A refinement of the Lecture Hall Theorem

Mireille Bousquet-Mélou
LaBRI, Université Bordeaux 1
351 cours de la Libération
33405 Talence Cedex, FRANCE
bousquet@labri.u-bordeaux.fr

Kimmo Eriksson
Department of Mathematics
KTH
S-100 44 Stockholm, SWEDEN
kimmo@nada.kth.se

Abstract

For $n \geq 1$, let $\mathcal{L}_n$ be the set of lecture hall partitions of length n, that is, the set of n-tuples of integers $\lambda = (\lambda_1, \ldots, \lambda_n)$ satisfying $0 \leq \frac{\lambda_1}{1} \leq \frac{\lambda_2}{2} \leq \cdots \leq \frac{\lambda_n}{n}$. Let $\lceil\lambda\rceil$ be the partition $(\lceil\lambda_1/1\rceil, \ldots, \lceil\lambda_n/n\rceil)$, and let $o(\lceil\lambda\rceil)$ denote the number of its odd parts. We show that the identity

$$\sum_{\lambda \in \mathcal{L}_n} q^{|\lambda|} u^{|\lceil\lambda\rceil|} v^{o(\lceil\lambda\rceil)} = \frac{(1+uvq)(1+uvq^2)\cdots(1+uvq^n)}{(1-u^2q^{n+1})(1-u^2q^{n+2})\cdots(1-u^2q^{2n})} \tag{1}$$

is equivalent to a refinement of Bott's formula for the affine Coxeter group $\widetilde{C}_n$, obtained by Macdonald [7] and Reiner [8]. The case $u = v = 1$ of the identity (1), called the Lecture Hall Theorem, was proved by us in [3], and then by Andrews [2].

In the present paper, we give two direct proofs of (1). The first one is rather short, but requires a little bit of q-calculus; the second one is the first truly bijective proof ever found in the domain of lecture hall partitions. Although we describe our bijection in completely combinatorial terms, it finds its origin in the algebraic context of Coxeter groups.

Both proofs are completely independent from all earlier proofs of the Lecture Hall Theorem.

1 Introduction

A *lecture hall partition* of length n is an integer sequence $\lambda = (\lambda_1, \ldots, \lambda_n)$ satisfying

$$0 \leq \frac{\lambda_1}{1} \leq \frac{\lambda_2}{2} \leq \cdots \leq \frac{\lambda_n}{n}.$$

We denote by $\mathcal{L}_n$ the set of lecture hall partitions of length n. These partitions, introduced in [3], have the following remarkable generating function:

$$\sum_{\lambda \in \mathcal{L}_n} q^{|\lambda|} = \frac{1}{(1-q)(1-q^3)\cdots(1-q^{2n-1})}, \tag{2}$$

where $|\lambda| = \lambda_1 + \cdots + \lambda_n$ is the *weight* of the partition λ. In other words, any integer has as many lecture hall partitions of length n as partitions into odd parts smaller than $2n$. This formula was first proved by Bousquet-Mélou and Eriksson [3] in two different ways: directly, and via Bott's formula for the Poincaré series of the affine Weyl group $\widetilde{C}_n$, which turns out to be equivalent to (2). A third proof was given by Andrews [2].

In 1995, Vic Reiner mentioned to us a three-variable refinement of Bott's formula, and asked whether it had a lecture hall counterpart. The present paper answers this question positively in the following terms.

Theorem 1.1 [Refined Lecture Hall Theorem] *Given a lecture hall partition λ, let $\lceil\lambda\rceil$ denote the partition $(\lceil\lambda_1/1\rceil,\ldots,\lceil\lambda_n/n\rceil)$. Furthermore, for any partition μ, let $o(\mu)$ denote the number of its odd parts. Then:*

$$\sum_{\lambda\in\mathcal{L}_n} q^{|\lambda|}u^{|\lceil\lambda\rceil|}v^{o(\lceil\lambda\rceil)} = \frac{(1+uvq)(1+uvq^2)\cdots(1+uvq^n)}{(1-u^2q^{n+1})(1-u^2q^{n+2})\cdots(1-u^2q^{2n})}.$$

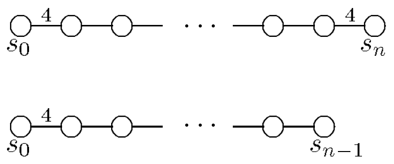
Again we will present two proofs, which are completely different from all earlier proofs of the old theorem. Let us start with an introduction to these proofs.

1.1 A direct proof (Section 2)

The statement of Theorem 1.1 raises the following question: given a partition $\mu=(\mu_1,\ldots,\mu_n)$, how many lecture hall partitions λ are there such that $\lceil\lambda\rceil=\mu$? More precisely, what is their weight generating function? It turns out that this question has a very simple answer (Proposition 2.2), which is the first step of our proof. The second step consists in summing on all possible μ, and resorts to a small amount of q-calculus.

1.2 A bijective proof inspired by algebra (Section 3)

Let us discuss how the refined lecture hall theorem was inspired by the refined version of Bott's formula mentioned to us by Vic Reiner (for general Coxeter group theory, see Humphreys [6]). The Coxeter group $\widetilde{C}_n$ and its parabolic finite subgroup C_n are respectively defined by the following Coxeter graphs:



The (refined) Poincaré series of $\widetilde{C}_n$ is:

$$\widetilde{C}_n(q,a,b) \overset{\text{def}}{=} \sum_{w\in\widetilde{C}_n} q^{\ell(w)}a^{\alpha(w)}b^{\beta(w)} = C_n(q,a)\frac{(1+bq)(1+bq^2)\cdots(1+bq^n)}{(1-abq^{n+1})(1-abq^{n+2})\cdots(1-abq^{2n})}, \tag{3}$$

where $\ell(w)$ is the *length* of w (i.e. the number of generators in a reduced word for w), $\alpha(w)$ and $\beta(w)$ are the number of occurrences of the generators s_0 and s_n respectively in a reduced word for w, and

$$C_n(q,a) \overset{\text{def}}{=} \sum_{w\in C_n} q^{\ell(w)}a^{\alpha(w)} = (1+aq)(1+aq^2)\cdots(1+aq^n)(1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}).$$

Formula (3) was proved by Macdonald [7] and later, independently, by Reiner [8]. Let $\widetilde{C}_n/C_n$ denote the set of minimal coset representatives. Then we can reformulate the above formula as

$$\widetilde{C}_n/C_n(q,a,b) \overset{\text{def}}{=} \sum_{w\in\widetilde{C}_n/C_n} q^{\ell(w)}a^{\alpha(w)}b^{\beta(w)} = \frac{(1+bq)(1+bq^2)\cdots(1+bq^n)}{(1-abq^{n+1})(1-abq^{n+2})\cdots(1-abq^{2n})}, \tag{4}$$

which after substitutions $ab=u^2$ and $b=uv$ is equal to the righthand expression in the refined lecture hall theorem.

In [3], we presented a bijection $h:\widetilde{C}_n/C_n\to\mathcal{L}_n$ that takes the length of the representative w to the weight of its image $\lambda=h(w)$, thereby proving equivalence between the lecture hall theorem and Bott's formula (corresponding to $a = b = 1$ in (3)). Later, when looking at Macdonald's and Reiner's refinement, we found that our bijection took $\alpha(w)$ to $(|\lceil\lambda\rceil| - o(\lceil\lambda\rceil))/2$ and $\beta(w)$ to $(|\lceil\lambda\rceil| + o(\lceil\lambda\rceil))/2$, thereby proving equivalence between the refined theorems.

