H_1^{(N)}(C)$ be the area generating function for animals of source C and type 1, defined as in Proposition 2.7:

$$H_1^{(N)}(C) = \sum_A u^{|A|_1} v^{|A|_2}.$$

Similarly, let $H_2^{(N)}(C)$ be the area generating function for animals of source C and type 2. The recurrence relations defining these series are now:

$$H_1^{(N)}(C) = u^{|C|} \sum_{D \subset C} H_2^{(N)}(D)$$

and

$$H_2^{(N)}(C) = v^{|C|} \sum_{D \subset \mathcal{N}(C)} H_1^{(N)}(D),$$

with $H_1^{(N)}(\emptyset) = H_2^{(N)}(\emptyset) = 1$. There are two ways of combining these identities. We can either write

$$H_1^{(N)}(C) = u^{|C|} \sum_{D \subset \mathcal{N}(C)} H_1^{(N)}(D) \sum_{B \subset C : D \subset \mathcal{N}(B)} v^{|B|} \tag{14}$$

or, alternatively,

$$H_2^{(N)}(C) = v^{|C|} \sum_{D \subset \mathcal{N}(C)} H_2^{(N)}(D) u^{|D|} (1+u)^{|\mathcal{N}(C) \setminus D|}.$$

Comparing this identity with (13) shows that

$$u^{|C|} H_2^{(N)}(C) = S_2^{(N)}(C)$$

if $t = uv$ and $x = 1+u$. Actually, this is again a consequence of the correspondence Φ between animals on the square and honeycomb lattices described in Section 2.2. We give in Section 5 a probabilistic interpretation of the recurrence relation (14).

3.3 Animals on the triangular lattice

Consider the cyclic oriented triangular lattice of Figure 15. The source of a directed animal is now a subset of the first two rows of the lattice.

Definition 3.3 — *Let $C, D \subset [N]$. A directed animal A* of source (C, D) *is a finite set of vertices such that*

- *the set of vertices of A lying on the first row is C,*
- *the set of vertices of A lying on the second row is D,*
- *any vertex of A lying on another row can be reached from a vertex of C or D through an oriented path having all its vertices in A.*

A cell *not belonging to the first two rows* is in one of the seven positions described in Figure 11. We denote $r(A)$ the number of cells not belonging to the first two rows and only supported on the right. For example, $r(A) = 0$ for the animal of Figure 15.

Notation. For $C \subset [N]$, the set $\{i+1 : i \in C\}$ is denoted C^+.

Lemma 3.4 — *For $C, D \subset [N]$, let*

$$T(C, D) = \sum_A t^{|A|} u^{r(A)}$$

be the generating function for animals of source (C, D), counted according to their area and number of cells only supported on the right. By convention, $T(\emptyset, \emptyset) = 1$. We have

$$T(C, D) = t^{|C|} \sum_{E \subset \mathcal{N}(D) \cup C^+} T(D, E) u^{|E \cap \mathcal{N}_\ell(C,D)|} \tag{15}$$

where $\mathcal{N}_\ell(C, D) = \{i \in D : i-1 \notin C \cup D\}$.

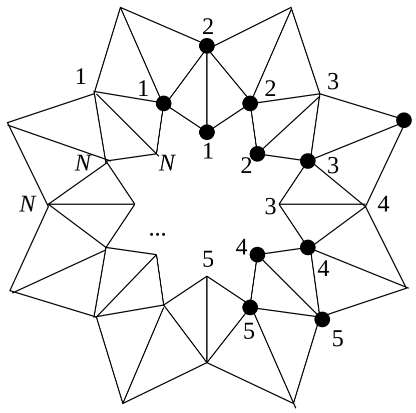


Figure 15: An animal of source $(\{1,2,4\},\{1,2,3,4,5\})$ on the cyclic triangular lattice.

4 One-dimensional gas models

Let $\mathcal{Y}=(Y_1,Y_2,\ldots,Y_N)$ be a vector of random variables taking their values in $\{0,1\}$. We consider that $\mathcal{Y}$ describes a gas model (i.e. a cell distribution) on $[N]$: an element i of $[N]$ is occupied by a cell (a molecule of gas) if and only if $Y_i=1$. For $C\subset[N]$ and $y\in\{0,1\}$, the notation $Y_C\equiv y$ means that $Y_i=y$ for all i in C. The gas model is completely determined by specifying the probabilities G_C, for $C\subset[N]$, defined by

$$G_C=P(Y_C\equiv 1, Y_{^cC}\equiv 0)$$

with ${}^cC=[N]\setminus C$. The *density* of the gas is the average number of molecules per site, i.e.

$$\frac{1}{N}\sum_{C\subset[N]}|C|G_C.$$

The model is *cyclically invariant* if $G_{C^+}=G_C$ for all C, with $C^+=\{i+1:i\in C\}$. In this case, the density is also the probability that the vertex labelled 1 is occupied. In the following lemma, we give the density of a cyclically invariant gas model with *nearest-neighbour interactions.*

Lemma 4.1 — *Let $\alpha>0$ and $\beta>0$. We consider the gas model given by*

$$P(Y_C\equiv 1, Y_{^cC}\equiv 0)=\frac{1}{Z}\alpha^{|C|}\beta^{|\bar{\mathcal{N}}_r(C)|}$$

where $\bar{\mathcal{N}}_r(C)=\{i\in C: i+1\notin C\}$ and

$$Z=\sum_{C\subset[N]}\alpha^{|C|}\beta^{|\bar{\mathcal{N}}_r(C)|}$$

is the partition function *of the model. Then the density is*

$$\alpha\frac{\lambda_1^{N-1}\lambda_1'+\lambda_2^{N-1}\lambda_2'}{\lambda_1^N+\lambda_2^N}$$

where $\lambda_{1,2}=(1+\alpha\pm\sqrt{\Delta})/2$ with $\Delta=(1-\alpha)^2+4\alpha\beta$ and $\lambda_i'=\partial\lambda_i/\partial\alpha$. Moreover,

$$\alpha\frac{\lambda_i'}{\lambda_i}=\frac{1}{2}\left(1\pm(\alpha-1)\Delta^{-1/2}\right).$$

Proof. This calculation is very classical in statistical physics (see [2] Chap. 2 for instance). We differentiate the partition function with respect to α to obtain:

$$\frac{\alpha}{NZ}\frac{\partial Z}{\partial \alpha} = \frac{1}{N}\sum_{C} |C| G_C.$$

The right hand-side of this identity is the density of the model, thus we only need to compute Z. Let V be the function defined by $V(0,0) = V(0,1) = 1$, $V(1,0) = \alpha\beta$ and $V(1,1) = \alpha$. For $y \in \{0,1\}^N$, we have

$$P(\mathcal{Y} = y) = \frac{1}{Z}\prod_{i=1}^{N} V(y_i, y_{i+1}),$$

with $y_{N+1} = y_1$, which gives

$$Z = \sum_{y}\prod_{i=1}^{N} V(y_i, y_{i+1}) = tr\left(V^N\right) = \lambda_1^N + \lambda_2^N$$

where λ_1 and λ_2 are the eigenvalues of the matrix

$$V = \begin{pmatrix} 1 & 1 \\ \alpha\beta & \alpha \end{pmatrix}.$$

■

4.1 Local transitions

Let $(p_1, p_2, p_3, p_4) \in [0,1]^4$, and suppose that a gas model $\mathcal{Y}$ is given.

