ALGEBRAIC GENERATING FUNCTIONS IN ENUMERATIVE COMBINATORICS, AND CONTEXT-FREE LANGUAGES

MIREILLE BOUSQUET-MÉLOU

ABSTRACT. Numerous families of simple discrete objects (words, trees, lattice walks...) are counted by a rational or algebraic generating function. Whereas it seems that objects with a rational generating function have a structure very similar to the structure of words of a regular language, objects with an algebraic generating function remain more mysterious. Some of them, of course, exhibit a clear "algebraic" structure, which recalls the structure of words of context-free languages. For many other objects, such a structure has not yet been discovered. We list several examples of this type, and discuss various methods for proving the algebraicity of a generating function.

1. Introduction

The general topic of this paper is the enumeration of simple discrete objects (words, trees, lattice walks...) and more specifically the *rational* or *algebraic* nature of the associated generating functions. Let $\mathcal{A}$ be a class of discrete objects equipped with a size:

$$\begin{array}{rcl} size: \mathcal{A} & \rightarrow & \mathbb{N} \\ A & \mapsto & |A| \end{array}$$

Assume that for all n,

$$\mathcal{A}_n := \{A \in \mathcal{A} : |A| = n\}$$

is finite. Let $a_n = |\mathcal{A}_n|$. The (ordinary) generating function of the objects of $\mathcal{A}$, counted by their size, is the following formal power series in the indeterminate t:

$$A(t) := \sum_{n\geq 0} a_n t^n = \sum_{A\in\mathcal{A}} t^{|A|}.$$

The purpose of enumerative combinatorics is to provide tools for finding a closed formula for the numbers a_n, or an expression for the generating function $A(t)$. In many cases, one is happy enough to find a recurrence relation defining the sequence a_n, or a functional equation defining $A(t)$. Enumerative problems naturally arise in various fields of mathematics and computer science, such as probability theory and the average case analysis of algorithms. Numerous interesting problems also arise from models in statistical physics, the most celebrated probably being the Ising model. When dealing in problems with a computer science of physics flavour, it is often sufficient, and more informative, to obtain the asymptotic behaviour of the numbers a_n rather than an exact formula.

Before defining the main classes of generating functions we are interested in, let us examine a few simple examples.

Example 1: a regular language
Let $\mathcal{L} = (a + bb)^*$, and let ℓ_n be the number of words of length n in $\mathcal{L}$. Clearly, $\ell_0 = \ell_1 = 1$,

---

Date: Dec. 6, 2004.
Key words and phrases. Enumerative combinatorics. Algebraic generating functions. Context-free languages.
MBM was partially supported by the European Commission's IHRP Programme, grant HPRN-CT-2001-00272, "Algebraic Combinatorics in Europe".

and for $n \geq 2$,

$$\ell_n = \ell_{n-1} + \ell_{n-2}. \tag{1}$$

Hence ℓ_n is the sequence of Fibonacci numbers. Solving the above recurrence relation gives

$$\ell_n = \frac{1}{\sqrt{5}} \left(\mu^{n+1} - \left(-\frac{1}{\mu} \right)^{n+1} \right) \quad \text{where} \quad \mu = \frac{1+\sqrt{5}}{2}.$$

Multiplying (1) by t^n, and summing over $n \geq 2$ gives

$$L(t) := \sum_{n \geq 0} \ell_n t^n = \frac{1}{1-t-t^2},$$

in accordance with the fact that the non-commutative generating function of $\mathcal{L}$ is

$$\frac{1}{1-a-bb}.$$

The closed form expression of ℓ_n implies that $\ell_n \sim \mu^{n+1}/\sqrt{5}$ as $n \to \infty$.

Example 2: ternary trees

Let $\mathcal{T}$ be the set of plane trees in which any vertex has either three children (ordered from left to right) or no child at all. In the former case, the vertex is said to be a node, and in the latter case, it is called a leaf. Let t_n be the number of such trees – called ternary trees – having n nodes. Then $t_0 = t_1 = 1$ and for $n \geq 2$,

$$t_n = \sum_{i,j,k \geq 0, i+j+k=n-1} t_i t_j t_k.$$

This recurrence relation is simply obtained by looking at the sizes of the three subtrees attached to the root. It translates into the following algebraic equation that defines the generating function $T(t) = \sum_{n \geq 0} t_n t^n$:

$$T(t) = 1 + tT(t)^3.$$

It is possible, but not very pleasant, to solve this equation and write $T(t)$ with radicals. More interestingly, the *Lagrange inversion formula* [47, p. 38] allows one to derive from the equation the following expression of t_n:

$$t_n = \frac{1}{3n+1} \binom{3n+1}{n} = \frac{1}{n+1} \binom{3n}{n}.$$

This formula may also be obtained by a purely combinatorial argument, through the encoding of ternary trees by certain *Lukasiewicz words*. The enumeration of these words is then performed via the *cycle lemma* [47, Ch. 5].