Moreover, Eriksson and Eriksson [5] recently gave a bijective proof of (4) by presenting a bijection $c : \mathcal{P}_n \to \widetilde{C}_n/C_n$, where $\mathcal{P}_n$ is the set of integer partitions ρ into distinct parts between 1 and n and arbitrarily many parts between $n+1$ and $2n$. This bijection takes the number of parts of ρ to $\beta(w)$, where $w = c(\rho)$, and the number of big parts of ρ (i.e. greater than n) to $\alpha(w)$.

Thus, the composition $f = h \circ c$ is a bijection between $\mathcal{P}_n$ and $\mathcal{L}_n$, explaining the refined lecture hall theorem. We have found that we can get rid of the Coxeter groups in-between. In Section 3 we describe combinatorially the map f and prove, combinatorially as well, that it is a bijection.

$$\begin{tikzcd} & \widetilde{C}_n/C_n \arrow[dr, "h"] & \\ \mathcal{P}_n \arrow[ur, "c"] \arrow[rr, "f = h \circ c"] & & \mathcal{L}_n \end{tikzcd}$$

2 A direct proof of the refined Lecture Hall Theorem

The statement of Theorem 1.1 suggests to study the following question: given a partition $\mu = (\mu_1, \ldots, \mu_n)$, how many lecture hall partitions λ such that $\lceil\lambda\rceil = \mu$ are there? More precisely, what is their weight generating function? It turns out that this question has a very nice answer, given by Proposition 2.2 below. It is amusing to note that this is somehow "the result we missed" when we first met lecture hall partitions, as we studied without success the analogous question obtained with $\lfloor\lambda\rfloor = \mu$ (with obvious notation), which has no simple answer...

Notations. We adopt the standard notations of q-calculus. If a and q are two indeterminates, and $n \geq 1$, then

$$(a;q)_n = (a)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}).$$

By convention, $(a)_0 = 1$. For $0 \leq k \leq n$, the q-analog of the binomial coefficient $\binom{n}{k}$ is

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q)_n}{(q)_k (q)_{n-k}}.$$

More generally, let $n \geq 0$, and let $(n_i)_{i\geq 0} = (n_0, n_1, n_2, \ldots)$ be a sequence of nonnegative integers summing to n. We denote by $\begin{bmatrix} n \\ (n_i)_{i\geq 0} \end{bmatrix}_q$ the q-analog of the multinomial coefficient $\begin{pmatrix} n \\ (n_i)_{i\geq 0} \end{pmatrix}$:

$$\begin{bmatrix} n \\ n_0, n_1, \ldots \end{bmatrix} = \begin{bmatrix} n \\ (n_i)_{i\geq 0} \end{bmatrix}_q = \frac{(q)_n}{\prod_i (q)_{n_i}}.$$

Actually, the analogs of multinomial coefficients that arise naturally in this paper are rather "$1/q$-analogs"; they are related to q-analogs by:

$$\begin{bmatrix} n \\ (n_i)_{i\geq 0} \end{bmatrix}_{1/q} = q^{-\sum_{i<j} n_i n_j} \begin{bmatrix} n \\ (n_i)_{i\geq 0} \end{bmatrix}_q.$$

Lemma 2.1 *For an n-tuple of nonnegative integers $\lambda = (\lambda_1, \dots, \lambda_n)$, let us define $\mu = (\mu_1, \dots, \mu_n)$ and $(r_1, \dots, r_n)$ by*

$$\forall i, \quad \lambda_i = i\mu_i - r_i \quad \text{and} \quad 0 \le r_i < i.$$

In other words, $\mu_i = \lceil \lambda_i / i \rceil$. Then λ is a lecture hall partition if and only if

$$0 \le \mu_1 \le \cdots \le \mu_n \quad \text{and} \quad r_{i+1} \le r_i \quad \text{whenever} \quad \mu_i = \mu_{i+1}.$$

Proof. Clearly, μ must be a partition. Moreover, the conditions for λ to be a lecture hall partition give:

$$\frac{r_{i+1}}{i+1} - \frac{r_i}{i} \le \mu_{i+1} - \mu_i.$$

This is satisfied without any additional condition on r_i and r_{i+1} if $\mu_i < \mu_{i+1}$. If $\mu_i = \mu_{i+1}$, we want

$$r_{i+1} \le \frac{i+1}{i} r_i = r_i + \frac{r_i}{i}.$$

As $r_i < i$, this is equivalent to $r_{i+1} \le r_i$.

■

Proposition 2.2 *Let $\mu = (\mu_1, \dots, \mu_n)$ be a partition of length n having n_i parts equal to i for $i \ge 0$. Let L_μ be the weight generating function for lecture hall partitions λ such that $\lceil \lambda \rceil = \mu$:*

$$L_\mu = \sum_{\lambda \in \mathcal{L}_n : \lceil \lambda \rceil = \mu} q^{|\lambda|}.$$

This polynomial has the following simple expression:

$$L_\mu = q^{\sum i\mu_i} \begin{bmatrix} n \\ (n_i)_{i \ge 0} \end{bmatrix}_{1/q}. \tag{5}$$

Proof. The fact that $\lceil \lambda \rceil = \mu$ means that there exist n integers $r_1, \dots, r_n$, with $0 \le r_i < i$, such that:

$$\forall i, \quad \lambda_i = i\mu_i - r_i.$$

The weight of λ is $\sum i\mu_i - \sum r_i$, and the r_i must satisfy the conditions of Lemma 2.1. The parts μ_i and μ_{i+1} are equal if and only if there exists an integer $k \ge 0$ and an integer $p \in [1, n_k - 1]$ such that $i = n_0 + n_1 + \cdots + n_{k-1} + p$. Hence, the condition on the r_i for λ to be a lecture hall partition is:

$$\forall\, k \ge 0, \quad 0 \le r_{n_0+n_1+\cdots+n_k} \le r_{n_0+n_1+\cdots+n_k-1} \le \cdots \le r_{n_0+n_1+\cdots+n_{k-1}+1} \le n_0 + n_1 + \cdots + n_{k-1}.$$

This condition is empty if $n_k = 0$, and gives $0 = r_{n_0} = \cdots = r_1$ for $k = 0$. The following well-known result (enumeration of partitions contained in a box of size $a \times b$):

$$\sum_{0 \le s_a \le s_{a-1} \le \cdots \le s_1 \le b} q^{-(s_1 + \cdots + s_a)} = \begin{bmatrix} a+b \\ a \end{bmatrix}_{1/q}$$

leads to the announced expression of L_μ.

■

Before going further, we note that the simplicity of this result enables us to enumerate lecture hall partitions λ according to the *complete structure* of $\lceil\lambda\rceil$, as shown by the following proposition.

