Counting paths on the slit plane

(extended abstract)

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Abstract

We present a method, based on functional equations, to enumerate paths on the square lattice that avoid a horizontal half-line. The corresponding generating functions are algebraic, and sometimes remarkably simple: for instance, the number of paths of length $2n+1$ going from $(0,0)$ to $(1,0)$ and avoiding the nonpositive horizontal axis (except at their starting point) is C_{2n+1}, the $(2n+1)$th Catalan number.

More generally, we enumerate exactly all paths of length n starting from $(0,0)$ and avoiding the nonpositive horizontal axis. We then obtain limit laws for the coordinates of their endpoint: in particular, the average abscissa of their endpoint grows like $\sqrt{n}$ (up to an explicit multiplicative constant), which shows that these paths are strongly repelled from the origin.

We derive from our results the distribution of the position where a random walk, starting from a given point, hits for the first time the horizontal half-line.

1 Introduction

In January 1999, Rick Kenyon posted on the "domino" mailing-list the following e-mail:

> "Take a simple random walk on $\mathbb{Z}^2$ starting on the y-axis at $(0,1)$, and stopping when you hit the nonpositive x-axis. Then the probability that you end at the origin is $1/2$.
> Since this result was obtained from a long calculation involving irrational numbers, I wonder if there is an easy proof? By way of comparison, if you start at $(1,0)$ the probability of stopping at the origin in $2-\sqrt{2}$."

This mail led Olivier Roques, a graduate student at LaBRI, to investigate the *number* of such walks of fixed length: he soon conjectured that exactly $4^n C_n$

walks of length $2n+1$ go from $(0,1)$ to $(0,0)$ without hitting the nonpositive x-axis before they reach their endpoint, where $C_n = \binom{2n}{n}/(n+1)$ is the nth Catalan number. Similarly, he conjectured that, if the starting point is chosen to be $(1,0)$, then the number of walks is even more remarkable, being C_{2n+1}. Let us mention that these conjectures directly imply Rick Kenyon's results.

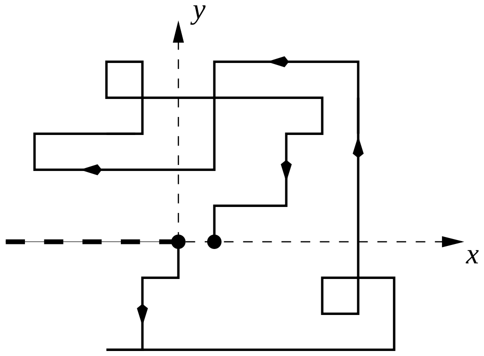

Figure 1: A path on the slit plane joining $(0,0)$ to $(1,0)$.

We attack them by studying the problem in its full generality: denoting by $a_{i,j}(n)$ the number of n-step walks starting from $(0,0)$, ending at (i,j), and avoiding the forbidden half-axis (*walks on the slit plane*), we give a closed form expression for the generating function

$$S(x,y;t) = \sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \sum_{n\geq 0} a_{i,j}(n) x^i y^j t^n,$$

which turns out to be algebraic. Our approach is based on a functional equation, which is trivial to establish, but tricky to solve (Section 4).

From this three-variate generating function, we can compute the generating function for walks ending at a prescribed position (i,j): this proves O. Roques's conjectures (Section 2), and allows us to study R. Kenyon's question for any starting point (Section 3). This gives the (bounded) solution of a discrete Dirichlet problem on the square lattice, with boundary conditions on the negative x-axis (see [12, Section 1.4] for definitions). We also study the asymptotic properties of n-step walks on the slit plane: for instance, we prove and quantify their *transience* (the point $(1,0)$ is only visited with probability $2-\sqrt{2}$). We show that their endpoint lies, on average, at distance $\sqrt{n}$ from the origin. More precisely, this endpoint, normalized by $\sqrt{n}$, converges in law towards an explicit distribution (Section 5).