Definition 4.2 — *The cell distribution* $\mathcal{X}$ induced by $\mathcal{Y}$ via the local transition (p_1, p_2, p_3, p_4) *is characterized by*

$$\forall x, y \in \{0,1\}^N \ s.t.\ P(\mathcal{Y} = y) \neq 0, \qquad P(\mathcal{X} = x/\mathcal{Y} = y) = \prod_{i=1}^{N} P(X_i = x_i/Y_i = y_i, Y_{i+1} = y_{i+1}) \tag{16}$$

where

$$P(X_i = 1/Y_i = y_i, Y_{i+1} = y_{i+1}) = \begin{cases} p_1 & if \ \ y_i = y_{i+1} = 0, \\ p_2 & if \ \ y_i = 1 \ \ and \ \ y_{i+1} = 0, \\ p_3 & if \ \ y_i = 0 \ \ and \ \ y_{i+1} = 1, \\ p_4 & if \ \ y_i = y_{i+1} = 1. \end{cases} \tag{17}$$

In other words, the X_i's are independent conditionally to the Y_j's, and X_i depends on the Y_j's by Y_i and Y_{i+1} only. We can visualize this transition on the two-rowed cyclic lattice of Figure 16. Let us call $\mathcal{L}_0$

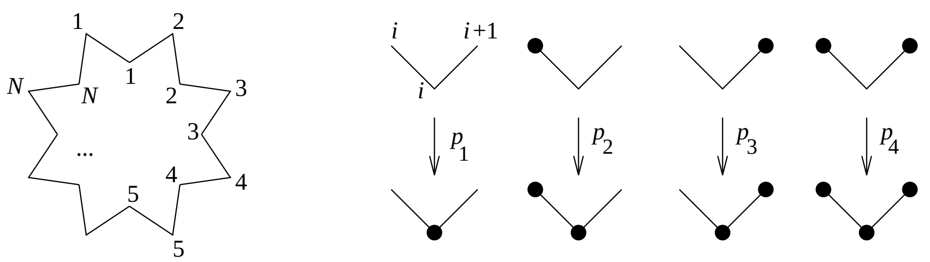


Figure 16: The two-rowed cyclic lattice and a local transition.

(resp. $\mathcal{L}_1$) the set of vertices of the internal (resp. external) row, and label the vertices in each row with $1, 2, \ldots, N$. Consider that $\mathcal{Y}$ describes a gas model on $\mathcal{L}_1$. Then the cell distribution induced by $\mathcal{Y}$ via (p_1, p_2, p_3, p_4) is obtained by adding a cell on a vertex i of $\mathcal{L}_0$

- with probability p_1 if the vertices i and $i+1$ of $\mathcal{L}_1$ are both empty,
- with probability p_2 if i is occupied and $i+1$ is empty,
- with probability p_3 if i is empty and $i+1$ is occupied,
- with probability p_4 if i and $i+1$ are both occupied.

The link between gas models and animals will be established by comparing the recurrence relations defining the generating function for animals of source C with the expression giving the probability that the cells of C are occupied in the model $\mathcal{X}$. For $C \subset [N]$, we denote $f_C = P(X_C \equiv 1)$ and $g_C = P(Y_C \equiv 1)$. Let E be a subset of $[N]$, identified with $\mathcal{L}_1$. We denote $\bar{\mathcal{N}}(E)$ the set of neighbours of E, $\bar{\mathcal{N}}_r(E)$ (resp. $\bar{\mathcal{N}}_\ell(E)$) the set of vertices of $\mathcal{L}_0$ that are only right (resp. left) neighbours of E, and $\bar{\mathcal{N}}_{r\ell}(E)$ the set of vertices of $\mathcal{L}_0$ that are simultaneously right and left neighbours of E. More precisely,

$$\bar{\mathcal{N}}(E) = E \cup \{i-1 : i \in E\}, \quad \bar{\mathcal{N}}_r(E) = \{i \in E : i+1 \notin E\}, \tag{18}$$
$$\bar{\mathcal{N}}_\ell(E) = \{i \notin E : i+1 \in E\} \quad \text{and} \quad \bar{\mathcal{N}}_{r\ell}(E) = \{i \in E : i+1 \in E\}.$$

Using the notations (11), we write

$$\begin{aligned} f_C &= \sum_{E \subset \mathcal{N}(C)} P(X_C \equiv 1 / Y_E \equiv 1, Y_{\mathcal{N}(C)\setminus E} \equiv 0) P(Y_E \equiv 1, Y_{\mathcal{N}(C)\setminus E} \equiv 0) \\ &= \sum_{E \subset \mathcal{N}(C)} p_1^{|C \cap^c \bar{\mathcal{N}}(E)|} p_2^{|C \cap \bar{\mathcal{N}}_r(E)|} p_3^{|C \cap \bar{\mathcal{N}}_\ell(E)|} p_4^{|C \cap \bar{\mathcal{N}}_{r\ell}(E)|} \sum_{B \subset \mathcal{N}(C)\setminus E} (-1)^{|B|} P(Y_{E\cup B} \equiv 1) \\ &= \sum_{D \subset \mathcal{N}(C)} (-1)^{|D|} g_D \sum_{E \subset D} (-1)^{|E|} p_1^{|C \cap^c \bar{\mathcal{N}}(E)|} p_2^{|C \cap \bar{\mathcal{N}}_r(E)|} p_3^{|C \cap \bar{\mathcal{N}}_\ell(E)|} p_4^{|C \cap \bar{\mathcal{N}}_{r\ell}(E)|}. \end{aligned} \tag{19}$$

We have used the inclusion-exclusion principle: if $A \cap E = \emptyset$, then

$$P(Y_E \equiv 1, Y_A \equiv 0) = \sum_{B \subset A} (-1)^{|B|} P(Y_{E\cup B} \equiv 1).$$

4.2 Stationary distributions

Definition 4.3 — *The cell distribution given by $\mathcal{Y}$ is said to be* stationary *for the local transition (p_1, p_2, p_3, p_4) if the induced distribution $\mathcal{X}$ is the same as $\mathcal{Y}$, that is, if $f_C = g_C$ for all C.*

We give some conditions on the p_i's that imply the existence and uniqueness of a stationary distribution for the local transition (p_1, p_2, p_3, p_4) and determine this distribution in the case where $p_1 p_4 (1-p_2)(1-p_3) = p_2 p_3 (1-p_1)(1-p_4)$.

