Enumeration of directed animals on an infinite family of lattices

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Abstract

We prove algebraic equations satisfied by the area generating function for directed animals on an infinite family of regular, non-planar, two-dimensional graphs.

1 Introduction

A *directed animal* A on an oriented graph having an origin O is a finite set of sites containing O such that each point of A is connected to O through an oriented path of the graph having all its vertices in A. The *area* of A is the number of its vertices. Typically, the graph in question is a regular lattice with the orientation of the bonds corresponding to some preferred direction. Examples are given in the next section.

Directed animals are geometrical entities whose properties have been extensively studied over the past fifteen odd years due to their interest in both combinatorics and statistical physics. Few exact results are known. In 1982, Dhar, Phani and Barma gave two conjectures on the number of directed animals on the square and triangular lattices [1]. These conjectures can be restated in the form of quadratic expressions for the corresponding generating functions, and were then proved in several ways [2, 3, 4, 5, 6]. Dhar [6] also solved the enumeration problem on a three-dimensional lattice through a correspondence with the hard hexagon model solved by Baxter [7]. The associated area generating function is again algebraic, but of degree 12 [8].

Directed animals on other lattices have been enumerated by computer with the aim of finding algebraic generating functions, with a marked lack of success other than for two families of decorated lattices [9, 10, 11]. These studies include many variations on the theme of animals, like bond-animals and trees (animals of cyclomatic index zero), but no algebraic generating function has been found for them. Recent work may provide some reasons for this [12].

The properties of directed animals have also been studied extensively. This was mostly done by computer enumeration and numerical techniques, although some rigorous results were also obtained. Properties associated with the shape of directed animals have been studied in *e.g.* [13, 14, 15, 16, 17, 18, 19]. Their perimeter and their cyclomatic index were studied in *e.g.* [11, 19, 20].

All the two-dimensional directed animals mentioned above seem to fall into the same universality class. Closely related to the study of directed animals, but distinguished by significantly different properties (and thus a very different universality class) are undirected animals (connected sets on a graph) also called sometimes *polyominoes* by combinatorialists. Surveys can be found be found in [21, 22, 23].

The most successful method for proving a formula for the generating function of directed animals has been to use, following Dhar [6], an equivalence to a hard particle gas model [10, 11]. The *heaps of pieces*

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approach [4, 21, 24] of Viennot works in a more intuitive manner for the square and triangular lattices, and has also been successful in other polyomino problems [25].

Using Dhar's method, we prove in this paper algebraic equations satisfied by the area generating functions for directed animals on an infinite family of regular lattices. We derive from these equations the asymptotic behaviour of the number of animals having k vertices, thus proving that all these models belong to the same universality class as the square lattice directed animal model. The lattices are defined in Section 2, in which we also state our results, and the proof is given in Section 3.

2 Results

We define an oriented lattice $\mathcal{L}_n$ indexed by an integer $n \geq 2$. The vertices of $\mathcal{L}_n$ are labeled by the elements of $\mathbb{N}^2$, and from each vertex (i, j) there are n emerging edges leading to the vertices $(i+r, j+1)$, for $0 \leq r < n$. The origin is $O = (0,0)$. Note that $\mathcal{L}_2$ is simply the oriented square lattice. More examples are shown in Figure 1.

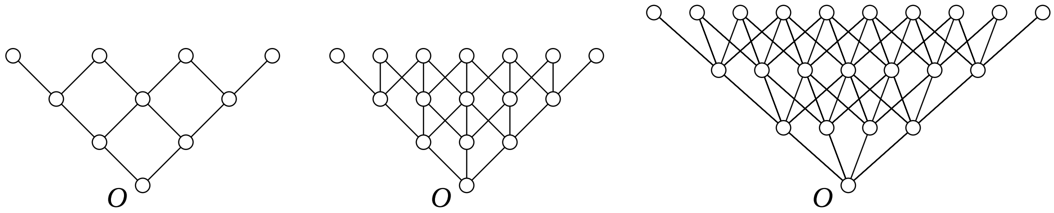

Figure 1: The lattices $\mathcal{L}_2$, $\mathcal{L}_3$ and $\mathcal{L}_4$ (all the edges are oriented upwards).