Using Stirling's formula, one finds

$$t_n \sim \frac{\sqrt{3}}{4\sqrt{\pi}} \left(\frac{27}{4} \right)^n n^{-3/2}.$$

Note that for $n \geq 0$,

$$2(2n+3)(n+1)t_{n+1} = 3(3n+2)(3n+1)t_n,$$

with the initial condition $t_0 = 1$, and this implies that the series $T(t)$ satisfies the linear differential equation:

$$6T(t) + 6(9t-1)T'(t) + t(27t-4)T''(t) = 0.$$

Example 3: loops in the plane

Let $\mathcal{W}$ be the set of walks in the discrete plane, formed from North, South, East and West steps, that start and end at the origin $(0,0)$. The length of such walks is necessarily even. Let w_n be the number of such walks (called *loops*) having $2n$ steps. Alternatively, w_n is the number of words on the alphabet $\{N, S, E, W\}$ having as many N's and S's, and as many E's as W's. By projecting the walk onto the two main diagonals of the plane, one finds

$$w_n = \binom{2n}{n}^2 \sim \frac{4^{2n}}{\pi n}.$$

These numbers satisfy the recurrence relation

$$(n+1)^2 w_{n+1} = 4(2n+1)^2 w_n$$

with the initial condition $w_0 = 1$. This gives the following linear differential equation satisfied by $W(t) = \sum_{n\geq 0} w_n t^n$:

$$4W(t) + (32t-1)W'(t) + t(16t-1)W''(t) = 0.$$

As we shall see later, the term n^{-1} in the asymptotic behaviour of w_n prevents the series $W(t)$ from satisfying a (non-trivial) polynomial equation of the form $P(t, W(t)) = 0$.

In enumerative combinatorics, there is a strong interest in the *nature* of the generating function for a class of objects. Before we try to explain why, let us define the three main types of formal power series that are usually considered: rational series, algebraic series, and D-finite series.

The formal power series $A(t)$ is *rational* if it can be written in the form

$$A(t) = \frac{P(t)}{Q(t)}$$

where $P(t)$ and $Q(t)$ are polynomials in t (see [46, Ch. 4]). In particular, the generating function of Example 1 is rational.

The series $A(t)$ is *algebraic* (over $\mathbb{Q}(t)$) if it satisfies a (non-trivial) polynomial equation [47, Ch. 6]:

$$P(t, A(t)) = 0.$$

The *degree* of $A(t)$ is the smallest possible degree of P (in its second variable). The generating function of ternary trees was shown in Example 2 to be algebraic (of degree 3).

The series $A(t)$ is *D-finite* if it satisfies a (non-trivial) linear differential equation [47, Ch. 6]:

$$P_k(t)A^{(k)}(t) + \cdots + P_1(t)A'(t) + P_0(t)A(t) = 0.$$

The generating function of loops in the plane was shown in Example 3 to be D-finite.

Why do combinatorialists like these families of series? Firstly, combinatorialists obey the general mathematical temptation of classifying everything that they see. Note that the three classes of series we have defined form a hierarchy, since every rational series is algebraic and every algebraic series is D-finite. Secondly, these three classes of series are rather well-behaved:

– they have interesting closure properties: to mention only the simplest ones, each of these families is closed under the sum and product of series,

– they can be *guessed* from sufficiently many of their first coefficients (for instance using the Maple package GFUN [42]),

– they are reasonably easy to handle via computer algebra (partial fraction expansions, elimination, resultants, Gröbner bases, GFUN...)

– the asymptotic behaviour of their coefficients is rather smooth, and can in general be determined automatically: typically, for a D-finite series,

$$a(n) \sim \alpha \mu^n n^\gamma (\log n)^j$$

where α, μ and γ are algebraic over $\mathbb{Q}$ and $j \in \mathbb{N}$. For algebraic series, $j = 0$ and $\gamma \in \mathbb{Q} \setminus \{-1, -2, \ldots\}$. Moreover, for rational series, the exponent γ belongs to $\mathbb{N}$. The word "typically" means that this is not exactly true... See [29] or [30] for more details on the algebraic case. The above description is especially incomplete in the case of D-finite series:

their coefficients may actually grow faster than any exponential when the differential equation they satisfy has an *irregular singular point*. However, the systematic study of the asymptotic behaviour of the coefficients remains attainable via the determination of singular expansions of the solutions (see [35, 51] for the theory, and the Maple packages DEtools and LREtools for its implementation).