We shall use a result which is well-known in the theory of Markov chains (see [20]). Let $\mathcal{Q} = (Q(D,C))_{C,D \subset [N]}$ be the *stochastic matrix* associated with the transition (p_1, p_2, p_3, p_4): if $\mathcal{Y}$ induces $\mathcal{X}$, then

$$P(X_C \equiv 1, X_{^cC} \equiv 0 / Y_D \equiv 1, Y_{^cD} \equiv 0) = Q(D, C).$$

This matrix is *irreducible* if for all C and D, there exists an integer $n \geq 1$ such that $Q^n(D,C) \neq 0$. In this case, the transition is said to be *irreducible* and has a unique stationary distribution $\mathcal{Y}$, which satisfies:

$$\forall C, \qquad P(Y_C \equiv 1, Y_{^cC} \equiv 0) \neq 0. \tag{20}$$

Lemma 4.4 — *Let us consider a local transition (p_1, p_2, p_3, p_4). It is irreducible in the two following cases:*

- $p_1 \neq 0$ *and* $(1-p_1)(1-p_2)(1-p_3)(1-p_4) \neq 0$,
- $p_4 \neq 1$ *and* $p_1 p_2 p_3 p_4 \neq 0$.

Proof. If $p_1 \notin \{0,1\}$, then $Q(\emptyset, C) \neq 0$ for all C. Moreover, if $p_i \neq 1$ for $i = 1 \ldots 4$, then $Q(D, \emptyset) \neq 0$ for all D. Therefore, $Q^2(D,C) \neq 0$ for all C and D. A similar argument holds in the second case.

■

Remark. The conditions given in the lemma are sufficient, but probably not necessary. Here are a few necessary conditions: $p_1 \neq 0$, $p_4 \neq 1$, $(p_1, p_4) \neq (1, 0)$, $(p_2, p_3) \neq (1, 0)$, etc...

Finding the stationary distribution of a *generic* local transition is not an easy task, and we have not been able to do it. However, we can characterize the irreducible local transitions whose stationary distribution $\mathcal{Y}$ induces a *markovian field* $(\mathcal{X}, \mathcal{Y})$ on the lattice $\mathcal{L}_0 \cup \mathcal{L}_1$.

Definition 4.5 — *A distribution of cells on the lattice $\mathcal{L}_0 \cup \mathcal{L}_1$, given by the random vectors $\mathcal{X}$ and $\mathcal{Y}$, is a* markovian field *if there exist some functions L_i and R_i, for $i \in [N]$, and a constant Z such that*

$$\forall x, y \in \{0,1\}^N \qquad P(\mathcal{X} = x, \mathcal{Y} = y) = \frac{1}{Z} \prod_{i=1}^{N} \left[L_i(y_i, x_i) R_i(x_i, y_{i+1}) \right]. \tag{21}$$

Remark. This is not exactly the usual definition of a markovian field: a random distribution of cells on a graph is said to be markovian if the state of a vertex i depends on the state of the other vertices by the state of its neighbours only. Here the graph is $\mathcal{L}_0 \cup \mathcal{L}_1$, and the markovian property means that, for all i,

$$P(X_i = x_i / \hat{\mathcal{X}}_i = \hat{x}_i, \mathcal{Y} = y) = P(X_i = x_i / Y_i = y_i, Y_{i+1} = y_{i+1})$$

and

$$P(Y_i = y_i / \mathcal{X} = x, \hat{\mathcal{Y}}_i = \hat{y}_i) = P(Y_i = y_i / X_{i-1} = x_{i-1}, X_i = x_i)$$

where $\hat{\mathcal{Z}}_i$ denotes $(Z_1, \ldots, Z_{i-1}, Z_{i+1}, \ldots, Z_N)$ if $\mathcal{Z} = (Z_1, \ldots, Z_N)$. One proves easily that a cell distribution given by (21) satisfies these identities. Conversely, a theorem of Hammersley and Clifford states that a markovian field such that $P(\mathcal{X} = x, \mathcal{Y} = y) \neq 0$ for all x and y can be written as (21) (see [17]).

Proposition 4.6 — *Let us consider an irreducible local transition (p_1, p_2, p_3, p_4). Its stationary distribution induces a markovian field on $\mathcal{L}_0 \cup \mathcal{L}_1$ if and only if*

$$p_1 p_4 (1 - p_2)(1 - p_3) = p_2 p_3 (1 - p_1)(1 - p_4). \tag{22}$$

In this case, the stationary distribution is given by

$$P(Y_D \equiv 1, Y_{^cD} \equiv 0) = \frac{1}{Z} \alpha^{|D|} \beta^{|\bar{\mathcal{N}}_r(D)|}$$

where

$$\alpha = \frac{p_1}{1 - p_4} \qquad \text{and} \qquad \beta = \frac{(1 - p_1)(1 - p_4)}{(1 - p_2)(1 - p_3)}$$

if $p_1 \neq 1$ and

$$\alpha = \frac{1}{1 - p_4} \qquad \text{and} \qquad \beta = \frac{p_4}{p_2 p_3}$$

otherwise, the set $\bar{\mathcal{N}}_r(D)$ being defined by (18).

The proof is given in the appendix.

Remark. As pointed out by B. Derrida, the case where $p_1 + p_4 = p_2 + p_3$ is much simpler than the case studied in the proposition above: the density of any cyclically invariant stationary distribution is $p_1/(1 + p_1 - p_4)$. This can be proved by computing directly $P(X_i = 1)$ in terms of Y_i and Y_{i+1}. However, this case is irrelevant for animal enumeration.

4.3 A special transition for the triangular lattice

Consider the three-rowed cyclic lattice of Figure 17. Let $\mathcal{L}_0$ (resp. $\mathcal{L}_1$, $\mathcal{L}_2$) denote the set of vertices lying in the internal (resp. middle, external) row. The vertices in each row are labelled with $1, 2, \ldots, N$.