Alternatively, one can construct $\mathcal{L}_n$ as follows. Start from the directed square lattice, and keep only one out of every $(n-1)$ rows, that is the first row, the row n , the row $2n-1$, and so on (a row is perpendicular to the preferred direction). Add an edge between two vertices of two consecutive rows if these vertices were linked by an oriented path in the original square lattice. This leaves the lattice $\mathcal{L}_n$. As the number of vertices at distance at most r from the origin grows like r^2, the lattice is said to be two-dimensional.

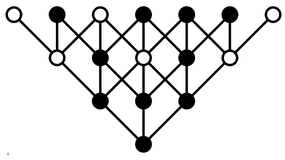

Figure 2: A directed animal on $\mathcal{L}_3$ (black vertices).

Let $n \geq 2$. We prove in this paper that the area generating function S_n for directed animals on $\mathcal{L}_n$, defined by

$$S_n = \sum_{\{A: A \text{ is an animal on } \mathcal{L}_n\}} t^{|A|}, \tag{1}$$

satisfies the following algebraic equation:

$$t^2(1+t)^{n-1}\left[1+(n+1)S_n\right]^{n+1} - \left[1+t+(n-1)S_n\right]^{n-1}(t-2S_n)^2 = 0. \tag{2}$$

Note that $S_n = 0$ is an irrelevant solution of this equation. Moreover, when n is odd, say $n = 2m+1$, the polynomial on the left-hand side of (2) can be factored and S_n satisfies the following simpler equation:

$$t(1+t)^m\left[1+(n+1)S_n\right]^{m+1} + \left[1+t+(n-1)S_n\right]^m(t-2S_n) = 0.$$

Hence, S_n is algebraic of degree (at most) n if n is even, and algebraic of degree (at most) $(n+1)/2$ if n is odd.

Here are the first few equations satisfied by the series S_n:

$$(1-3t)(S_2+1)S_2 - t = 0,$$
$$(1-4t-4t^2)(2S_3+1)S_3 - t(1+t) = 0,$$
$$(27-135t-275t^2-125t^3)(S_4+1)S_4^3 + (9-54t-110t^2-50t^3)S_4^2 + (1+t)(1-12t-10t^2)S_4 - t(1+t)^2 = 0,$$
$$2(4-27t-54t^2-27t^3)(2S_5+1)S_5^2 + (1+t)(1-12t-9t^2)S_5 - t(1+t)^2 = 0.$$

All other (non-trivial) exactly known results for generating functions for directed animals on two-dimensional lattices (square, triangular, decorated square and triangular [11]) have been expressible as an algebraic equation, quadratic in the generating function. Furthermore, in each case the generating function diverges near the critical point $t = 1/\mu$ with a behaviour like $(1-\mu t)^{-\frac{1}{2}}$ for some lattice dependent μ. This means that the number of animals with k sites grows like $k^{-\frac{1}{2}}\mu^k$. Numerical analysis of other two-dimensional lattices, for which one does not have explicit generating functions, has indicated the same asymptotic behaviour (*e.g.* [9]).

This common behaviour indicates that all these models belong to the same universality class, and it might be expected that the directed animals on $\mathcal{L}_n$ exhibit the same behaviour despite the higher degree of the algebraic equation satisfied by S_n. This is true and can be proved as follows.