Our method can be applied to similar problems: in particular, we can change the "forbidden" axis into the half-line $x=y$, $x\leq 0$, as explained in the full version of the paper [3].

Let us mention that two simple proofs of the C_{2n+1} result have been given recently [2, 4], the former being bijective. Moreover, the series $S(1,1;t)$ has already been studied in the literature [12, Chap.2] but, to our knowledge, only asymptotic results had been obtained so far. See also [6] for numerical simulations.

2 Main results

2.1 The complete generating function

We consider paths (or *walks*) on the square lattice made of four kinds of steps: north, east, south and west. Let $n \geq 0$, and $(i,j) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(k,0) : k < 0\}$. We denote by $a_{i,j}(n)$ the number of walks of length n on the square lattice that start from $(0,0)$, end at (i,j), and never touch the horizontal half-axis $\{(k,0) : k \leq 0\}$ once they have left their starting point: we call them *paths on the slit plane*. Fig. 1 shows such a walk, with $(i,j) = (1,0)$ and $n = 59$. We denote by $a(n)$ the number of paths of length n on the slit plane, starting from $(0,0)$, regardless of their endpoint.

Let $S(x,y;t)$ be the generating function for paths on the slit plane, counted by their length and the position of their endpoint:

$$\begin{aligned} S(x,y;t) &\equiv S(x,y) = \sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \sum_{n\geq 0} a_{i,j}(n) x^i y^j t^n, \\ &= 1 + t(x+y+\bar{y}) + t^2(x^2 + 2xy + 2x\bar{y} + \bar{x}y + \bar{x}\bar{y} + y^2 + \bar{y}^2) \\ &+ (5x + x^3 + 4y + 4\bar{y} + y^3 + \bar{y}^3 + 3xy^2 + 3x\bar{y}^2 + 3x^2y + 3x^2\bar{y} \\ &+ \bar{x}^2y + \bar{x}^2\bar{y} + 2\bar{x}y^2 + 2\bar{x}\bar{y}^2)t^3 + O(t^4) \end{aligned}$$

with $\bar{x} = x^{-1}, \bar{y} = y^{-1}$. We shall prove that this series is algebraic of degree 8 over the field of rational functions in x, y and t.

Theorem 1 *The generating function $S(x,y;t)$ for paths on the slit plane is*

$$S(x,y;t) = \frac{\left(1 - 2t(1+\bar{x}) + \sqrt{1-4t}\right)^{1/2} \left(1 + 2t(1-\bar{x}) + \sqrt{1+4t}\right)^{1/2}}{2[1 - t(x + \bar{x} + y + \bar{y})]}.$$

This series is algebraic of degree 8*. When $x = y = 1$, it specializes to*

$$S(1,1;t) = \sum_{n\geq 0} a(n)t^n = \frac{(1+\sqrt{1-4t})^{1/2}(1+\sqrt{1+4t})^{1/2}}{2(1-4t)^{3/4}},$$

so that the asymptotic growth of the number of paths of length n on the slit plane is

$$a(n) \sim \frac{\sqrt{1+\sqrt{2}}}{2\Gamma(3/4)}\, 4^n n^{-1/4}.$$

In other words, the probability that a random walk on the square lattice, starting from $(0,0)$, has not met the forbidden half-line after n steps is asymptotic to $cn^{-1/4}$ with $c = \sqrt{1+\sqrt{2}}/2/\Gamma(3/4)$. The decay in $n^{-1/4}$ was known [10, 12, p.71], but the details of the asymptotic behaviour seem to be new.