Let $\mathcal{U} = (U_1, \ldots, U_N)$, $\mathcal{V} = (V_1, \ldots, V_N)$, $\mathcal{X} = (X_1, \ldots, X_N)$ and $\mathcal{Y} = (Y_1, \ldots, Y_N)$ be four vectors of random variables taking their values in $\{0, 1\}$. Consider that they respectively describe a gas model on $\mathcal{L}_0$,

$\mathcal{L}_1$, $\mathcal{L}_1$ and $\mathcal{L}_2$. Let $(p_1, p_2) \in [0,1]^2$ with $p_1 \neq 1$. We say that $\mathcal{U}$ and $\mathcal{V}$ are *induced by $\mathcal{X}$ and $\mathcal{Y}$ via the transition* (p_1, p_2) if for all $u, v, x, y \in \{0,1\}^N$ such that $P(\mathcal{X} = x, \mathcal{Y} = y) \neq 0$,

$$P(\mathcal{U} = u, \mathcal{V} = v / \mathcal{X} = x, \mathcal{Y} = y) = \prod_{i=1}^{N} [P(U_i = u_i / Y_{i+1} = y_{i+1}) P(V_i = v_i / X_i = x_i, Y_i = y_i, Y_{i+1} = y_{i+1})]$$

where

$$P(U_i = 1 / Y_{i+1} = y_{i+1}) = \begin{cases} 1 & \text{if } \; y_{i+1} = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$P(V_i = 1 / X_i = x_i, Y_i = y_i, Y_{i+1} = y_{i+1}) = \begin{cases} p_1 & \text{if } \; x_i = 1 \; \text{ and } \; y_i = y_{i+1} = 0, \\ p_2 & \text{if } \; x_i = 1, \; y_i = 1 \; \text{ and } \; y_{i+1} = 0, \\ 0 & \text{otherwise.} \end{cases}$$

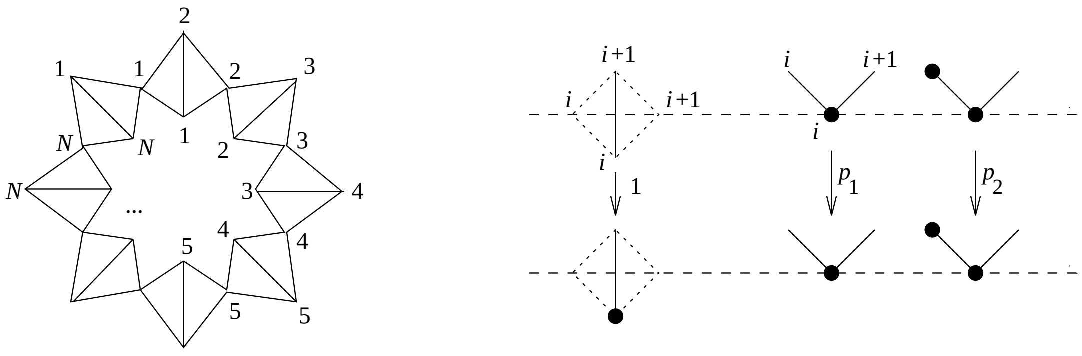


Figure 17: The three-rowed cyclic lattice and the transition (p_1, p_2).

The link between this transition and triangular lattice animals will be established by comparing the probability that some given vertices of $\mathcal{L}_0 \cup \mathcal{L}_1$ are occupied in the distribution $(\mathcal{U}, \mathcal{V})$ with the recurrence relations defining the generating functions for animals. For $C, D \subset [N]$, let $f_{C,D} = P(U_C \equiv 1, V_D \equiv 1)$ and $g_{C,D} = P(X_C \equiv 1, Y_D \equiv 1)$. Let $C^+ = \{i+1 : i \in C\}$ and $\mathcal{N}_\ell(C, D) = \{i \in D : i-1 \notin C \cup D\}$. We have:

$$\begin{aligned} f_{C,D} &= P(V_D \equiv 1, X_D \equiv 1, Y_{C^+} \equiv 0) \\ &= \sum_{A \subset \mathcal{N}_\ell(C,D)} P(V_D \equiv 1 / X_D = 1, Y_A \equiv 1, Y_{C^+ \cup \mathcal{N}(D) \setminus A} \equiv 0) P(X_D = 1, Y_A \equiv 1, Y_{C^+ \cup \mathcal{N}(D) \setminus A} \equiv 0) \\ &= \sum_{A \subset \mathcal{N}_\ell(C,D)} p_1^{|D|} \left(\frac{p_2}{p_1}\right)^{|A|} \sum_{B \subset C^+ \cup \mathcal{N}(D) \setminus A} (-1)^{|B|} P(X_D \equiv 1, Y_{A \cup B} \equiv 1) \\ &= p_1^{|D|} \sum_{E \subset \mathcal{N}(D) \cup C^+} (-1)^{|E|} g_{D,E} \left(1 - \frac{p_2}{p_1}\right)^{|E \cap \mathcal{N}_\ell(C,D)|}. \end{aligned} \tag{23}$$

Definition 4.7 — *The cell distribution given by $(\mathcal{X}, \mathcal{Y})$ is said to be* stationary *for the transition (p_1, p_2) if the induced distribution $(\mathcal{U}, \mathcal{V})$ is the same as $(\mathcal{X}, \mathcal{Y})$, that is, if $f_{C,D} = g_{C,D}$ for all C and D.*

The study of the existence and uniqueness of a stationary distribution is a bit more delicate than above and will not be described in details. The transition is no longer irreducible since some states $(C, D) \in [N]^2$ are *transient*. However, there is a *unique* set of states on which the transition is irreducible (at least when $p_1(1-p_1)(1-p_2) \neq 0$) and this implies the existence and uniqueness of the stationary distribution, given by the following proposition.

Proposition 4.8 — *Assume that $p_1(1-p_1)(1-p_2)\neq 0$. The markovian field given by*

$$P(X_D\equiv 1, X_{^cD}\equiv 0, Y_E\equiv 1, Y_{^cE}\equiv 0)=\begin{cases}\frac{1}{Z}\left(\frac{1}{p_1}\right)^{|^cE\cap D^+|}\left(\frac{p_2}{1-p_2}\right)^{|E\cap ^cD^+|} & \text{if } E\subset D\\ 0 & \text{otherwise}\end{cases}$$

where

$$Z=\sum_{E\subset D\subset[N]}\left(\frac{1}{p_1}\right)^{|^cE\cap D^+|}\left(\frac{p_2}{1-p_2}\right)^{|E\cap ^cD^+|},$$

is stationary for the transition (p_1,p_2). Moreover, there exists a constant Z' such that, for $E\subset\mathcal{L}_2$,

$$P(Y_E\equiv 1, Y_{^cE}\equiv 0)=\frac{1}{Z'}\alpha^{|E|}\beta^{|\bar{\mathcal{N}}_r(E)|}$$

with $\alpha=p_1/(1+p_1)$ and $\beta=1/((1+p_1)(1-p_2))$.

Proof. The proof is a simple calculation very similar to what is done in the appendix.

5 Directed animals and gas models

In this section, all animals and gas models live on cyclic lattices having N vertices in each row, for a given integer N.

The correspondence between directed animals and hard particle models described by Dhar implies, in the square lattice case, that

$$S_0^{(N)}(t)=-\rho(-t,0,0,0), \tag{24}$$

where $S_0^{(N)}(t)$ is the area generating function for one-source animals on the cyclic square lattice and $\rho(-t,0,0,0)$ is the density of the gas whose distribution is stationary for the local transition $(-t,0,0,0)$. This density can be computed by combining Lemma 4.4, Proposition 4.6 and Lemma 4.1. We generalize this result in two different ways: we show that the position generating function and the area and perimeter generating function for directed animals on the square lattice can also be expressed in terms of the density of a gas whose distribution is stationary for a given local transition. Unfortunately, we could not compute this density in the most general case, but only in a particular case to which Section 6 is devoted.