Let T_n be the series defined by

$$T_n = \frac{t - 2S_n}{1 + (n+1)S_n}. \tag{3}$$

Since $S_n = t + o(t)$, we have $T_n = -t + o(t)$. Moreover, (2) implies that $f_n(t) = f_n(T_n)$ where $f_n(u)$ is the polynomial $u^2(1+u)^{n-1}$. A study of f_n (see Figure 3) shows that T_n is an analytic function of t for $-2/(n+1) < t < t_n$ where t_n is the positive solution of

$$f_n(t_n) = f_n\left(-\frac{2}{n+1}\right) = 4\frac{(n-1)^{n-1}}{(n+1)^{n+1}}.$$

Moreover, T_n cannot be continuously defined in the neighbourhood of t_n. As $t_n < 2/(n+1)$, the smallest singularity of T_n is t_n. When $t \longrightarrow t_n^-$, then $T_n \longrightarrow -2/(n+1)^+$.

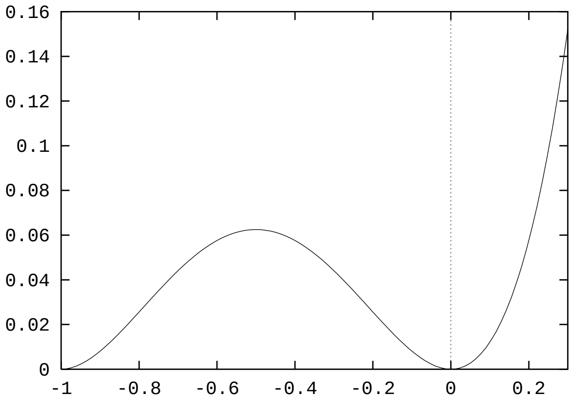

Figure 3: A typical ($n=3$) graph of $f_n(u) = u^2(1+u)^{n-1}$.

Inverting (3) leads to

$$S_n = \frac{t - T_n}{2 + (n+1)T_n},$$

which proves that t_n is also the smallest singularity of S_n. Applying Taylor's formula around $-2/(n+1)$ and t_n gives

$$f_n\left(v - \frac{2}{n+1}\right) = \frac{4(n-1)^{n-1}}{(n+1)^{n+1}} + \frac{v^2}{2} f_n''\left(-\frac{2}{n+1}\right) + o(v^2)$$

and

$$f_n(t_n - u) = \frac{4(n-1)^{n-1}}{(n+1)^{n+1}} - uf_n'(t_n) + o(u).$$

This shows that as $t \longrightarrow t_n^-$,

$$T_n = -\frac{2}{n+1} + \sqrt{\frac{-2f_n'(t)}{f_n''(-2/(n+1))}(t_n - t) + o(t_n - t)}.$$

Hence, up to a multiplicative constant (which can be made explicit),

$$S_n \sim (t_n - t)^{-1/2}$$

(meaning that $\sqrt{t_n - t}\, S_n$ tends to a constant when t tends to t_n). Thus the number of k-site animals on $\mathcal{L}_n$ is asymptotically equivalent to $A_n k^{-\frac{1}{2}} \mu_n^k$ as expected, with

$$\frac{(1+\mu_n)^{n-1}}{\mu_n^{n+1}} = 4\frac{(n-1)^{n-1}}{(n+1)^{n+1}}. \tag{4}$$

Let $v_n = \mu_n/(n+1)$. We can derive from (4) that $(v_n)_n$ is an increasing positive sequence. Let v be its limit in $\mathbb{R} \cup \{+\infty\}$. From (4) we derive $\exp(v^{-1} + 2) = 4v^2$, which gives $v = 1.79556...$ Hence $\mu_n \sim 1.79556...n$ as $n \longrightarrow \infty$.

3 Proof

We use a similar argument to that used by Dhar in [6]: we prove that the generating function S_n is the negative of the density of the hard particle model of activity $-t/(1+t)$ on the lattice formed with the first two rows of $\mathcal{L}_n$.