2.2 Paths ending at a prescribed ordinate

We would like to enumerate paths on the slit plane ending at a prescribed point (i, j). Let

$$S_{i,j}(t) \equiv S_{i,j} = \sum_{n \geq 0} a_{i,j}(n) t^n$$

be the corresponding generating function. This series is obtained by extracting the coefficient of $x^i y^j$ from $S(x, y; t)$. As i and j belong to $\mathbb{Z}$, rather than $\mathbb{N}$, this is not an obvious task. In particular, let us underline that the algebraicity of $S(x, y; t)$ does *not* imply that $S_{i,j}(t)$ is algebraic. This is clearly shown by the enumeration of paths starting from $(0, 0)$ in the (unslit) plane. The complete generating function is $1/[1 - t(x + \bar{x} + y + \bar{y})]$. It is rational, hence algebraic. However, for $i, j \geq 0$, the coefficient of $x^i y^j$ in this series is

$$\sum_{m \geq 0} \binom{2m+i+j}{m} \binom{2m+i+j}{m+i} t^{2m+i+j},$$

which is transcendental: the coefficient of t^n grows like $4^n/n$, up to a multiplicative constant, revealing a logarithmic singularity in the generating function that implies its transcendence (see [9] for a discussion on the possible singularities of an algebraic series).

In constrast, we shall prove that for any i and j, the series $S_{i,j}(t)$ is algebraic.

Our first step will be to extract from $S(x, y; t)$ the coefficient of y^j. Before we state our result, let us introduce a few notations.

Notations. Given a ring $\mathbb{L}$ and n indeterminates $x_1, \ldots, x_n$, we denote by

- $\mathbb{L}[x_1, \ldots, x_n]$ the ring of polynomials in $x_1, \ldots, x_n$ with coefficients in $\mathbb{L}$,
- $\mathbb{L}[[x_1, \ldots, x_n]]$ the ring of formal power series in $x_1, \ldots, x_n$ with coefficients in $\mathbb{L}$,

and if $\mathbb{L}$ is a field, we denote by

- $\mathbb{L}(x_1, \ldots, x_n)$ the field of rational functions in $x_1, \ldots, x_n$ with coefficients in $\mathbb{L}$.

It will be convenient to express the series $S_{i,j}(t)$ in terms of the following power series in t:

$$u = \frac{\sqrt{1+4t} - 1}{\sqrt{1-4t} + 1} = \sum_{n \geq 0} (2.4^n C_n - C_{2n+1}) t^{2n+1}.$$

Note that u is quartic over $\mathbb{Q}(t)$:

$$t = \frac{u(1-u^2)}{(1+u^2)^2}.$$

This equation allows us to write any rational function in t as a rational function in u. Moreover, the generating function of Catalan numbers,

$$C(t) = \frac{1 - \sqrt{1-4t}}{2t} = \sum_{n \geq 0} \frac{1}{n+1} \binom{2n}{n} t^n,$$

satisfies

$$C(t) = \frac{1+u^2}{1-u}, \quad C(-t) = \frac{1+u^2}{1+u} \quad \text{and} \quad tC(t)C(-t) = u,$$

so that $\mathbb{Q}(u) = \mathbb{Q}(t, \sqrt{1-4t}, \sqrt{1+4t})$. Also, let

$$\Delta(x) = [1 - x(C(t)-1)]\,[1 + x(C(-t)-1)] = \left(1 - xu\,\frac{1+u}{1-u}\right)\left(1 - xu\,\frac{1-u}{1+u}\right).$$

Corollary 2 *Let $j \in \mathbb{Z}$. The generating function for paths on the slit plane ending at ordinate j is*

$$S_j(x,t) = \sum_{i\in\mathbb{Z}, n\geq 0} a_{i,j}(n)x^i t^n = \frac{M(x)^j}{\sqrt{\Delta(x)}}$$

where

$$M(x) \equiv M(x;t) = \frac{1 - t(x+\bar{x}) - \sqrt{[1-t(x+\bar{x}+2)]\,[1-t(x+\bar{x}-2)]}}{2t}.$$