5.1 Position generating function for square lattice animals

Let us consider the local transition $(q_1, q_1q_2, q_1q_3, q_1q_2q_3)$ and its stationary distribution. For $C\subset[N]$, let $\mathcal{N}_{r\ell}(C)=\{i\in C: i-1\in C\}$. Using the notations (11), we have, according to (19):

$$\begin{aligned}g_C &= \sum_{D\subset\mathcal{N}(C)}(-1)^{|D|}g_D\sum_{E\subset D}(-1)^{|E|}q_1^{|C|}q_2^{|E\cap\mathcal{N}_\ell(C)|}q_3^{|E\cap\mathcal{N}_r(C)|}(q_2q_3)^{|E\cap\mathcal{N}_{r\ell}(C)|}\\ &= q_1^{|C|}\sum_{D\subset\mathcal{N}(C)}(-1)^{|D|}g_D(1-q_2)^{|D\cap\mathcal{N}_\ell(C)|}(1-q_3)^{|D\cap\mathcal{N}_r(C)|}(1-q_2q_3)^{|D\cap\mathcal{N}_{r\ell}(C)|}\\ &= q_1^{|C|}\sum_{D\subset\mathcal{N}(C)}(q_2q_3-1)^{|D|}g_D\left(\frac{q_2-1}{q_2q_3-1}\right)^{|D\cap\mathcal{N}_\ell(C)|}\left(\frac{q_3-1}{q_2q_3-1}\right)^{|D\cap\mathcal{N}_r(C)|}.\end{aligned}$$

Comparing this identity with (12) gives the following result.

Proposition 5.1 — *Let $C\subset[N]$. The position generating function for animals of source C is*

$$S_1^{(N)}(C)=\left(\frac{p_4}{p_1}-1\right)^{|C|}g_C,$$

where g_C is the probability that the vertices of C are occupied when the cell distribution is stationary for the transition (p_1, p_2, p_3, p_4), with

$$p_1 = \frac{tuv}{1-u-v}, \quad p_2 = \frac{tu(1-u)}{1-u-v}, \quad p_3 = \frac{tv(1-v)}{1-u-v} \quad \text{and} \quad p_4 = \frac{p_2 p_3}{p_1}.$$

In particular, the position generating function for one-source directed animals on the cyclic square lattice is

$$\sum_A t^{|A|} u^{r(A)} v^{\ell(A)} = \frac{1-u-v}{uv} \rho(p_1, p_2, p_3, p_2 p_3/p_1)$$

where $\rho(p_1, p_2, p_3, p_4)$ denotes the density of the gas whose distribution is stationary for the transition (p_1, p_2, p_3, p_4).

Remark. The special case $u = v = 1$ is nothing but Dhar's result, given by (24).

Using the ideas of Section 2.1, we can express several generating functions for one-source directed animals (i.e. animals of source $\{1\}$) in terms of the density of some gas models.

Corollary 5.2

- *The generating function for one-source directed animals on the cyclic square lattice according to their area and number of cells only supported on the right is*

$$\sum_A t^{|A|} u^{r(A)} = -\rho(p_1, p_2, 0, 0)$$

where $p_1 = -t$ and $p_2 = t(u-1)$.

- *The loop generating function for one-source directed animals on the cyclic square lattice is*

$$\sum_A t^{|A|} w^{b(A)} = (w-2)\rho(p_1, p_2, p_2, p_2^2/p_1)$$

where

$$p_1 = \frac{t}{w-2} \quad \text{and} \quad p_2 = \frac{t(w-1)}{w-2}.$$

- *In particular, the area-generating function for one-source directed animals having no loops (trees) is*

$$\sum_A t^{|A|} = -2\rho(p, -p, -p, p)$$

where $p = -t/2$.

- *The area and perimeter generating function for one-source directed bond-animals on the cyclic square lattice is*

$$\sum_A t^{|A|} x^{p(A)} = \rho(p_1, p_2, p_2, p_2^2/p_1)$$

where $p_1 = x^2$ and $p_2 = x(t+x)$.

5.2 Area and perimeter generating functions

Let us consider the local transition $(q_1 q_2, q_1, q_1, q_1)$ and its stationary distribution. With the notations of Section 4, we have, according to (19):

$$g_C = \sum_{D \subset \mathcal{N}(C)} (-1)^{|D|} g_D \sum_{E \subset D} (-1)^{|E|} q_1^{|C|} q_2^{|C \cap^c \bar{\mathcal{N}}(E)|}.$$

But

$$\sum_{E \subset D} (-1)^{|E|} q_2^{|C \cap^c \bar{\mathcal{N}}(E)|} = \sum_{E \subset D} (-1)^{|E|} (1 + (q_2 - 1))^{|C \cap^c \bar{\mathcal{N}}(E)|}$$

$$= \sum_{E \subset D} (-1)^{|E|} \sum_{B \subset C \cap^c \bar{\mathcal{N}}(E)} (q_2 - 1)^{|B|}$$

$$= \sum_{B \subset C} (q_2 - 1)^{|B|} \sum_{E \subset D \cap^c \mathcal{N}(B)} (-1)^{|E|}$$

$$= \sum_{B \subset C : D \subset \mathcal{N}(B)} (q_2 - 1)^{|B|},$$

the sum being over all subsets B of C such that $D \subset \mathcal{N}(B)$. Therefore,

$$g_C = q_1^{|C|} \sum_{D \subset \mathcal{N}(C)} (-1)^{|D|} g_D \sum_{B \subset C : D \subset \mathcal{N}(B)} (q_2 - 1)^{|B|}.$$

Comparing this identity with (14) gives the following result.

Proposition 5.3 — *Let $C \subset [N]$. The area generating function for animals of source C and type 1 on the cyclic honeycomb lattice is*

$$H_1^{(N)}(C) = (-1)^{|C|} g_C,$$

where g_C is the probability that the vertices of C are occupied when the cell distribution is stationary for the transition (p_1, p_2, p_2, p_2), with $p_1 = -u(1+v)$ and $p_2 = -u$.

Using the ideas of Section 2.2, we can express several generating functions for one-source directed animals in terms of the density of some gas models.

Corollary 5.4
- *The area generating function for one-source directed animals of type 1 on the cyclic honeycomb lattice is*

$$\sum_A u^{|A|_1} v^{|A|_2} = -\rho(p_1, p_2, p_2, p_2)$$

  where $p_1 = -u(1+v)$ and $p_2 = -u$.