Let $n \geq 2$ be fixed. For $N \geq 2$, consider the lattice $\mathcal{L}_n^{(N)}$, which is similar to $\mathcal{L}_n$ but has a finite width N and cyclic boundary conditions. In other words, its set of vertices is $[N] \times \mathbb{N}$ where $[N] = \mathbb{Z}/N\mathbb{Z} = \{1, 2, \ldots, N\}$, and the edges still go from (i, j) to $(i + r, j + 1)$ for $0 \leq r < n$. The lattice $\mathcal{L}_3^{(8)}$ is drawn in Figure 4 (edges being oriented away from the center). As indicated on this figure, the vertices are labeled with $1, 2, \ldots, N$ on each row. A subset of vertices of a row will often be denoted by the set of corresponding labels.

We consider in this section animals that may have a *source* formed of several vertices. Let $C \subset [N]$ be a subset of vertices of the first row. A directed animal A *of source* C is a finite set of vertices containing C such that any vertex of A can be reached from a vertex of C through an oriented path having all its vertices in A (see Figure 4).

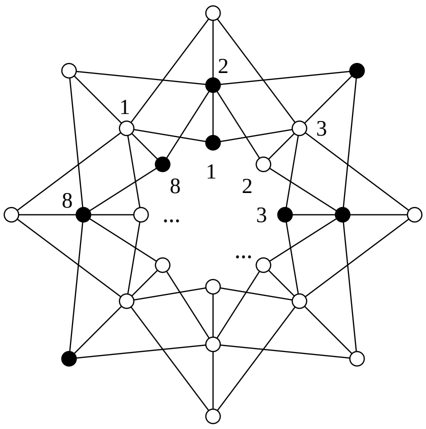

Figure 4: The cyclic lattice $\mathcal{L}_3^{(8)}$ and an animal of source $\{1, 3, 8\}$.

Let $S_C^{(N)}$ be the area generating function for animals of source C on $\mathcal{L}_n^{(N)}$. We clearly have

$$\lim_{N\to\infty} S_{\{1\}}^{(N)} = S_n \tag{5}$$

where S_n is the generating function for animals on $\mathcal{L}_n$, defined by (1). Moreover, removing the bottom row of an animal gives another animal. This remark provides a finite set of recurrence relations defining the series $S_C^{(N)}$, for $C \subset [N]$:

$$S_C^{(N)} = t^{|C|} \sum_{D\subset \mathcal{N}(C)} S_D^{(N)} \tag{6}$$

where $S_\emptyset^{(N)} = 1$ and $\mathcal{N}(C)$ is the set of "upper" neighbours of C:

$$\mathcal{N}(C) = \bigcup_{i\in C} \{i, i+1, \ldots, i+n-1\}.$$

(Remember that $i + N = i$ in $\mathbb{Z}/N\mathbb{Z}$.)

Now, consider the lattice $R_0 \cup R_1$ formed with the first two rows of $\mathcal{L}_n^{(N)}$ (see Figure 5), and suppose that a distribution of cells is given on the exterior row R_1. For $D \subset R_1$, let g_D be the probability that all the vertices of D are occupied, and let G_D be the probability that D is exactly the set of occupied vertices. From the cell distribution on R_1 we derive a cell distribution on the inner row R_0, obtained as follows: a vertex i lying on R_0 will be occupied by a cell

- with probability p if all its neighbours (on R_1) are empty ($0 < p < 1$),
- with probability 0 if at least one of its neighbours is occupied.

This transition is schematised in Figure 5, in which a black vertex denotes an occupied site.