Proposition 2.3 *Let $S_n(X_0, X_1, X_2, \ldots)$ be the following power series:*

$$S_n(X_0, X_1, X_2, \ldots) = \sum_{\lambda \in \mathcal{L}_n} X_{\lceil \frac{\lambda_1}{1} \rceil} X_{\lceil \frac{\lambda_2}{2} \rceil} \ldots X_{\lceil \frac{\lambda_n}{n} \rceil}.$$

Then

$$S_n(X_0, X_1, X_2, \ldots) = (X_0 + X_1 + X_2 + \cdots)^n.$$

Proof. We have:

$$\begin{aligned} S_n(X_0, X_1, X_2, \ldots) &= \sum_{\mu} \sum_{\lambda \in \mathcal{L}_n : \lceil\lambda\rceil = \mu} X_{\mu_1} X_{\mu_2} \ldots X_{\mu_n} \\ &= \sum_{(n_i)_{i\geq 0} : \sum_i n_i = n} \binom{n}{(n_i)_{i\geq 0}} X_0^{n_0} X_1^{n_1} \ldots X_i^{n_i} \ldots \\ &= (X_0 + X_1 + X_2 + \cdots)^n. \end{aligned}$$

■

Corollary 2.4 *We have proved the specialization $q = 1$ of Theorem 1.1:*

$$\sum_{\lambda \in \mathcal{L}_n} u^{|\lceil\lambda\rceil|} v^{o(\lceil\lambda\rceil)} = \frac{(1+uv)^n}{(1-u^2)^n}.$$

Proof. In Proposition 2.3, replace X_{2i} by u^{2i} and X_{2i+1} by $u^{2i+1}v$.

■

Let us now proceed on the way to the proof of Theorem 1.1. Thanks to Proposition 2.2, we have reduced it to the following identity:

$$\sum_{\mu = (\mu_1, \ldots, \mu_n)} u^{|\mu|} v^{o(\mu)} L_\mu = \frac{(-uvq)_n}{(u^2 q^{n+1})_n},$$

where L_μ is the polynomial in q given by (5). We begin with a proposition that explains combinatorially the form of the numerator $(-uvq)_n$. Its statement is very similar to that of Proposition 2.2.

Notation. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a lecture hall partition. We denote by $\lceil\lambda\rceil_\bullet$ the "half" of the partition $\lceil\lambda\rceil$. More precisely:

$$\lceil\lambda\rceil_\bullet = \left(\left\lfloor \frac{1}{2} \left\lceil \frac{\lambda_1}{1} \right\rceil \right\rfloor, \left\lfloor \frac{1}{2} \left\lceil \frac{\lambda_2}{2} \right\rceil \right\rfloor, \ldots, \left\lfloor \frac{1}{2} \left\lceil \frac{\lambda_n}{n} \right\rceil \right\rfloor \right).$$

Proposition 2.5 *Let $\eta = (\eta_1, \ldots, \eta_n)$ be a partition of length n. Let $\tilde{L}_\eta$ be the generating function for lecture hall partitions λ such that $\lceil\lambda\rceil_\bullet = \eta$:*

$$\tilde{L}_\eta = \sum_{\lambda \in \mathcal{L}_n : \lceil\lambda\rceil_\bullet = \eta} q^{|\lambda|} u^{|\lceil\lambda\rceil|} v^{o(\lceil\lambda\rceil)}.$$

This polynomial has the following simple expression:

$$\tilde{L}_\eta = (-uvq)_n u^{2|\eta|} L_{2\eta}$$

where 2η is the partition $(2\eta_1, \ldots, 2\eta_n)$ and L_μ is the polynomial in q defined in Proposition 2.2.

Proof. We shall use the famous q-binomial theorem [1, p.36]:

$$(1+x)(1+xq)\cdots(1+xq^{m-1}) = \sum_{k=0}^{m} x^k q^{\binom{k}{2}} \begin{bmatrix} m \\ k \end{bmatrix}_q .$$

It can be easily proved by counting partitions into distinct parts smaller than m.

Suppose that the partition η has m_i parts equal to i, for $i \geq 0$. Let λ be a lecture hall partition such that $\lceil \lambda \rceil_\bullet = \eta$. If the partition $\mu = \lceil \lambda \rceil = (\mu_1, \ldots, \mu_n)$ has n_i parts equal to i, then $n_{2i} + n_{2i+1} = m_i$. Moreover, we have

$$|\mu| = 2|\eta| + \sum_i n_{2i+1},$$

$$o(\mu) = \sum_i n_{2i+1}$$

and

$$\sum_i i\mu_i = 2\sum_i i\eta_i + \sum_i (m_0 + \cdots + m_i) n_{2i+1} - \sum_i \binom{n_{2i+1}}{2}.$$

Thus:

$$\begin{aligned}
\tilde{L}_\eta &= \sum_{\mu : \lfloor \mu_i/2 \rfloor = \eta_i} u^{|\mu|} v^{o(\mu)} L_\mu \\
&= u^{2|\eta|} q^{2\sum i\eta_i} \sum_{(n_i): n_{2i}+n_{2i+1}=m_i} (uv)^{\sum n_{2i+1}} q^{\sum (m_0+\cdots+m_i) n_{2i+1} - \sum_i \binom{n_{2i+1}}{2}} \begin{bmatrix} n \\ (n_i)_{i\geq 0} \end{bmatrix}_{1/q} \\
&= u^{2|\eta|} q^{2\sum i\eta_i} \begin{bmatrix} n \\ (m_i)_{i\geq 0} \end{bmatrix}_{1/q} \prod_i \sum_{n_{2i+1}=0}^{m_i} (uvq^{m_0+\cdots+m_i})^{n_{2i+1}} q^{-\binom{n_{2i+1}}{2}} \begin{bmatrix} m_i \\ n_{2i+1} \end{bmatrix}_{1/q} \\
&= u^{2|\eta|} q^{2\sum i\eta_i} \begin{bmatrix} n \\ (m_i)_{i\geq 0} \end{bmatrix}_{1/q} \prod_{i\geq 0} (1+uvq^{m_0+\cdots+m_i}) \cdots (1+uvq^{m_0+\cdots+m_{i-1}+1}) \\
&= u^{2|\eta|} q^{2\sum i\eta_i} \begin{bmatrix} n \\ (m_i)_{i\geq 0} \end{bmatrix}_{1/q} (1+uvq)(1+uvq^2)\cdots(1+uvq^n) \\
&= (-uvq)_n u^{2|\eta|} L_{2\eta}.
\end{aligned}$$

We have used the q-binomial theorem (actually, the $1/q$-binomial theorem) to derive the fourth identity.

■

We now come to the last step of our calculation. Proposition 2.5 reduces Theorem 1.1 to the following identity:

$$\Phi_n(u,q) \stackrel{\text{def}}{=} \sum_{\eta=(\eta_1,\ldots,\eta_n)} u^{2|\eta|} L_{2\eta} = \frac{1}{(u^2q^{n+1})_n},$$

where L_μ is the polynomial in q given by (5). Note that $\Phi_n(u,q)$ is the generating function for lecture hall partitions λ such that $\lceil \lambda \rceil$ has only even parts.