- *The area and perimeter generating function for one-source directed animals on the cyclic square lattice is*

$$\sum_A t^{|A|} x^{p(A)} = 1 - x - \rho(p_1, p_2, p_2, p_2)$$

  where

$$p_1 = 1 - x - t \quad \text{and} \quad p_2 = 1 - x.$$

- *The area and perimeter generating function for one-source directed animals on the cyclic honeycomb lattice is*

$$\sum_A t^{|A|} x^{p(A)} = 1 - x - \rho(p_1, p_2, p_2, p_2)$$

  where

$$p_1 = 1 - x(1+t) - t^2 \quad \text{and} \quad p_2 = 1 - x(1+t).$$

We summarize in the following table the links between animals generating functions and local transitions.

Lattice	Site/Bond	Parameters	Transition
square	site	$\|A\|$, $r(A)$ and $\ell(A)$	$p_1, p_2, p_3, p_2 p_3/p_1$
square	site	$\|A\|$ and $r(A)$	$p_1, p_2, 0, 0$
square	site	$\|A\|$ and $b(A)$	$p_1, p_2, p_2, p_2^2/p_1$
square	bond	$\|A\|$ and $p(A)$	idem
square	bond	$\|A\|$	$1, p, p, p^2$
square	site	$\|A\|$ and $p(A)$	p_1, p_2, p_2, p_2
honeycomb	site	$\|A\|$ and $p(A)$	idem
honeycomb	site	$\|A\|$	$p(1-p), p, p, p$

5.3 Animals on the triangular lattice

Let us consider the transition (p_1, p_2) defined in Section 4.3 and its stationary distribution given by Proposition 4.8. According to (23), it satisfies

$$g_{C,D} = p_1^{|D|} \sum_{E \subset \mathcal{N}(D) \cup C^+} (-1)^{|E|} g_{D,E} \left(1 - \frac{p_2}{p_1}\right)^{|E \cap \mathcal{N}_\ell(C,D)|}.$$

Comparing with (15) gives the following result.

Proposition 5.5 — *Let $C, D \subset [N]$. The generating function for triangular lattice animals of source (C, D), counted according to their area and number of cells only supported on the right, is*

$$T(C,D) = (-p_1)^{|C|}(-1)^{|D|} g_{C,D}$$

where $g_{C,D}$ is the probability that all vertices of C and D are occupied when the cell distribution is stationary for the transition (p_1, p_2), with $p_1 = -t$ and $p_2 = t(u-1)$.

6 Explicit results

We give here the generating functions for directed animals on the square and triangular lattices according to their area and number of cells only supported on the right. These results are new and provide the generating functions for animals on decorated lattices, as well as the mean number of loops in animals of given area. We found no "combinatorial" proof of these formulas.

6.1 The square lattice

Proposition 6.1 — *The generating function for directed animals on the square lattice according to their area and number of cells only supported on the right is*

$$S_1(t,u,1) = \frac{1}{2}\left(\left(1 - \frac{4t}{(1+t)(1+t-tu)}\right)^{-1/2} - 1\right).$$

Equivalently,

$$\left(1 - t(2+u) + t^2(1-u)\right)\left(S_1(t,u,1) + (S_1(t,u,1))^2\right) = t.$$

Proof. Combine the first item of Corollary 5.2 with Lemma 4.4, Proposition 4.6 and Lemma 4.1, and let N tend to infinity.

■

Remark. According to Lemma 2.5, $uS_1(t,u,1)$ is the generating function for directed animals according to their area and number of right neighbours.

From Proposition 2.3, we obtain the generating function for animals on the decorated square lattices.

Corollary 6.2 — *The area generating function for animals on the $n-$decorated square lattice is*

$$S_{d,n}(x,y) = S_1\left(x\frac{1-y^{n+1}}{1-y}, y^n\frac{1-y}{1-y^{n+1}}, 1\right)$$

where $S_1(t,u,1)$ is given in the proposition above. In particular,

$$\left(1 - 3x + x^2\frac{1-x^n}{1-x}\right)\left(S_{d,n}(x,x) + (S_{d,n}(x,x))^2\right) = x.$$

Remark. This result has already been proved by Ali [1]. His proof is also inspired by Dhar's method, but one of its steps involves a two-dimensional Ising model, whereas we only need a one-dimensional model here.

From Proposition 2.4, we compute the total number of loops in animals of given area.

Corollary 6.3 — *The generating function for the total number of loops in animals of given area is*

$$S_m(t) = \sum_A b(A) t^{|A|} = \frac{1}{2}\left(1 - \frac{1 - 4t + t^2 + 4t^3}{(1+t)^{1/2}(1-3t)^{3/2}}\right).$$

This implies that, in animals of area n,

- *the mean number of cells supported only on the right (left) is asymptotically* $4n/9$,
- *the mean number of loops is asymptotically* $n/9$.

6.2 The triangular lattice

Proposition 6.4 — *The generating function for directed animals on the triangular lattice according to their area and number of cells only supported on the right is*

$$T_1(t, u, 1, 1, 1) = \frac{1}{2}\left(\left(1 - \frac{4t}{1 + t - tu}\right)^{-1/2} - 1\right).$$

Equivalently,

$$(1 - 3t - tu)\left(T_1(t, u, 1, 1, 1) + (T_1(t, u, 1, 1, 1))^2\right) = t.$$

Proof. We apply Proposition 5.5 with $C = \emptyset$ and $D = \{1\}$: the generating function for one-source directed animal on the cyclic triangular lattice is the opposite of the probability that a given vertex of $\mathcal{L}_2$ is occupied, when the distribution of cells on $\mathcal{L}_1 \cup \mathcal{L}_2$ is stationary for the transition $(-t, t(u-1))$. We combine Proposition 4.8 and Lemma 4.1 and let N tend to infinity to conclude the proof.

■

Remark. Comparing with Proposition 1.2 shows that the parameters "number of cells only supported on the right" and "number of cells only supported at the center" have the same distribution on animals of given area. This is not obvious at all, and it would be nice to find a more direct proof.

From Proposition 2.10, we obtain the generating function for animals on the decorated triangular lattices.

Corollary 6.5 — *The area generating function for animals on the* $n-$*decorated triangular lattice is*

$$T_{d,n}(x, y) = T_1\left(x\frac{1 - y^{n+1}}{1 - y}, y^n \frac{1 - y}{1 - y^{n+1}}, 1, 1, 1\right),$$

where $T_1(t, u, 1, 1, 1)$ *is given in the proposition above. In particular,*

$$T_{d,n}(x, x) = T_0(x) = \frac{1}{2}\left((1 - 4x)^{-1/2} - 1\right),$$

which means that this simplified area generating function for animals on the n-decorated triangular lattice does not depend on n.

From Proposition 2.11, we compute the total number of loops in animals of given area.

Corollary 6.6 — *The generating function for the total number of loops in animals of given area is*

$$T_m(t) = \sum_A b(A) t^{|A|} = \frac{1}{2}\left(1 - \frac{1 - 6t + 6t^2}{(1 - 4t)^{3/2}}\right).$$

This implies that, in animals of area n,

- *the mean number of cells supported only on the right (on the left, at the center) is* $\frac{n(n-1)}{2(2n-1)}$,
- *the mean number of loops is* $\frac{(n-1)(n-2)}{2(2n-1)}$.