Finally, and most importantly in this paper, there are also *combinatorial reasons* why we like to be able to determine in which class of series the generating function we consider lies: there is a general intuition as to what a class of objects with a rational or algebraic generating function looks like. This is described in the next two sections[1].

2. Rational generating functions

The combinatorial intuition associated with rational generating functions is easy to describe, since it essentially coincides with the notion of regular (or: rational) languages [34, 41]. It is generally believed that, if a class $\mathcal{A}$ of objects has a rational generating function, then the structure of the objects is similar to the structure of the words of a regular language. In particular, they can be constructed recursively using a finite-state automaton. Informally, we may say that

"A class of objects has a rational generating function if these objects have a *linear structure*: that is, if all objects of size n can be constructed by expanding all objects of size $n-1$ in a *finite* number of ways."

Again, this is not a complete description, since it is usually necessary to introduce several families of objects, $\mathcal{A} = \mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k$ (one per state of the automaton).

Another way to describe this intuition is to say that the objects of $\mathcal{A}$ are in bijection with certain paths on a finite directed graph, or that they can be counted using the *transfer matrix method* [46, Ch. 4]. This intuition has proved so right in the past that I do not know of any rational generating function (counting combinatorial objects) that would not be $\mathbb{N}$-rational[2]. Moreover, it is usually rather easy to realize that a class of objects has a rational generating function (a few minutes for a well-trained combinatorialist?).

A typical example is that of integer compositions. Numerous examples are presented in [46, Ch. 4]. We shall not discuss further the rather simple case of rational generating functions. However, there are many interesting questions regarding positive rational series in several variables (see I. Gessel's lecture at the 50th Séminaire Lotharingien, available from his web page [31]).

3. Algebraic generating functions

By analogy with the rational case, one may think that the objects of a class $\mathcal{A}$ counted by an algebraic generating function have the same structure as the words of a non-ambiguous context-free (or: algebraic) language. In more combinatorial terms, we may say, informally again, that

"A class of objects has an algebraic generating function if these objects have an *algebraic structure*: that is, if they admit a recursive description based on the *concatenation* of smaller objects of the same type."

A typical example is that of ternary trees, which consist of three smaller trees. This intuition was translated in the 80's into a methodology for proving the algebraicity of the generating function of some classes of objects. The principle was the following: in order to prove *in a satisfactory way* that the generating function of a class $\mathcal{A}$ of objects is algebraic, one should establish a size-preserving bijection between these objects and the words of a non-ambiguous context-free language $\mathcal{L}$. From a non-ambiguous grammar generating $\mathcal{L}$, one

[1] I do not believe there currently exists such an intuition for D-finite series.

[2] A series is $\mathbb{N}$-rational if it is the generating function of a regular language.

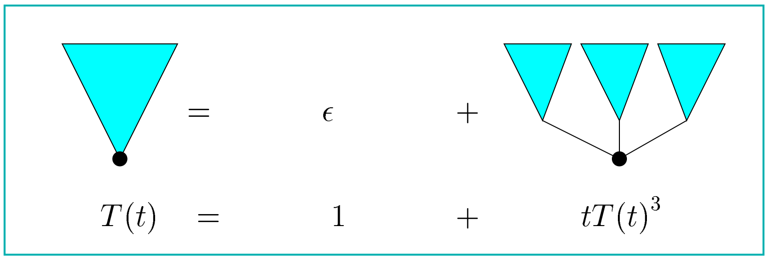


FIGURE 1. Ternary trees have an algebraic generating function.

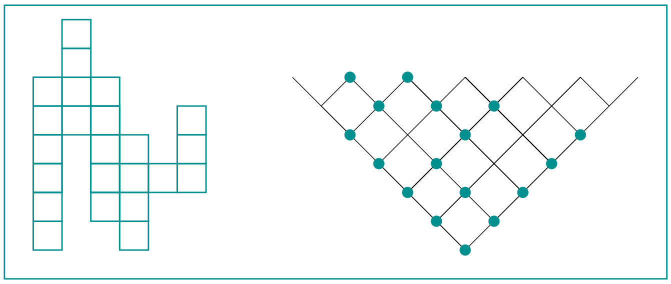


FIGURE 2. Column-convex polyominoes and directed animals have an algebraic structure.

can then write a system of algebraic equations defining the generating function of $\mathcal{L}$, or, equivalently, of the objects of $\mathcal{A}$. This approach was called "Schützenberger's methodology" by X. G. Viennot and the Bordeaux school, and provided satisfactory explanations for the algebraicity of the generating function of numerous classes of objects [50]. In particular, it helped to clarify the algebraic nature of many families of *polyominoes* and *animals* (see Figure 2 and [23, 24, 25]). In some cases, the algebraic structure of the objects was rather clear, but in other instances, as for directed animals, it took a few years before this structure was elucidated [6, 7]. Note that, very often, the algebraic structure can be read directly from the objects: to take but a simple example, it is not necessary to encode plane trees by Dyck words to realize that their generating function is algebraic (Figure 3). This observation was formalized, many years later, into the notion of "object grammars" [26].