We are going to write a recurrence relation for $\Phi_n(u,q)$ by removing the zero parts from η. Suppose that η has exactly m non-zero parts ($m = m_1 + m_2 + \cdots$) and let $\gamma = (\gamma_1, \ldots, \gamma_m)$ be the partition of length m defined by $\gamma_{m-i} = \eta_{n-i} - 1$ for $0 \leq i < m$. Then for $i \geq 0$, γ has p_i parts equal to i, with $p_i = m_{i+1}$. Moreover, we have the following identities:

$$|\eta| = |\gamma| + m,$$

$$\sum_i i\eta_i = \frac{m(2n+1-m)}{2} + (n-m)|\gamma| + \sum_i i\gamma_i,$$

and

$$\begin{bmatrix} n \\ (m_i)_{i\geq 0} \end{bmatrix}_{1/q} = \begin{bmatrix} n \\ m \end{bmatrix}_{1/q} \begin{bmatrix} m \\ (p_i)_{i\geq 0} \end{bmatrix}_{1/q}.$$

This gives

$$\begin{aligned} \Phi_n(u,q) &= \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_{1/q} u^{2m} q^{m(2n+1-m)} \sum_{\gamma=(\gamma_1,\ldots,\gamma_m)} (uq^{n-m})^{2|\gamma|} q^{2\sum_i i\gamma_i} \begin{bmatrix} m \\ (p_i)_{i\geq 0} \end{bmatrix}_{1/q} \\ &= \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_{1/q} u^{2m} q^{m(2n+1-m)} \Phi_m(uq^{n-m}, q), \end{aligned}$$

with the convention $\Phi_0(u,q) = 1$. By induction, this formula completely defines the series $\Phi_n(u,q)$. We wish to prove that $1/(u^2q^{n+1})_n$ satisfies the same formula. Denoting $x = u^2q^n$ and using $\left[{n \atop m}\right]_{1/q} = q^{-m(n-m)}\left[{n \atop m}\right]_q$, we end up with the following identity:

$$\frac{1}{(xq)_n} = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{x^m q^m}{(xq^{n-m+1})_m}.$$

This can be easily proved by counting partitions $\mu = (\mu_1, \ldots, \mu_p)$ such that $1 \leq \mu_1 \leq \ldots \leq \mu_p \leq n$, classifying them according to the number m of parts μ_i such that $\mu_i + i \leq n+1$. Alternatively, it is a consequence of the q-analog of the Chu-Vandermonde summation [1, p. 37].

This concludes our first proof of the refined lecture hall theorem.

■

3 A bijective proof of the refined Lecture Hall Theorem

Let $\mathcal{P}_n$ be the set of integer partitions into distinct parts between 1 and n and arbitrarily many parts between $n+1$ and $2n$. As before, let $\mathcal{L}_n$ be the set of lecture hall partitions of length n. The generating function identity

$$\sum_{\lambda\in\mathcal{L}_n} q^{|\lambda|} u^{|\lceil\lambda\rceil|} v^{o(\lceil\lambda\rceil)} = \frac{(1+uvq)(1+uvq^2)\cdots(1+uvq^n)}{(1-u^2q^{n+1})(1-u^2q^{n+2})\cdots(1-u^2q^{2n})}$$

says that there is a bijective correspondence $f : \mathcal{P}_n \longrightarrow \mathcal{L}_n$ between partitions in $\mathcal{P}_n$ and $\mathcal{L}_n$ such that if $f(\rho) = \lambda$, then

- the weight is preserved, i.e. $|\rho| = |\lambda|$;
- every part in $\{n+1, \ldots, 2n\}$ in ρ contributes with two to the weight of $\lceil\lambda\rceil$;
- every part in $\{1, \ldots, n\}$ in ρ contributes with one to the weight of $\lceil\lambda\rceil$, and the number of these "small" parts is the number of odd parts in $\lceil\lambda\rceil$.

In this perspective, it seems a natural idea that the correspondence works inductively, so that if we add a new largest part (greater than n) to ρ, the only effect on the partition $\lceil\lambda\rceil$ would be that one of its parts is increased by two, the weight of λ being increased by the size of the new part of ρ.

We shall now present a bijection $f : \mathcal{P}_n \longrightarrow \mathcal{L}_n$ with the above properties.

3.1 Pictorial description of the bijection $f : \mathcal{P}_n \longrightarrow \mathcal{L}_n$

Our bijection works by induction on the number of parts larger than n in $\rho \in \mathcal{P}_n$. Actually, what we construct is an n by n triangle $T(\rho)$ of integers, whose diagonal elements are the parts of $\lceil\lambda\rceil$ and whose column sums are the parts of the lecture hall partition λ:

$$T(\rho) = \begin{array}{cccc} t_{11} & t_{12} & \dots & t_{1n} \\ & t_{22} & \dots & t_{2n} \\ & & \ddots & \vdots \\ & & & t_{nn} \end{array}$$

where $\lceil\lambda\rceil = (t_{11}, t_{22}, \dots, t_{nn})$ and $\lambda_i = t_{1i} + t_{2i} + \dots + t_{ii}$.

Let $\rho = (\rho_1, \dots, \rho_r)$ where the parts come in increasing order. If no part of ρ is larger than n, let $T(\rho)$ be the n by n triangle of zeros and ones where ρ_{r-k} ones are placed at the bottom of column $n-k$ for $k = 0, \dots, r-1$.

Otherwise, let $\rho' = (\rho_1, \dots, \rho_{r-1})$. From the triangle $t' = T(\rho')$ and the part ρ_r we construct the triangle $t = T(\rho)$ as follows. We know that $\rho_r = n + m$ for some $m \in \{1, \dots, n\}$. Remove the mth *hook* (i.e., the mth row and column) from $T(\rho')$, yielding an $n-1$ by $n-1$ triangle. Add a new nth column by setting $t_{nn} = t'_{mm} + 2$, $t_{in} = t'_{im} + 2$ for $i = 1, \dots, m-1$, and $t_{in} = t'_{m,i+1} + 1$ for $i = m, \dots, n-1$. The transformation of t' into t is schematized in Fig. 1.

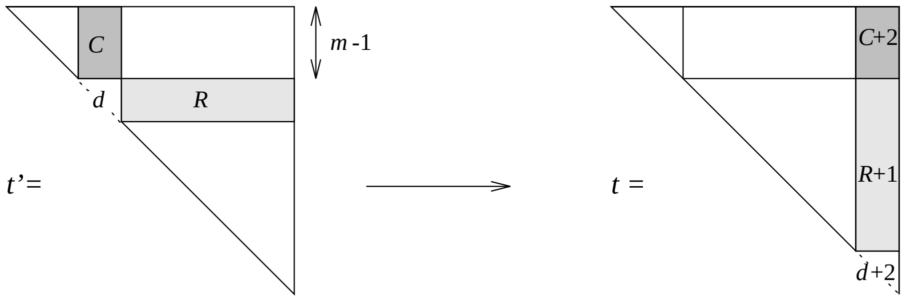


Figure 1: Straightening of the mth hook.

Example
Let $n = 4$ and let $\rho = (2, 4, 5, 6, 6, 8) \in \mathcal{P}_4$. We successively construct the triangles associated with $\rho^{(0)} = (2, 4)$, $\rho^{(1)} = (2, 4, 5)$, $\rho^{(2)} = (2, 4, 5, 6)$, and so on until we reach $\rho = \rho^{(4)}$. We first obtain, for $\rho^{(0)}$,

$$T(\rho^{(0)}) = \begin{array}{cccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ & 0 & 1 & 1 \\ & & 1 & 1 \\ & & & 1 \end{array}, \qquad \lambda^{(0)} = (0, 0, 2, 4), \quad \lceil\lambda^{(0)}\rceil = (0, 0, 1, 1).$$

Since the next part in ρ is $\rho_3 = 5 = 4 + 1$, we have boldfaced row 1 and column 1 in the above picture to signal that they will be removed and replaced in Step 1:

$$T(\rho^{(1)}) = \begin{array}{cccc} 0 & \mathbf{1} & 1 & 1 \\ & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ & & 1 & 2 \\ & & & 2 \end{array}, \qquad \lambda^{(1)} = (0, 2, 3, 6), \quad \lceil\lambda^{(1)}\rceil = (0, 1, 1, 2).$$

In Step 2 we use $\rho_4 = 6 = 4 + 2$. Hence we now do the manoeuvre with row 2 and column 2:

$$T(\rho^{(2)}) = \begin{array}{cccc} 0 & \mathbf{1} & 1 & 3 \\ & \mathbf{1} & \mathbf{2} & \mathbf{2} \\ & & 2 & 2 \\ & & & 3 \end{array}, \qquad \lambda^{(2)} = (0, 2, 5, 10), \quad \lceil \lambda^{(2)} \rceil = (0, 1, 2, 3).$$