Remarks. 1. Uniform generation of random animals suggests that they are in general very "thin" [10]. Corollaries 6.3 and 6.6 give a measure of thinness: if random animals were compact, then they would have lots of loops whereas only one fourth of the cells are loops.

2. The generating functions for animals on the decorated square lattice and the results of Corollaries 6.3 and 6.6 were conjectured by Andrew Conway [7, 8]. He rightly says that results should be easier to prove once the answer is known... Let us advertise another of his conjectures, which is related to the mean perimeter of square lattice animals of given area, and could maybe be derived from Section 5.2:

$$\sum_A p(A)t^{|A|} = \frac{1}{2t(1+t)}\left(\frac{1-3t+2t^2+t^3-3t^4}{(1-3t)^{3/2}(1+t)^{1/2}} - 1 - t - t^2\right).$$

Appendix: proof of Proposition 4.6.

1. Let us consider a markovian field on $\mathcal{L}_0 \cup \mathcal{L}_1$. Suppose that it is induced by the stationary distribution of an irreducible local transition (p_1, p_2, p_3, p_4). Since the stationary distribution is unique, it is cyclically invariant, and so is the induced markovian field: there exist two functions L and R such that (21) is satisfied with $L_i = L$ and $R_i = R$ for all i. Let $y \in \{0,1\}^N$. We have

$$\begin{aligned} P(\mathcal{Y}=y) &= \sum_x P(\mathcal{X}=x, \mathcal{Y}=y) \\ &= \frac{1}{Z}\sum_x \prod_i [L(y_i,x_i)R(x_i,y_{i+1})] \\ &= \frac{1}{Z}\prod_i [L(y_i,0)R(0,y_{i+1}) + L(y_i,1)R(1,y_{i+1})] \end{aligned}$$

and this probability is positive, according to (20). This gives

$$P(\mathcal{X}=x/\mathcal{Y}=y) = \prod_i \frac{L(y_i,x_i)R(x_i,y_{i+1})}{L(y_i,0)R(0,y_{i+1}) + L(y_i,1)R(1,y_{i+1})}.$$

Comparing with (16) and (17) leads to

$$\frac{L(0,1)R(1,0)}{L(0,0)R(0,0)+L(0,1)R(1,0)} = p_1, \qquad \frac{L(1,1)R(1,0)}{L(1,0)R(0,0)+L(1,1)R(1,0)} = p_2,$$

$$\frac{L(0,1)R(1,1)}{L(0,0)R(0,1)+L(0,1)R(1,1)} = p_3, \qquad \frac{L(1,1)R(1,1)}{L(1,0)R(0,1)+L(1,1)R(1,1)} = p_4.$$

A simple calculation shows that these identities imply (22).

2. Conversely, assume that (22) is satisfied. Since the transition is irreducible, $p_1 \neq 0$ and $p_4 \neq 1$. Let us first assume that $p_1 \neq 1$. Then $(1-p_2)(1-p_3) \neq 0$. Otherwise, according to (22), we would have $p_2p_3(1-p_1)(1-p_4) = 0$, and thus $p_2p_3 = 0$, i.e. $(p_2,p_3) = (0,1)$ or $(1,0)$, and the transition would not be irreducible. Let us consider the cell distribution on $\mathcal{L}_1$ given by

$$P(Y_D \equiv 1, Y_{^cD} \equiv 0) = \frac{1}{Z}\left(\frac{p_1}{1-p_4}\right)^{|D|}\left(\frac{(1-p_1)(1-p_4)}{(1-p_2)(1-p_3)}\right)^{|\bar{\mathcal{N}}_r(D)|}. \tag{25}$$

Let $\mathcal{X}$ be induced by $\mathcal{Y}$ via the local transition (p_1,p_2,p_3,p_4). Let $F_{C,D} = P(X_C \equiv 1, X_{^cC} \equiv 0, Y_D \equiv 1, Y_{^cD} \equiv 0)$. We have

$$F_{C,D} = P(X_C \equiv 1, X_{^cC} \equiv 0 / Y_D \equiv 1, Y_{^cD} \equiv 0)P(Y_D \equiv 1, Y_{^cD} \equiv 0).$$

According to (16) and (17), and using

$$\frac{p_4}{1-p_4} = \frac{p_2p_3(1-p_1)}{p_1(1-p_2)(1-p_3)},$$

we obtain

$$F_{C,D} = \frac{1}{Z}(1-p_1)^{|{}^c\bar{\mathcal{N}}(D)|}(1-p_2)^{|\bar{\mathcal{N}}_r(D)|}(1-p_3)^{|\bar{\mathcal{N}}_\ell(D)|}(1-p_4)^{|\bar{\mathcal{N}}_{r\ell}(D)|}\left(\frac{p_1}{1-p_1}\right)^{|C|}\left(\frac{p_2(1-p_1)}{p_1(1-p_2)}\right)^{|C\cap\bar{\mathcal{N}}_r(D)|}$$

$$\left(\frac{p_3(1-p_1)}{p_1(1-p_3)}\right)^{|C\cap\bar{\mathcal{N}}_\ell(D)|}\left(\frac{p_2p_3(1-p_1)^2}{p_1^2(1-p_2)(1-p_3)}\right)^{|C\cap\bar{\mathcal{N}}_{r\ell}(D)|}\left(\frac{p_1}{1-p_4}\right)^{|D|}\left(\frac{(1-p_1)(1-p_4)}{(1-p_2)(1-p_3)}\right)^{|\bar{\mathcal{N}}_r(D)|}$$

where the sets $\bar{\mathcal{N}}(D), \bar{\mathcal{N}}_r(D), \bar{\mathcal{N}}_\ell(D)$ and $\bar{\mathcal{N}}_{r\ell}(D)$ are defined by (18). Now, note that

$$|\bar{\mathcal{N}}_r(D)| = |\bar{\mathcal{N}}_\ell(D)| \quad , \quad |\bar{\mathcal{N}}(D)| = |D| + |\bar{\mathcal{N}}_r(D)| \quad , \quad |\bar{\mathcal{N}}_{r\ell}(D)| = |D| - |\bar{\mathcal{N}}_r(D)|, \tag{26}$$

$$|C\cap\bar{\mathcal{N}}_r(D)| + |C\cap\bar{\mathcal{N}}_{r\ell}(D)| = |C\cap D| \quad \text{and} \quad |C\cap\bar{\mathcal{N}}_\ell(D)| + |C\cap\bar{\mathcal{N}}_{r\ell}(D)| = \left|D\cap C^+\right|$$

where $C^+ = \{i+1 : i \in C\}$. We finally obtain

$$F_{C,D} = \frac{(1-p_1)^N}{Z}\left(\frac{p_1}{1-p_1}\right)^{|C|+|D|}\left(\frac{p_2(1-p_1)}{p_1(1-p_2)}\right)^{|C\cap D|}\left(\frac{p_3(1-p_1)}{p_1(1-p_3)}\right)^{|D\cap C^+|}.$$