So, is it always true that objects with an algebraic generating function are in bijection with words of a non-ambiguous context-free language? In other words, is every algebraic generating function $\mathbb{N}$-algebraic? Well, maybe not.

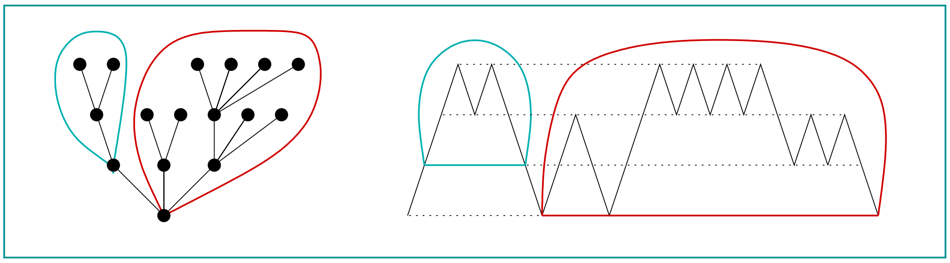
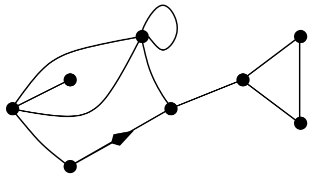


FIGURE 3. The decomposition of a plane tree into two subtrees matches the decomposition of a Dyck word (or path) into two factors.



$M = T - tT^3$ with
$T = 1 + 3tT^2$

FIGURE 4. Planar maps, counted by the number of edges, have an algebraic generating function.

Historically, one of the first combinatorial examples suggesting that things may be more tricky than in the rational case was the example of *planar maps*. A planar map is a proper embedding of a planar graph in the sphere, defined up to a continuous deformation (Figure 4). Maps are usually *rooted*, meaning that one edge is distinguished and oriented. It was proved in the early 60's by Tutte [48] that the generating function $M(t)$ of planar maps, counted by their number of edges, satisfies

$$M(t) = T(t) - tT(t)^3 \qquad (2)$$

where $T(t)$ is the only formal power series in t such that

$$T(t) = 1 + 3tT(t)^2. \qquad (3)$$

Tutte's result raised two questions:

– do we definitely need "minus" signs to describe algebraic generating functions arising from combinatorics? In particular, is $M(t)$ $\mathbb{N}$-algebraic?

– are there combinatorial interpretations of the above pair of equations?

I do not know, at the moment, the answer to the first question. I would actually be happy to learn about techniques for proving that an algebraic series is, or is not, $\mathbb{N}$-algebraic. Is there a characterization of $\mathbb{N}$-algebraic series, as there is a characterization of $\mathbb{N}$-rational series [5, 45]?

The second question came from the fact that Tutte's proof did not consist of a direct combinatorial explanation of these equations. He derived them by *guessing* the solution of another functional equation that was easier to establish (see Section 5 for details). The first combinatorial explanation was given in 1980 by Cori and Vauquelin [22]. They describe a set of trees $\mathcal{T}$ naturally counted by the series $T(t)$, and a size-preserving bijection between planar maps and a subset $\mathcal{S}$ of $\mathcal{T}$. Then, they show that the trees of size n lying in $\mathcal{T} \setminus \mathcal{S}$ are in bijection with 3-tuples of trees of $\mathcal{T}$, of total size $n-1$. This explains combinatorially the system (2–3). Another, simpler explanation was given much more recently. Again, it is based on a bijection (due to Schaeffer) between planar maps and certain trees, called *balanced blossoming trees* [43, 44]. The series $T(t)$ counts all blossoming trees, and a bijective argument borrowed from [19] shows that the *unbalanced* blossoming trees are counted by $tT(t)^3$.

At this stage, we have encountered an algebraic generating function that is likely not to be $\mathbb{N}$-algebraic. This suggests that context-free languages may not encapsulate all algebraic series. Still, from a purely combinatorial point of view, this is not really annoying: the important point for combinatorialists is to be able to provide *a direct combinatorial explanation* of an algebraic system that defines the generating function.