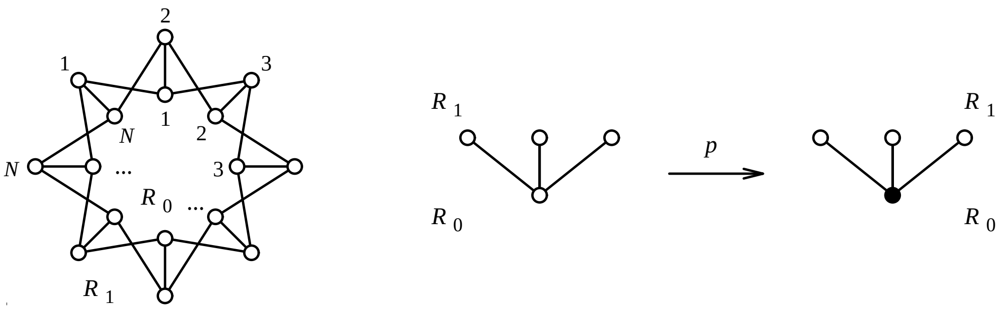

Figure 5: The lattice $R_0 \cup R_1$ and the transition from R_1 to R_0 (for $n = 3$).

For $C \subset R_0$, we define f_C to be the probability that the vertices of C are occupied, and F_C to be the probability that C is exactly the set of occupied vertices of R_0. We clearly have

$$f_C = p^{|C|}\mathrm{Prob}\,(\mathcal{N}(C)\ \text{ empty}).$$

Using the inclusion-exclusion principle, this can be rewritten as follows:

$$f_C = p^{|C|} \sum_{D\subset\mathcal{N}(C)} (-1)^{|D|} g_D. \tag{7}$$

We can also express F_C in terms of G_D:

$$F_C = \left(\frac{p}{1-p}\right)^{|C|} \sum_{D\subset R_1\setminus\mathcal{N}(C)} (1-p)^{N-|\bar{\mathcal{N}}(D)|} G_D \tag{8}$$

where $\bar{\mathcal{N}}(D)$ is the set of "lower" neighbours of D:

$$\bar{\mathcal{N}}(D) = \bigcup_{i \in D} \{i, i-1, \ldots, i-n+1\}.$$

Note that $|\bar{\mathcal{N}}(D)| = |\mathcal{N}(D)|$ for all $D \subset [N]$.

The cell distribution on R_1 is said to be *stationary* if it is the same as the induced cell distribution on R_0. The theory of Markov chains implies that our transition has a unique stationary distribution. For this distribution, Equation (7) becomes:

$$g_C = p^{|C|} \sum_{D \subset \mathcal{N}(C)} (-1)^{|D|} g_D.$$

Comparing this equation with (6) shows that, when $t = -p$,

$$S_C^{(N)} = (-1)^{|C|} g_C.$$

In particular, the generating function $S_{\{1\}}^{(N)}$ for one-source directed animals on $\mathcal{L}_n^{(N)}$ is the negative of the *density* of the stationary distribution (with $t = -p$). The density is, by definition, $\rho_N = g_{\{1\}}$. According to (5), the generating function for directed animals on $\mathcal{L}_n$ is

$$S_n = -\rho_\infty = -\lim_{N \to \infty} \rho_N.$$

The stationary distribution is easy to describe in this case: as in [6], it is the marginal distribution of the hard particle distribution of activity $p/(1-p)$ on $R_0 \cup R_1$. More precisely, one can easily check, using (8), that the distribution given by

$$G_D = \frac{1}{Z_N} \left(\frac{p}{1-p}\right)^{|D|} (1-p)^{|\bar{\mathcal{N}}(D)|}, \tag{9}$$

with

$$Z_N = \sum_{D \subset R_1} \left(\frac{p}{1-p}\right)^{|D|} (1-p)^{|\bar{\mathcal{N}}(D)|},$$

is stationary. Thus we need to compute the density ρ_N of this one-dimensional gas-model, or at least its limit ρ_∞.