Comparing with (21) shows that the cell distribution given by (25) induces a markovian field on $\mathcal{L}_0 \cup \mathcal{L}_1$ given by

$$L_i(y,x) = \begin{cases} p_1/(1-p_1) & \text{if } (y,x) = (0,1), \\ p_2/(1-p_2) & \text{if } (y,x) = (1,1), \\ 1 & \text{otherwise,} \end{cases}$$

and

$$R_i(x,y) = \begin{cases} p_1/(1-p_1) & \text{if } (x,y) = (0,1), \\ p_3/(1-p_3) & \text{if } (x,y) = (1,1), \\ 1 & \text{otherwise.} \end{cases}$$

We have now to show that the distribution given by (25) is indeed stationary. For $C \subset \mathcal{L}_0$, we have

$$P(X_C \equiv 1, X_{^cC} \equiv 0) = \sum_D F_{C,D}.$$

Using the notations (11) and $\mathcal{N}_{r\ell}(C) = \{i \in C : i-1 \in C\}$, we write

$$|D| = |D \cap {}^c\mathcal{N}(C)| + |D\cap\mathcal{N}_r(C)| + |D\cap\mathcal{N}_\ell(C)| + |D\cap\mathcal{N}_{r\ell}(C)|,$$

and use identities similar to (26) to obtain

$$P(X_C \equiv 1, X_{^cC} \equiv 0) = \frac{1}{Z}\left(\frac{p_1}{1-p_4}\right)^{|C|}\left(\frac{(1-p_1)(1-p_4)}{(1-p_2)(1-p_3)}\right)^{|\mathcal{N}_r(C)|}.$$

Since $|\mathcal{N}_r(C)| = |\bar{\mathcal{N}}_r(C)|$ for all $C \subset [N]$, this proves that the distribution given by (25) is stationary for the transition (p_1, p_2, p_3, p_4).

We have now to study the case $p_1 = 1$. According to (22), we have $p_4(1-p_2)(1-p_3) = 0$. Since the transition is irreducible, $p_4 \neq 0$. Hence, $p_2 = 1$ or $p_3 = 1$. Suppose that $p_2 = 1$ (the case $p_3 = 1$ is symmetric). Then $p_3 \neq 0$, otherwise the transition would not be irreducible. One checks that the distribution of cells given by

$$P(Y_D \equiv 1, Y_{^cD} \equiv 0) = \frac{1}{Z}(1-p_4)^{-|D|}\left(\frac{p_4}{p_3}\right)^{|\bar{\mathcal{N}}_r(D)|}$$

is stationary for the transition $(1,1,p_3,p_4)$ and induces on $\mathcal{L}_0 \cup \mathcal{L}_1$ a markovian field given by (21) where, for all i, $L_i(1,1) = L_i(1,0) = R_i(1,1) = R_i(1,0) = p_4/(1-p_4)$, $L_i(0,0) = p_4(1-p_3)/(p_3(1-p_4))$, $R_i(0,0) = 0$ and the two other parameters are equal to 1.

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References

[1] A. A. Ali, *Enumeration of directed site animals on the decorated square lattices*, Phys. A **202** (1994) 520–528.

[2] R. J. Baxter, Exactly solved models in statistical mechanics, Academic Press (1982).

[3] J. Bétréma and J.-G. Penaud, *Animaux et arbres guingois*, Theoret. Comput. Sci. **117** (1993) 67–89.

[4] J. Bétréma and J.-G. Penaud, *Modèles avec particules dures, animaux dirigés et séries en variables partiellement commutatives*, Report 93–18, LaBRI, Université Bordeaux 1, France (1993).

[5] J. Blease, *Series expansions for directed bond-percolation problem*, J. Phys. C: Solid. State Phys. **10** (1977) 917–924.

[6] M. Bousquet-Mélou, *Percolation models and animals*, European J. Combin. **17** (1996) 343–369.

[7] A. R. Conway, *On the behaviour of self-avoiding walks and related problems*, Ph.D. Thesis, University of Melbourne, Australia (1993).

[8] A. R. Conway, *Some exact results for moments of 2D directed animals*, preprint 1995.

[9] A. R. Conway, R. Brak and A. J. Guttmann, *Directed animals on two-dimensional lattices*, J. Phys. A: Math. Gen. **26** (1993) 3085–3091.

[10] A. Denise and J.-G. Penaud, *Génération aléatoire d'animaux dirigés*, Actes des Journées "Polyominos et pavages", P. Aigrain ed., Lyon (1992) 61–69.

[11] D. Dhar, *Equivalence of the two-dimensional directed site animal problem to Baxter's hard square lattice gas model*, Phys. Rev. Lett. **49** (1983) 959–962.

[12] D. Dhar, *Exact solution of a directed-site animals-enumeration problem in three dimensions*, Phys. Rev. Lett. **51** (1983) 853–856.

[13] D. Dhar, M. K. Phani and M. Barma, *Enumeration of directed site animals on two-dimensional lattices*, J. Phys. A: Math. Gen. **15** (1982) L279–L284.

[14] D. Gouyou-Beauchamps and G. Viennot, *Equivalence of the two-dimensional directed animal problem to a one-dimensional path problem*, Adv. in Appl. Math. **9** (1988) 334–357.

[15] A. J. Guttmann, *On the number of lattice animals embeddable in the square lattice*, J. Phys. A: Math. Gen. **15** (1982) 1987–1990.

[16] A. J. Guttmann and I. G. Enting, *On the solvability of some statistical mechanical systems*, to appear in *Phys. Rev. Lett.*

[17] X. Guyon, Champs aléatoires sur un réseau. Modélisations, statistique et application. Techniques stochastiques, Masson (1993).

[18] D. A. Klarner and R. L. Rivest, *A procedure for improving the upper bound for the number of n-ominoes*, Canad. J. Math. **13** (1973) 585–602.

[19] J.-P. Nadal, B. Derrida and J. Vannimenus, *Directed lattice animals in 2 dimensions: numerical and exact results*, J. Physique **43** (1982) 1561–1574.

[20] E. Seneta, *Non-negative matrices and Markov chains*, 2^{nd} ed., Springer Series in Statistics, Springer-Verlag (1981).

[21] D. Stauffer and A. Aharony, *Introduction to percolation theory*, 2^{nd} ed., Taylor and Francis, London (1992).

[22] X. G. Viennot, *Heaps of pieces I: Basic definitions and combinatorial lemmas*, Combinatoire énumérative (G. Labelle and P. Leroux eds.), Lecture Notes in Math. **1234** (1986) 321–350.