In Step 3 we use $\rho_5 = 6 = 4 + 2$, so once again we work with row 2 and column 2:

$$T(\rho^{(3)}) = \begin{array}{cccc} 0 & 1 & 3 & \mathbf{3} \\ & 2 & 2 & \mathbf{3} \\ & & 3 & \mathbf{3} \\ & & & \mathbf{3} \end{array}, \qquad \lambda^{(3)} = (0, 3, 8, 12), \quad \lceil \lambda^{(3)} \rceil = (0, 2, 3, 3).$$

Finally, in Step 4 we have $\rho_6 = 8 = 4 + 4$. We remove row 4 and column 4:

$$T(\rho) = T(\rho^{(4)}) = \begin{array}{cccc} 0 & 1 & 3 & 5 \\ & 2 & 2 & 5 \\ & & 3 & 5 \\ & & & 5 \end{array}, \qquad \lambda = \lambda^{(4)} = (0, 3, 8, 20), \quad \lceil \lambda \rceil = (0, 2, 3, 5).$$

Note that it is not necessary to evaluate $\lambda^{(i)}$ at each step.

The rest of this section is devoted to the proof of the following theorem, which implies Theorem 1.1.

Theorem 3.1 *Let $\mathcal{P}_n$ be the set of integer partitions into distinct parts between 1 and n and arbitrarily many parts between $n+1$ and $2n$. The correspondence*

$$\begin{array}{rccl} f: & \mathcal{P}_n & \longrightarrow & \mathbb{N}^n \\ & \rho & \longmapsto & \lambda = (\lambda_1, \ldots, \lambda_n) \end{array}$$

defined by $\lambda_i = t_{1i} + \cdots + t_{ii}$, where $(t_{ij}) = T(\rho)$, induces a bijection (also denoted f) from $\mathcal{P}_n$ to the set $\mathcal{L}_n$ of lecture hall partitions. Moreover, $\lceil \lambda \rceil = (t_{11}, \ldots, t_{nn})$. Finally, let $\rho^{(0)}$ be obtained by removing from ρ all parts larger than n; then

$$|\lambda| = |\rho|, \qquad |\lceil \lambda \rceil| = 2\ell(\rho) - \ell(\rho^{(0)}) \qquad \text{and} \qquad o(\lceil \lambda \rceil) = \ell(\rho^{(0)}), \tag{6}$$

where the length (i.e. the number of parts) of a partition μ is denoted $\ell(\mu)$.

In subsection 3.2, we shall prove that $\lambda = f(\rho)$ is a lecture hall partition. In subsection 3.3, we shall prove that every lecture hall partition is obtained exactly once and show how to reverse the construction. There is one simple statement in Theorem 3.1: properties (6), which we can prove at once. They are implied by the following lemma.

Lemma 3.2 *Let ρ be a partition of $\mathcal{P}_n$. Let $\rho^{(0)}$ be obtained by removing from ρ all parts larger than n. Let $t = T(\rho)$. We have:*

$$\sum_{1 \le i \le j \le n} t_{ij} = |\rho|, \qquad \sum_{1 \le i \le n} t_{ii} = 2\ell(\rho) - \ell(\rho^{(0)}), \quad \text{and} \quad |\{i : t_{ii} \text{ is odd }\}| = \ell(\rho^{(0)}).$$

Proof. We prove these properties by induction on the number of parts larger than n in ρ, using the recursive definition of the correspondence T.

If ρ has no part larger than n, then all properties are immediate. Otherwise, with the notations used above, we assume that $t' = T(\rho')$ has the desired properties. We observe that

$$\sum_{1 \le i \le j \le n} t_{ij} = \sum_{1 \le i \le j \le n} t'_{ij} + n + m = |\rho'| + \rho_r = |\rho|,$$

and

$$\sum_{1 \le i \le n} t_{ii} = \sum_{1 \le i \le n} t'_{ii} + 2 = 2\ell(\rho') - \ell(\rho^{(0)}) + 2 = 2\ell(\rho) - \ell(\rho^{(0)}).$$

Finally, we note that the number of odd diagonal entries does not change when we transform t' into t.

■

Remark. We have introduced intermediate objects (triangles) between partitions of $\mathcal{P}_n$ and lecture hall partitions. This might seem a bit mysterious. However, these triangles have a simple meaning in terms of Coxeter groups: as mentioned in the introduction, our bijection f is the composition $h \circ c$ of two bijections, where c maps $\mathcal{P}_n$ onto $\widetilde{C}_n/C_n$ and h maps $\widetilde{C}_n/C_n$ onto $\mathcal{L}_n$. Given $\rho \in \mathcal{P}_n$, the triangle $T(\rho)$ gives the *inversion counts* (as defined in [3]) of the element $c(\rho)$.

3.2 T-triangles

Notation. We use the symbol $\equiv$ to denote the congruence of integers modulo 2.

Definition 3.3 *A* T-triangle *of size n is a triangular array $t = (t_{ij})_{1 \le i \le j \le n}$ of nonnegative integers satisfying the four following properties.*

P1 *In every column, all entries are either equal to the diagonal element or one less: $t_{jj} - t_{ij} \in \{0, 1\}$ for all indices $i < j$.*

P2 *For $i < k < j$, if $t_{ik} \equiv t_{kj}$ then $t_{ij} \equiv t_{ik}$ $(\equiv t_{kj})$.*

P3 *The diagonal terms are weakly increasing: $0 \le t_{11} \le \cdots \le t_{nn}$.*

P4 *For $i < j$, if $t_{ii} = t_{jj}$ then $t_{ij} = t_{ii}$ $(= t_{jj})$.*

Examples. The triangles $T(\rho^{(i)})$ constructed in the previous subsection are T-triangles.

These triangles enjoy many properties that can be derived from their definition: for instance, we can prove that their rows are weakly increasing, and this property could replace **P4**. But one of their most interesting properties is their connection to lecture hall partitions.

Lemma 3.4 *Let $t = (t_{ij})_{1 \le i \le j \le n}$ be a T-triangle, and define $\lambda = (\lambda_1, \ldots, \lambda_n)$ by $\lambda_i = t_{1i} + \cdots + t_{ii}$. Then λ is a lecture hall partition, and $t_{ii} = \lceil \lambda_i / i \rceil$ for all i.*
More precisely, a nonnegative triangle satisfying **P1** *and* **P2** *satisfies* **P3** *and* **P4** *if and only if the n-tuple λ defined above is a lecture hall partition.*

Proof. Let t be a T-triangle. According to **P1**, $\lambda_i = i\mu_i - r_i$, with $\mu_i = t_{ii}$ and $r_i = |\{k < i : t_{ki} = t_{ii} - 1\}|$. We use Lemma 2.1 to prove that λ is a lecture hall partition. According to **P3**, we have $0 \le \mu_1 \le \cdots \le \mu_n$. If $\mu_i = \mu_{i+1}$, then, according to **P4**, we have $t_{i,i+1} = t_{ii} = t_{i+1,i+1}$. According to **P2**, for all $k < i$ such that $t_{k,i+1} = t_{i+1,i+1} - 1$, we have $t_{ki} = t_{ii} - 1$. This implies that $r_i \ge r_{i+1}$ and hence, λ is a lecture hall partition.