The generating function $S_j^+(x;t)$ for paths on the slit plane ending at ordinate j and at a positive abscissa is of the form

$$S_j^+(x;t) = \sum_{i>0, n\geq 0} a_{i,j}(n)x^i t^n = f(x,u) + \frac{g(x,u)}{\sqrt{\Delta(x)}},$$

where $f(x,u)$ and $g(x,u)$ are Laurent polynomials in x with coefficients in $\mathbb{Q}(u)$. Similarly, the generating function $S_j^-(x;t)$ for paths on the slit plane ending at ordinate j and at a non-positive abscissa is of the form

$$S_j^-(x;t) = \sum_{i\leq 0, n\geq 0} a_{i,j}(n)x^i t^n = S_j^-(x;t) = f(x,u) + g(x,u)\sqrt{\Delta(\bar{x})},$$

with the same conditions on $f(x,u)$ and $g(x,u)$. The series S_j^+ and S_j^- can be computed explicitely. For instance,

$$\begin{aligned}
S_0^+(x;t) &= \frac{1}{\sqrt{\Delta(x)}} - 1,\\
S_1^+(x;t) &= \frac{1}{2}\left[\frac{1-t(x+\bar{x})}{t\sqrt{\Delta(x)}} - \frac{1}{t(1+u^2)} + \bar{x}\right],\\
S_1^-(x;t) &= \frac{1}{2}\left[\frac{1}{t(1+u^2)} - \bar{x} - \frac{\sqrt{\Delta(\bar{x})}}{u}\right].
\end{aligned}$$

Sketch of the proof. We start from the expression of $S(x,y;t)$ given in Theorem 1:

$$S(x,y;t) = \frac{t\sqrt{\Delta(\bar{x})}}{u[1-t(x+\bar{x}+y+\bar{y})]},$$

and convert the denominator $1/[1-t(x+\bar{x}+y+\bar{y})]$ into partial fractions of y.

■

2.3 Paths ending at a prescribed position

The series $S_j^+(x;t)$ and $S_j^-(x;t)$ can now be expanded in x (resp. $\bar{x}$), and their form implies the following result.

Corollary 3 *For all i and j, the generating function $S_{i,j}(t) = \sum_n a_{i,j}(n)t^n$ for paths on the slit plane ending at (i,j) belongs to $\mathbb{Q}(u) = \mathbb{Q}(t, \sqrt{1-4t}, \sqrt{1+4t})$, and can be computed explicitely. It is either quadratic, or quartic over $\mathbb{Q}(t)$. In particular,*

$$S_{0,1}(t) = \frac{u}{1-u^2} = \frac{1-\sqrt{1-16t^2}}{8t} = \sum_{n\geq 0} 4^n C_n t^{2n+1},$$

and

$$S_{1,0}(t) = \frac{u(1+u^2)}{1-u^2} = \frac{2-\sqrt{1-4t}-\sqrt{1+4t}}{4t} = \sum_{n\geq 0} C_{2n+1} t^{2n+1},$$

as conjectured by O. Roques. Some other values are

$$S_{-1,1}(t) = \frac{u^2}{1-u^2} = \frac{\sqrt{1+4t}-\sqrt{1-4t}-4t}{8t} = \frac{1}{2}\sum_{n\geq 1} C_{2n} t^{2n},$$

$$\begin{aligned} S_{1,1}(t) &= \frac{u^2(2-u^2)}{(1-u^2)^2} = \frac{1-24t^2+4t\sqrt{1+4t}-4t\sqrt{1-4t}-\sqrt{1-16t^2}}{32t^2} \\ &= \sum_{n\geq 1}(4^{n-1}C_n + C_{2n}/2)t^{2n}. \end{aligned}$$

Proof. Expanding in x (or $\bar{x}$) the series $S_0^+(x;t)$, $S_1^+(x;t)$ and $S_1^-(x;t)$ given in the previous corollary provides the expressions of $S_{0,1}$, $S_{1,0}$, $S_{-1,1}$ and $S_{1,1}$. In theory, the series $S_{i,j}$ could be rational, but this can be ruled out by a simple asymptotic argument on the numbers $a_{i,j}(n)$.