We are actually going to solve a more general model, depending on two variables a and b, and given by

$$G_D = \frac{1}{Z_N} a^{|D|} b^{|\bar{\mathcal{N}}(D)|}.$$

The partition function is

$$Z_N = \sum_{D \subset [N]} a^{|D|} b^{|\bar{\mathcal{N}}(D)|},$$

and the density is

$$\rho_N = \frac{1}{N} \frac{1}{Z_N} \sum_D |D| a^{|D|} b^{|\bar{\mathcal{N}}(D)|} = \frac{a}{N Z_N} \frac{\partial Z_N}{\partial a}.$$

In what follows, the state of the vertices $i, i+1, \ldots, i+n-1$ is described by a vector $\sigma_i \in \{0,1\}^{n-1}$ for all $i \in [N]$. The partition function can then be rewritten as

$$Z_N = \sum_{\sigma_1, \ldots, \sigma_N} \left(\prod_{i=1}^{N} V(\sigma_i, \sigma_{i+1}) \right)$$

where σ_i runs over $\{0,1\}^{n-1}$ for all $i \leq N$, and $V = (V(\sigma,\tau))_{\sigma,\tau}$ is a square matrix defined as follows: if $\sigma = (s_1, \ldots, s_{n-1})$ and $\tau = (t_2, \ldots, t_n)$ (note the different numbering schemes), then

$$V(\sigma,\tau) = \begin{cases} 0 & \text{if } (s_2,\ldots,s_{n-1}) \neq (t_2,\ldots,t_{n-1}), \\ ab & \text{if } (s_2,\ldots,s_{n-1}) = (t_2,\ldots,t_{n-1}) \text{ and } s_1 = 1, \\ b & \text{if } (s_2,\ldots,s_{n-1}) = (t_2,\ldots,t_{n-1}),\ s_1 = 0 \text{ and } \tau \neq (0,0,\ldots,0), \\ 1 & \text{otherwise.} \end{cases}$$

Going back to Z_N, we have

$$Z_N = \operatorname{tr}\left(V^N\right) = \lambda_1^N + \cdots + \lambda_{2^{n-1}}^N$$

where $\lambda_1, \ldots, \lambda_{2^{n-1}}$ are the eigenvalues of the matrix V. The density of the model is thus

$$\rho_N = a \frac{\sum_i \lambda_i^{N-1} \lambda_i'}{\sum_i \lambda_i^N}$$

where λ_i' denotes $\partial \lambda_i / \partial a$. When N tends to infinity, it tends to

$$\rho_\infty = a \frac{\lambda'}{\lambda} \tag{10}$$

where λ is the dominant eigenvalue of V. The characteristic polynomial of V, denoted $P(x)$, can be calculated exactly:

$$P(x) = x^{2^{n-1}-n} \left(x^n - x^{n-1}(1+ab) + a(1-b) \sum_{k=0}^{n-2} x^k b^{n-1-k} \right).$$

Equivalently,

$$P(x) = \frac{x^{2^{n-1}-n}}{b-x} \left[ab^n(1-b) - x^{n-1}(x-1)(x-b-ab) \right].$$

Thus the dominant eigenvalue λ of V satisfies $\lambda \neq b$ and

$$ab^n(1-b) = \lambda^{n-1}(\lambda-1)(\lambda-b-ab).$$

Differentiating with respect to a leads to

$$\lambda' = \frac{\lambda(\lambda-1)(\lambda-b)}{a\left[(n-1)(\lambda-1)(\lambda-b-ab) + \lambda(2\lambda-1-b-ab)\right]}. \tag{11}$$

We finally obtain the limit of the density by combining (10) and (11):

$$\rho_\infty = \frac{(\lambda-1)(\lambda-b)}{(n-1)(\lambda-1)(\lambda-b-ab) + \lambda(2\lambda-1-b-ab)}.$$

In particular, the density of the model given by (9) tends to

$$\rho_\infty = \frac{\lambda+p-1}{(n+1)\lambda-(n-1)}$$

when $N \to \infty$, where λ satisfies

$$p^2(1-p)^{n-1} = \lambda^{n-1}(\lambda-1)^2.$$

As $S_n = -\rho_\infty$ when $t = -p$, Equation (2) follows.

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