Conversely, assume that **P1** and **P2** hold, and that λ is a lecture hall partition. According to **P1**, $t_{ii} = \lceil \lambda_i / i \rceil$ and hence **P3** holds. We now prove **P4** *ab absurdo*: assume $t_{ii} = t_{jj}$ and $t_{ij} = t_{jj} - 1$. Then by **P2**, for all $k < i$ such that $t_{ki} = t_{ii} - 1$, we have $t_{kj} = t_{jj} - 1$. This implies that $r_j > r_i$ and contradicts the fact that λ is a lecture hall partition.

■

The main result in this subsection is the following characterization.

Proposition 3.5 *Let $\rho \in \mathcal{P}_n$. Then $T(\rho)$ is a T-triangle. In particular, $\lambda = f(\rho)$ is a lecture hall partition, and $\lceil\lambda\rceil$ is given by the diagonal of $T(\rho)$.*

This proposition is the consequence of the next four lemmas.

Lemma 3.6 *Let $\rho \in \mathcal{P}_n$ and $t = T(\rho)$.*
a) Property **P1** *holds: for $1 \le i \le j \le n$, we have $t_{jj} - t_{ij} \in \{0,1\}$.*
b) Moreover, if all parts of ρ are smaller than or equal to $n+m$, with $m \ge 1$, then for $m \le i \le j \le n$, we have $t_{ij} - t_{ii} \in \{0,1\}$. In other words, the entries in a row $i \ge m$ are equal to the diagonal term or one more.

Proof. The proof works again by induction on the number of parts larger than n in ρ. We use the same notations as in the "pictorial" definition of T.

If ρ has no part larger than n, then the properties are clear. Otherwise, we assume they are true for ρ'. This immediately implies that property a) is true in the $n-1$ first columns of t, and in the top part of its last column. Moreover, by property b), the entries of R are equal to d or $d+1$, where $d = t'_{mm}$. This proves that property a) is also satisfied in the bottom part of the last column of t.
We now want to prove property b), and, more precisely, that the bottom entries in the last column of t will not ruin this property, i.e., that $t_{in} - t_{ii} \in \{0,1\}$ for $m \le i < n$. But $t_{in} - t_{ii} = t'_{m,i+1} + 1 - t'_{i+1,i+1}$, and this belongs to $\{0,1\}$ because t' satisfies property a).

■

Lemma 3.7 *Let $\rho \in \mathcal{P}_n$. Then $t = T(\rho)$ satisfies* **P2***.*

Proof. The lemma is true if ρ has no part larger than n. Otherwise, let us assume it is true for the partition ρ'.

We use the notations of Definition 3.3. We need to study separately several cases, depending on the positions of i, j, k with respect to m and n. The assumption that t' satisfies **P2** immediately implies that **P2** holds for the triangle t when $k < m$ or $m \le k < n$ and $j < n$. We are left with the case $m \le k < n$, $j = n$, which we split into two cases, depending on whether $i < m$ or $i \ge m$.

Assume $i < m$ and $t_{ik} \equiv t_{kn}$. What can we say about t_{in}? We have $t'_{i,k+1} \not\equiv t'_{m,k+1}$. As t' satisfies **P2**, this shows that t'_{im} cannot be congruent to $t'_{m,k+1}$. Hence $t'_{im} \equiv t'_{i,k+1}$ i.e. $t_{in} \equiv t_{ik}$.

Now, assume $i \ge m$ and $t_{ik} \equiv t_{kn}$. We have $t'_{i+1,k+1} \not\equiv t'_{m,k+1}$. As t' satisfies **P2**, this shows that $t'_{m,i+1}$ cannot be congruent to $t'_{i+1,k+1}$. Hence $t'_{m,i+1} \equiv t'_{m,k+1}$ i.e. $t_{in} \equiv t_{kn}$.

■

Lemma 3.8 *Let $\rho \in \mathcal{P}_n$. Then $t = T(\rho)$ satisfies* **P3***.*

Proof. This is true if ρ has no part larger than n. Otherwise, assuming that t' satisfies **P3**, we just have to prove that $t_{nn} \ge t_{n-1,n-1}$, i.e. that $t'_{mm} + 2 \ge t'_{nn}$. But according to Lemma 3.6, $t'_{mm} + 2 \ge t'_{mn} + 1 \ge t'_{nn}$.
■

Lemma 3.9 *Let $\rho \in \mathcal{P}_n$. Then $t = T(\rho)$ satisfies* **P4***.*

Proof. This is true if ρ has no part larger than n. Otherwise, assuming that t' satisfies **P4**, we just have to prove **P4** in the case $j = n$ (with the notations used in Definition 3.3). Assume $t_{ii} = t_{nn}$. As t' satisfies **P3**, this shows that $m \le i$. Hence, $t_{in} = t'_{m,i+1} + 1$. By **P1**, $t'_{m,i+1} + 1 \ge t'_{i+1,i+1}$, i.e. $t_{in} \ge t_{ii} = t_{nn}$. As t satisfies **P1**, we also have $t_{in} \le t_{nn}$, which finally gives $t_{in} = t_{nn}$.

■

3.3 The reverse bijection

We now wish to prove that we can reverse our algorithm: we want to start from a lecture hall partition λ, construct a (unique) T-triangle t whose column sums give λ, and then reverse the construction of Fig. 1 by transforming the last column of t into a new hook, repeating this transformation until we find a triangle containing only zeroes and ones.

Our first aim will be to prove that lecture hall partitions and T-triangles are in one-to-one correspondence. We shall use the following reformulation of property **P2**.

Lemma 3.10 *Let $t = (t_{ij})_{1\le i\le j\le n}$ be an integer triangle. Then t satisfies* **P2** *if and only if there exists a permutation σ in the symmetric group $\mathcal{S}_n$ such that*

$$\forall i < j, \quad t_{ij} \equiv 0 \Longleftrightarrow \sigma_i < \sigma_j. \tag{7}$$

If such a permutation exists, then it is unique.

Proof. Assume there exists σ such that (7) holds. Let $i < k < j$. If $t_{ik} \equiv t_{kj} \equiv 0$ then $\sigma_i < \sigma_k < \sigma_j$, hence $\sigma_i < \sigma_j$ and $t_{ij} \equiv 0$. Similarly, if $t_{ik} \equiv t_{kj} \equiv 1$ then $\sigma_i > \sigma_k > \sigma_j$, hence $\sigma_i > \sigma_j$ and $t_{ij} \equiv 1$. Thus t satisfies **P2**.

Conversely, let t satisfy **P2**. It is clear that (7) defines at most one permutation σ. Let $\prec$ be the binary relation on $\{1, 2, \ldots, n\}$ defined by

$$i \prec j \Longleftrightarrow \begin{cases} i = j, \\ \text{or } i < j \text{ and } t_{ij} \equiv 0, \\ \text{or } i > j \text{ and } t_{ji} \equiv 1. \end{cases}$$

The relation $\prec$ is reflexive and antisymmetric. Moreover, property **P2** implies that $\prec$ is transitive. Hence, this relation orders completely the set $\{1, 2, \ldots, n\}$. Defining $\sigma \in \mathcal{S}_n$ by $\sigma_1^{-1} \prec \sigma_2^{-1} \prec \cdots \prec \sigma_n^{-1}$ provides a permutation such that (7) holds.

∎

Remark. Property (7) implies that a T-triangle t is completely characterized by the corresponding permutation σ and the partition μ formed with the diagonal terms $t_{11}, \ldots, t_{nn}$. Consequently, we could have described the map f in terms of pairs (μ, σ) (satisfying some properties) instead of T-triangles. We found that the "pictorial" description was simpler to remember, and hence chose the T-triangle version.