∎

Using our results, we can actually enumerate walks starting from *any* point of the nonpositive x-axis. This will be useful in the following section.

Corollary 4 *Let $a_{i,j}^{[k]}(n)$ be the number of paths of length n that go from $(-k,0)$ to (i,j) and do not meet the nonpositive horizontal axis once they have left their starting point. Let*

$$D_{i,j}(x;t) = \sum_{k,n\geq 0} a_{i,j}^{[k]}(n) x^k t^n$$

be the corresponding generating function. Then the series $D_{i,j}(x;t)$ can be computed from the partial sections

$$S_{i,j}^+(x;t) = \sum_{k\geq 0} x^k S_{i+k,j}(t)$$

through the following relation, valid for any i, j:

$$S_{i,j}^+(x;t) = D_{i,j}(x;t) + xS_{i,j}^+(x;t)D_{1,0}(x;t). \tag{1}$$

Proof. Let P be a path joining $(0,0)$ to $(i+k,j)$ in the slit plane. Such paths are counted by $S^+_{i,j}(x;t)$. If P does not meet the segment $[1,k]=\{(\ell,0):1\leq \ell\leq k\}$, then P is isomorphic to a path joining $(-k,0)$ to (i,j) in the slit plane. This gives the term $D_{i,j}(x;t)$ in the right-hand side of (1). Otherwise, let $(\ell,0)$ be the leftmost point of $[1,k]$ that belongs to P. We can split P into two factors: from $(0,0)$ to the last time $(\ell,0)$ is reached, we get a path that is isomorphic to a path going from $(-\ell+1,0)$ to $(1,0)$ in the slit plane; this corresponds to the coefficient of $x^{\ell-1}$ in $D_{1,0}(x;t)$. The second factor goes from $(0,\ell)$ to $(i+k,j)$ and is isomorphic to a path going from $(0,0)$ to $(i+k-\ell,j)$ in the slit plane; it corresponds to the coefficient of $x^{k-\ell}$ in $S^+_{i,j}(x;t)$. This completes the proof of Relation (1).

■

2.4 Walks on the slit plane are transient

It is well-known that the random walk on the square lattice is *recurrent*: any given point (i,j) of the lattice is visited by the walk with probability 1. In more enumerative terms, the proportion of walks of length n visiting (i,j) tends to 1 as n goes to infinity. This is no longer the case for walks on the slit plane.

Corollary 5 *Let $b(n)$ denote the number of walks of length n on the slit plane that visit the point $(1,0)$. Then, as n goes to infinity,*

$$\frac{b(n)}{a(n)}\rightarrow 2-\sqrt{2}<1.$$

Proof. A walk visiting $(1,0)$ can be seen in a unique way as the concatenation w_1w_2 of a walk w_1 going from $(0,0)$ to $(1,0)$, and a walk w_2 starting from $(1,0)$, that not only avoids the horizontal half-axis but also the point $(1,0)$ itself. This implies that

$$\sum_n b(n)t^n = S_{1,0}(t)S(1,1;t),$$

and the result follows, as $S_{1,0}(1/4)=2-\sqrt{2}$.

■

3 The hitting distribution of a half-line

The above results allow us to solve a number of probabilistic questions "à la Kenyon". Let (i,j) be a point of $\mathbb{Z}^2$ not belonging to the forbidden half-axis. A random walk starting from (i,j) hits this half-axis with probability 1. The probability that the *first* hitting point is $(0,0)$ is

$$p_{i,j}=\sum_{n\geq 0}\frac{a_{i,j}(n)}{4^n}=S_{i,j}(1/4).$$

More precisely, $a_{i,j}(n)/4^n$ is the probability that this event occurs after n steps. Corollary 3 implies that $p_{i,j} \in \mathbb{Q}[\sqrt{2}]$. In particular, we find $p_{0,1} = 1/2$ and $p_{1,0} = 2 - \sqrt{2}$, as stated in R. Kenyon's e-mail.

More generally, given $k \geq 0$, one can ask about the probability $p_{i,j}^{[k]}$ that the first hitting point is $(-k, 0)$. Using the notations of Corollary 4,

$$\sum_{k\geq 0} p_{i,j}^{[k]} x^k = D_{i,j}(x; 1/4),$$

and this series can be computed explicitely.

Let us, for instance, derive the distribution of the hitting abscissa for a walk starting from $(1,0)$. Using Relation (1) with $(i,j) = (1,0)$, we express $D_{1,0}(x;t)$ in terms of $S_{1,0}^{+}(x;t) = S_0^{+}(x;t)/x$:

$$D_{1,0}(x;t) = \frac{1}{x} \frac{S_0^{+}(x;t)}{1 + S_0^{+}(x;t)}.$$

The value of $S_0^{+}(x;t)$ is given in Corollary 2. Hence the probability distribution of the hitting abscissa for random walks starting from $(1,0)$ is

$$\begin{aligned} \sum_{k\geq 0} p_{1,0}^{[k]} x^k &= D_{1,0}(x;1/4) = \frac{1}{x} \frac{S_0^{+}(x;1/4)}{1 + S_0^{+}(x;1/4)} \\ &= \frac{1}{x} \left[1 - \sqrt{(1-x)\left(1 - x\left(\sqrt{2}-1\right)^2\right)} \right]. \end{aligned}$$

Observe that the smallest singularity of this series is at $x = 1$, and that $p_{1,0}^{[k]}$ decays like $k^{-3/2}$ as $k \to \infty$. This asymptotic behaviour actually holds for all starting points:

Proposition 6 *For any starting point (i,j), the probabilities $p_{i,j}^{[k]}$ decay like $k^{-3/2}$ as k goes to infinity.*

The proof is based on Corollaries 2 and 4. Equivalently, the study of the dominant singularity of $D_{i,j}(x;1/4)/(1-x)$ shows that the probability that the hitting abscissa is smaller than $-k$ decays like $k^{-1/2}$. This is related to the (already known) fact that the probability that a random walk starting from the origin reaches a point at distance k of the origin before it hits the horizontal half-line also decays like $k^{-1/2}$ [10, 12, Chap.2].

4 Derivation of the complete generating function

4.1 Functional equations

We obtain a functional equation for the series $S(x,y;t)$ defined in Section 2.1 by saying that a path of length n is obtained by adding a step to another path of length $n-1$. However, when generating paths via this procedure, one must be careful not to produce paths ending on the forbidden half-line. This gives:

$$S(x,y) = 1 + t(x+\bar{x}+y+\bar{y})S(x,y) - t\bar{x} - tS_{1,0} - 2tS_1^-(x),$$

where, as above,

$$S_{1,0} = \sum_{n\geq 0} a_{1,0}(n)t^n \quad \text{ and } \quad S_1^-(x) = \sum_{i\leq 0, n\geq 0} a_{i,1}(n)x^i t^n.$$

That is,

$$[1 - t(x+\bar{x}+y+\bar{y})]\, S(x,y) = 1 - t(\bar{x} + S_{1,0} + 2S_1^-(x)). \tag{2}$$

Observe that this equation is equivalent to a recurrence relation defining the numbers $a_{i,j}(n)$ by induction on n: hence, this equation completely determines the series $S(x,y)$, and in particular its sections $S_{1,0}$ and $S_1^-(x)$.