Corollary 3.11 *The map*

$$g : t = (t_{ij})_{1\le i\le j\le n} \longmapsto \lambda = (\lambda_1, \ldots, \lambda_n)$$

where $\lambda_i = t_{1i} + \cdots + t_{ii}$ establishes a one-to-one correspondence between T-triangles and lecture hall partitions.

Proof. We have already proved (Lemma 3.4) that $\lambda = g(t)$ is a lecture hall partition. Let us write $\lambda_i = i\mu_i - r_i$ with $0 \le r_i < i$. Then t and the permutation σ associated with t by the above lemma can be recovered from λ via the following properties:

$$\forall j, \quad |\{i < j : \sigma_i < \sigma_j\}| = \begin{cases} r_j & \text{if } \mu_j \equiv 1, \\ j - 1 - r_j & \text{if } \mu_j \equiv 0, \end{cases}$$

and

$$\forall i < j, \quad t_{jj} = \mu_j, \quad t_{jj} - t_{ij} \in \{0, 1\} \quad \text{and} \quad t_{ij} \equiv 0 \Longleftrightarrow \sigma_i < \sigma_j. \tag{8}$$

Therefore, in order to prove that g is one-to-one, we have to prove that, given any lecture hall partition λ, the triangle t defined by these properties is always a T-triangle.

By construction, **P1** holds. Property **P2** holds because it is equivalent to the existence of σ (Lemma 3.10). Finally, we can check that the column sums of t give λ, which is a lecture hall partition. This implies (Lemma 3.4) that t also satisfies **P3** and **P4**.

∎

Example. Let $n = 5$ and $\lambda = (0, 1, 2, 6, 12)$. We find $\mu = \lceil\lambda\rceil = (0, 1, 1, 2, 3)$ and $r = (0, 1, 1, 2, 3)$. This gives $\sigma = 15324$ and, using (8),

$$t = \begin{array}{ccccc} 0 & 0 & 0 & 2 & 2 \\ & 1 & 1 & 1 & 3 \\ & & 1 & 1 & 2 \\ & & & 2 & 2 \\ & & & & 3 \end{array}.$$

We can check that t is indeed a T-triangle and that $g(t) = \lambda$.

According to Corollary 3.11, what we now want to prove is that every T-triangle t corresponds to exactly one partition $\rho \in \mathcal{P}_n$ via the transformation T.

If $t = (t_{ij})_{1\leq i\leq j\leq n}$ is such that $t_{nn} \leq 1$, then the only candidate for ρ is formed by the positive column sums of t.

If $t_{nn} \geq 2$, we want to reverse the construction of Fig.1 by transforming the nth column of t into the mth hook of a new triangle t'. Clearly, once m is chosen, the construction of t' is non ambiguous. In particular, $t'_{im} = t_{in} - 2$ for $1 \leq i < m$, $t'_{mm} = t_{nn} - 2$ and $t'_{mi} = t_{i-1,n} - 1$ for $m < i \leq n$. The following lemma implies that there is at most one possible choice for m.

Lemma 3.12 *Let t be a T-triangle of size n. Let $1 \leq m \leq n$, and let t' be obtained by reversing the construction of Fig.1: we remove the nth column of t and transform it into a hook inserted in mth position. The triangle t' satisfies property* **P3** *if and only if:*

P3' *for all $i \leq n$,* $\begin{cases} t_{ii} < t_{nn} - 2 & \Longrightarrow \quad i < m \\ t_{ii} > t_{nn} - 2 & \Longrightarrow \quad i \geq m. \end{cases}$

The triangle t' satisfies property **P4** *if and only if:*

P4' *for all $i \leq n$ such that $t_{ii} = t_{nn} - 2$, $t_{in} = t_{nn} \Longleftrightarrow i < m$.*

Proof. The first assertion is obvious. With the notations of Definition 3.3, we see that **P4** holds when $i \neq m$ and $j \neq m$. Thus **P4** will hold for the whole triangle t' if and only if:

$$\forall i < m \text{ such that } t'_{ii} = t'_{mm},\ t'_{im} = t'_{mm}$$

and

$$\forall i \geq m \text{ such that } t'_{i+1,i+1} = t'_{mm},\ t'_{m,i+1} = t'_{mm}.$$

These two conditions are equivalent to **P4'**. Fig. 2 shows a typical pair (t, m) satisfying **P3'** and **P4'**.

■

The above lemma shows that the unique choice of m that is likely to give a T-triangle t' is the following:

$$m(t) = \min\{i : (t_{ii} > t_{nn} - 2) \text{ or } (t_{ii} = t_{nn} - 2 \text{ and } t_{in} = t_{nn} - 1)\}. \tag{9}$$

Lemma 3.13 *Let t be a T-triangle such that $t_{nn} \geq 2$. The (unique) integer m such that properties* **P3'** *and* **P4'** *hold is $m = m(t)$.*

Proof. The choice $m = m(t)$ clearly fulfills property **P3'**. To prove that it also fulfills **P4'**, we have to prove that, if i is such that $t_{ii} = t_{nn} - 2$ and $t_{in} = t_{nn}$, then for all $k < i$ such that $t_{kk} = t_{ii} = t_{nn} - 2$, we also have $t_{kn} = t_{nn}$.

Assume $k < i$, $t_{kk} = t_{ii} = t_{nn} - 2$ and $t_{in} = t_{nn}$. According to **P4**, $t_{ki} = t_{ii}$. According to **P2**, this implies that $t_{kn} = t_{nn}$.

■

We can now describe the reverse algorithm that transforms a T-triangle t into the unique partition $\rho \in \mathcal{P}_n$ such that $T(\rho) = t$.

If $t_{nn} \leq 1$, let $r = |\{i : t_{ii} = 1\}|$. We take $\rho = (\rho_1, \ldots, \rho_r)$ where ρ_{r-k} is the sum of the entries of column $n - k$, for $k = 0, \ldots, r - 1$.

Otherwise, let $m = m(t)$ be defined by (9) and apply the construction of Fig.2 by inserting the new hook in row and column m. We shall see below that this provides a new T-triangle t' such that $m(t') \leq m(t)$. Let ρ' be the unique partition of $\mathcal{P}_n$ such that $T(\rho') = t'$. Add to ρ' a new part equal to $n + m$ to obtain ρ.

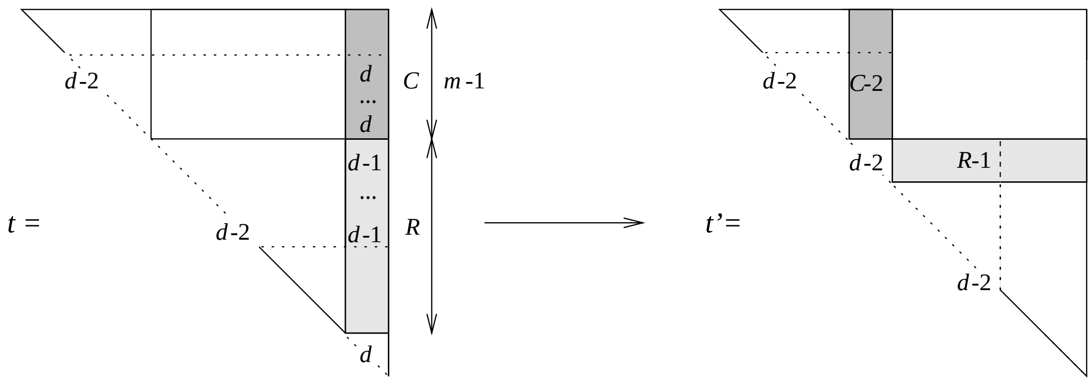


Figure 2: The reverse algorithm: insertion of a new hook in row and column m.