However, we found some difficulties working with Eq. (2), because $S(x,y)$ involves simultaneously positive and negative exponents of y. Instead, we are going to work with the series

$$T(x,y;t) \equiv T(x,y) = \sum_{i,j,n} a_{i,j}(n)x^i y^{|j|} t^n,$$

which, by symmetry of the model, contains as much information as $S(x,y)$ itself. Again, one writes easily a functional equation for the series $T(x,y)$:

$$[1 - t(x+\bar{x}+y+\bar{y})]\, T(x,y) = 1 - t[\bar{x} + S_{1,0} + 2S_1^-(x) + (\bar{y}-y)S_0(x)], \tag{3}$$

with

$$S_0(x) = \sum_{i\geq 0, n\geq 0} a_{i,0}(n)x^i t^n.$$

The combinatorial part of the proof is now achieved. What we shall do from now on to solve Eq. (3) resorts to algebra.

4.2 First application of the kernel method

Let us call $K(x,y) := [1 - t(x+\bar{x}+y+\bar{y})]$ the *kernel* of the functional equation (3). The principle of the kernel method is to cancel the kernel so as to obtain certain relations between the sections that occur on the right-hand side of the equation. See [7, 8, 11] for early uses of this method, and [1, 5] for more recent developments.

Here, the kernel $K(x,y)$, as a rational function of y, it has two roots. One of them is a formal power series in t with coefficients in $\mathbb{Z}[x,\bar{x}]$, and will be denoted $M(x)$:

$$M(x) = \frac{1 - t(x+\bar{x}) - \sqrt{(1-t(x+\bar{x}))^2 - 4t^2}}{2t} = t + (x+\bar{x})t^2 + (x^2+3+\bar{x}^2)t^3 + O(t^4).$$

Observe that $T(x, M(x))$ is a well-defined series belonging to $\mathbb{Z}[x, \bar{x}][[t]]$. Let us replace y by $M(x)$ in Eq. (3): the kernel vanishes, and we obtain

$$0 = 1 - t\left[\bar{x} + S_{1,0} + 2S_1^-(x) + \left(\frac{1}{M(x)} - M(x)\right) S_0(x)\right]. \tag{4}$$

4.3 From linear to algebraic equations

The series $M(x)$ is defined as one of the roots of the kernel $K(x,y) = 1 - t(x + \bar{x} + y + \bar{y})$. We eliminate $M(x)$ between (4) and the equation $K(x, M(x)) = 0$ and obtain

$$\begin{aligned}[1-t(x+\bar{x}+2)][1-t(x+\bar{x}-2)]S_0(x)^2 - 4t^2 S_1^-(x)^2 \\ +4t(1 - t\bar{x} - tS_{1,0})S_1^-(x) - (1 - t\bar{x} - tS_{1,0})^2 = 0.\end{aligned}$$

Now, remember that $S_0(x)$ is a formal power series in t and x, while $S_1^-(x)$ is a formal power series in t and $\bar{x}$. Thanks to the absence of terms involving simultaneously a series in x and a series in $\bar{x}$, we can easily extract the nonnegative part of the above equation, that is, the terms in x^i with $i \geq 0$. This forces us, however, to introduce the generating function $S_{0,1}$ for paths on the slit plane ending at $(0,1)$. We obtain:

$$[1-t(x+\bar{x}+2)][1-t(x+\bar{x}-2)]S_0(x)^2 - 4t^2 S_{1,0}S_{0,1} - (1-2tS_{0,1})^2 - (1-t\bar{x}-tS_{1,0})^2 + 1 = 0. \tag{5}$$

The extraction of the negative part yields, after dividing by $4t$:

$$-tS_1^-(x)^2 + (1 - t\bar{x} - tS_{1,0})S_1^-(x) + S_{0,1}[tS_{0,1} + tS_{1,0} - 1] = 0. \tag{6}$$

4.4 Second application of the kernel method

Let us focus on Eq. (5). Its kernel $[1 - t(x + \bar{x} + 2)][1 - t(x + \bar{x} - 2)]$, as a rational function of x, has four roots. Two of them are formal power series in t. With the notations used in Section 2.2, these two roots are

$$C(t) - 1 = u\,\frac{1+u}{1-u} \quad \text{and} \quad 1 - C(-t) = u\,\frac{1-u}{1+u}.$$

Replacing x by these roots in Eq. (5) provides two relations between the unknown functions $S_{1,0}$ and $S_{0,1}$, from which we compute:

$$S_{0,1} = \frac{u}{1-u^2} \quad \text{and} \quad S_{1,0} = \frac{u(1+u^2)}{1-u^2}.$$

4.5 The complete solution

We now replace $S_{1,0}$ and $S_{0,1}$ by their values in Eqs. (5) and (6). This yields, with the notations of Section 2.2,

$$S_0(x) = \frac{1}{\sqrt{\Delta(x)}}$$

and

$$S_1^-(x) = \frac{1}{2}\left[\frac{1}{t(1+u^2)} - \bar{x} - \frac{\sqrt{\Delta(\bar{x})}}{u}\right].$$

Having computed $S_0(x)$, $S_1^-(x)$ and $S_{1,0}$, we can now express the series $S(x,y)$ and $T(x,y)$ using Eqs. (2) and (3). In particular, we obtain

$$S(x,y;t) = \frac{t\sqrt{\Delta(\bar{x})}}{u[1-t(x+\bar{x}+y+\bar{y})]}$$

which completes the proof of Theorem 1.

■

5 The limit distribution of the endpoint

We wish to describe the distribution of the endpoint of a random path in the slit plane. More precisely, when all paths of length n are taken equally likely, the endpoint becomes a random variable (X_n, Y_n). With the notations used in the previous sections,

$$\Pr((X_n, Y_n) = (i,j)) = \frac{[x^i y^j t^n]S(x,y;t)}{[t^n]S(1,1;t)} = \frac{a_{i,j}(n)}{a(n)}.$$

By expanding the series $S(x,y;t)$, we can plot the histograms of X_n and Y_n. This suggests to normalize by $\sqrt{n}$ (Fig. 2) and that the normalized random variable $(X_n/\sqrt{n}, Y_n/\sqrt{n})$ converges in distribution.

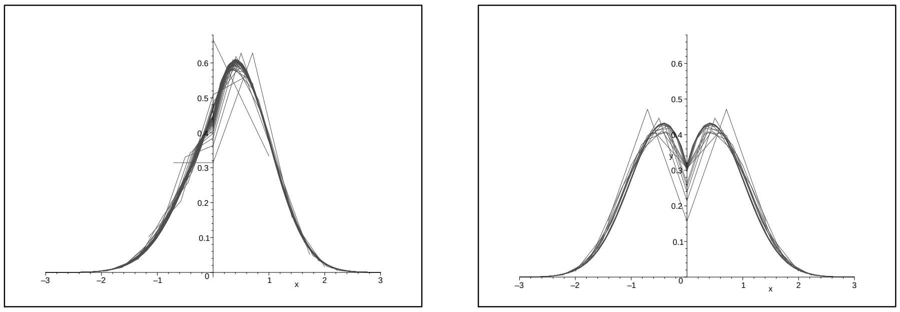

Figure 2: The convergence of $\sqrt{n}\,\Pr(X_n = i)$ and $\sqrt{n}\,\Pr(Y_n = i)$ against $i/\sqrt{n}$, for $n = 2, 3, \ldots, 10, 20, 30, \ldots, 100$.

We have, indeed, proved the existence of a limit law, and obtained a (big) expression for its density. However, comparing with the corresponding continuous problem (a Brownian motion conditioned so that it avoids a half-line) suggests that this expression can be simplified. Let us simply state a result on the average abscissa of the endpoint:

$$< X_n > \sim \frac{\Gamma(3/4)}{\Gamma(1/4)}\sqrt{n}.$$

The behaviour in $\sqrt{n}$ had been observed numerically [6], and is also known to hold in the corresponding continuous model.

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