Here comes at last the proposition that will complete the proof of Theorem 3.1.

Proposition 3.14 *Let t be a T-triangle. Iterating the above algorithm provides the unique partition $\rho \in \mathcal{P}_n$ such that $T(\rho) = t$.*

Proof. If $t_{nn} \leq 1$, then the properties that define T-triangles show that t only contains zeroes and ones, the ones lying at the bottom of the rightmost columns of t, and that the positive column sums $\rho_1, \ldots, \rho_r$ satisfy $0 < \rho_1 < \cdots < \rho_r$. Hence $T(\rho)$ is exactly the triangle t.

Otherwise, let us check that the triangle t' constructed by the reverse algorithm is a T-triangle and that $n + m$ is at least equal to the largest part of ρ': the latter property is automatically fulfilled if ρ' has no part larger than n; otherwise, we need to prove that $m(t') \leq m$.

The integer m has been chosen so that **P3** and **P4** hold (Lemmas 3.12 and 3.13). Property **P1** clearly holds in columns $j \leq m$. For $m < j \leq n$, we only have to check that $t'_{jj} - t'_{mj} \in \{0, 1\}$. If $t'_{jj} = t_{nn} - 2$, then m has been chosen so that $t'_{mj} = t_{nn} - 2 = t'_{jj}$. If $t'_{jj} = t_{nn} - 1$, there is no problem as $t'_{mj} = t_{j-1,n} \in \{t_{nn} - 1, t_{nn} - 2\}$. Finally, if $t'_{jj} = t_{nn}$, i.e. $t_{j-1,j-1} = t_{nn}$, then by **P4**, $t_{j-1,n} = t_{nn}$, and thus $t'_{mj} = t_{j-1,n} - 1 = t_{nn} - 1 = t'_{jj} - 1$ and **P1** holds.

Property **P2** clearly holds for $k \neq m$. Now assume $i < m < j$ and $t'_{im} \equiv t'_{mj}$, i.e. $t_{in} \not\equiv t_{j-1,n}$. As t satisfies **P2**, this implies that $t_{i,j-1} \equiv t_{in}$, i.e. $t'_{ij} \equiv t'_{im}$.

Hence, we have proved that t' is a T-triangle.

Finally, as $t'_{mm} = t_{nn} - 2$, it is clear that $m(t') \leq m$ if $t'_{nn} < t_{nn}$, i.e. if $t_{n-1,n-1} < t_{nn}$. If $t_{n-1,n-1} = t_{nn}$, then $t'_{mm} = t'_{nn} - 2$, but $t'_{mn} = t_{n-1,n} = t_{nn} - 1 = t'_{nn} - 1$, and according to the definition of $m(t')$, this shows that $m(t') \leq m$.

■

Example

Let $n = 5$ and $\lambda = (0, 1, 2, 6, 12)$. We have already seen that this lecture hall partition corresponds to the following T-triangle:

$$t = \begin{array}{ccccc} 0 & 0 & 0 & 2 & 2 \\ & 1 & 1 & 1 & 3 \\ & & 1 & 1 & 2 \\ & & & 2 & 2 \\ & & & & 3 \end{array}.$$

Two of the diagonal entries t_{ii} are equal to $t_{nn} - 2$, and we find $m(t) = 3$. Inserting the new hook in row and column 3 gives the array

$$t^{(1)} = \begin{array}{ccccc} 0 & 0 & 0 & 0 & 2 \\ & 1 & 1 & 1 & 1 \\ & & 1 & 1 & 1 \\ & & & 1 & 1 \\ & & & & 2 \end{array}.$$

Next, we find $m(t^{(1)}) = 2$, and

$$t^{(2)} = \begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 \\ & & 1 & 1 & 1 \\ & & & 1 & 1 \\ & & & & 1 \end{array}.$$

This triangle only contains zeroes and ones: its pre-image by T is the partition $\rho^{(2)} = (1, 2, 3)$, from which we obtain the pre-image of $t^{(1)}$, which is $\rho^{(1)} = (1, 2, 3, 7)$ and then the pre-image of t, which is $\rho = (1, 2, 3, 7, 8) \in \mathcal{P}_5$. We can check that $T(\rho) = t$.

Final comments

- The original lecture hall theorem asserts that there exists a weight-preserving bijection between partitions into odd parts smaller than $2n$ and lecture hall partitions of length n. To our knowledge, no direct bijection between these two classes of partitions has been found. However, there is a simple bijection between partitions into odd parts smaller than $2n$ and partitions of $\mathcal{P}_n$ ("as long as two parts smaller than $n+1$ are equal, merge them"), which we can compose with $f : \mathcal{P}_n \longrightarrow \mathcal{L}_n$ to obtain a bijection with the desired property. For instance, using the above example, we see that the image of the partition into odd parts $(1^{11}, 3, 7)$ is the lecture hall partition $(0, 1, 2, 6, 12)$.

- In [3], we proved the following refinement of the lecture hall theorem:

$$\sum_{\lambda \in \mathcal{L}_n} x^{|\lambda|_e} y^{|\lambda|_o} = \frac{1}{(1-x)(1-x^2y)(1-x^3y^2)\cdots(1-x^ny^{n-1})} \tag{10}$$

  where $|\lambda|_e = \lambda_n + \lambda_{n-2} + \lambda_{n-4} + \cdots$ and $|\lambda|_0 = \lambda_{n-1} + \lambda_{n-3} + \lambda_{n-5} + \cdots$ are respectively the even and odd weights of λ. Thus we have two distinct refinements of the lecture hall theorem: identity (10) above and Theorem 1.1 of this paper. These refinement do not seem to be compatible. For instance,

$$\sum_{\lambda \in \mathcal{L}_3} x^{|\lambda|_e} y^{|\lambda|_o} u^{|\lceil \lambda \rceil|} = \frac{1 + xu + x^2u + x^2yu^2 + x^3yu^2 + x^4yu^3}{(1-x^3u)(1-x^3y^2u^2)(1-x^4y^2u^3)},$$

  and the numerator of this rational function is irreducible. The proof of the lecture hall theorem given in [3] was based on the refinement of the weight $|\lambda|$ by $|\lambda|_e$ and $|\lambda|_o$, and hence, could not be extended, at least in a straightforward way, to prove the identity of the present paper.

- In [4], we have generalized the lecture hall theorem to a whole family of identities of the type:

$$\sum_{\lambda \in \mathcal{L}_a} q^{|\lambda|} = \frac{1}{(1-q^{e_1})(1-q^{e_2})\cdots(1-q^{e_n})},$$

  where

$$\mathcal{L}_a = \mathcal{L}_{(a_1,\dots,a_n)} = \left\{ (\lambda_1, \dots, \lambda_n) : \frac{\lambda_1}{a_1} \le \cdots \le \frac{\lambda_n}{a_n} \right\}.$$

  The proof of these identities uses again the even and odd weights of partitions. We do not know whether these identities have refinements similar to Theorem 1.1.

Acknowledgements. We are grateful to Vic Reiner, who mentioned to us the refinement of Bott's formula that is the starting point of this paper. We also thank Christian Krattenthaler and Gilles Schaeffer for their comments, respectively on the q-calculus part and on the bijective part of the paper.

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