This is, however, not the end of the story: the truth is that many classes of objects simply refuse to show their algebraic structure, even though they do have an algebraic generating function. More precisely, in the past few years, I have kept stumbling across classes of objects for which I was able to prove, with some difficulty, that the generating function is algebraic, without being able to exhibit neither a recursive construction of these objects based on concatenation, nor a bijection with other objects that were clearly algebraic. Some of the most striking examples of this type are presented in the next section.

4. Why are these objects "algebraic"?

All the classes of objects listed in this section have been proved to possess an algebraic generating function by an *ad hoc* method. Unless explicitly stated, no combinatorial explanation (based on bijections and algebraic decompositions) has been given for the algebraicity of these generating functions. Several of them have, moreover, nice and simple coefficients, which are not understood combinatorially either. This raises challenging combinatorial problems.

4.1. Kreweras' words and walks on the quarter plane

Let $\mathcal{L}_0$ be the set of words u on the alphabet $\{a, b, c\}$ satisfying the following two conditions:

- (i) $|u|_a = |u|_b = |u|_c$,
- (ii) for every prefix v of u, $|v|_a \geq |v|_b$ and $|v|_a \geq |v|_c$.

These words encode certain walks on the plane: these walks start and end at $(0, 0)$, are made of three types of steps, $a = (1, 1)$, $b = (-1, 0)$ and $c = (0, -1)$, and never leave the first quadrant of the plane, defined by $x, y \geq 0$. The pumping lemma [34, Theorem 4.7], applied to the word $a^n b^n c^n$, shows that the language $\mathcal{L}_0$ is not context-free. However, its generating function *is* algebraic. Denoting by $\ell_{0,0}(3n)$ the number of words of $\mathcal{L}_0$ of length $3n$, one has

$$L_0(t) = \sum_{n\geq 0} \ell_{0,0}(3n) t^{3n} = \frac{W}{2t}\left(1 - \frac{W^3}{4}\right),$$

where $W \equiv W(t)$ is the unique power series in t satisfying

$$W = t(2 + W^3).$$

Moreover, the number of such words is remarkably simple:

$$\ell_{0,0}(3n) = \frac{4^n}{(n+1)(2n+1)}\binom{3n}{n}.$$

The latter formula was proved in 1965 by Kreweras, in a fairly complicated way [36]. The algebraicity of the generating function was recognized by Gessel [32]. This rather mysterious result has attracted the attention of several combinatorialists since its publication [10, 12, 32, 38, 39].

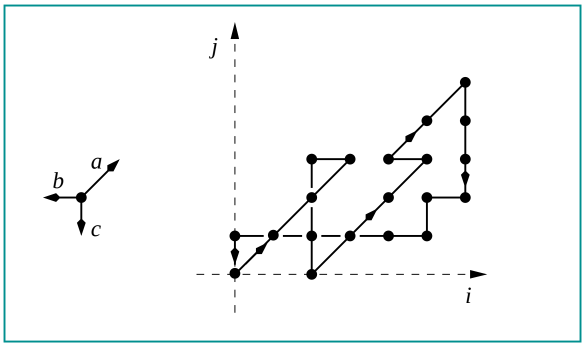


FIGURE 5. Kreweras' walks in a quadrant.

The language $\mathcal{L}$ formed by the words satisfying condition (ii) above is not context-free either (the pumping lemma again), but it also has an algebraic generating function:

$$L(t) = 2\,\frac{(1/W-1)\sqrt{1-W^2}}{1-3t} - \frac{1}{t}.$$

More generally, let us denote by $\ell_{i,j}(n)$ the number of words u of $\mathcal{L}$ of length n such that $|u|_a - |u|_b = i$ and $|u|_a - |u|_c = j$. Define the associated three-variable generating function by

$$L(u,v;t) = \sum_{i,j,n} \ell_{i,j}(n) u^i v^j t^n.$$

Then

$$L(u,v;t) = \frac{(1/W-\bar u)\sqrt{1-uW^2} + (1/W-\bar v)\sqrt{1-vW^2}}{uv - t(u+v+u^2v^2)} - \frac{1}{uvt}$$

where $\bar u = 1/u$ and $\bar v = 1/v$.

Note that it is not true that walks in the quarter plane always have an algebraic generating function: for instance, the number of *square lattice walks* (with North, South, East and West steps) of size $2n$ that start and end at $(0,0)$ and always remain in the quarter plane is

$$\frac{1}{(2n+1)(2n+2)}\binom{2n+2}{n+1}^2 \sim \frac{4^{2n+1}}{\pi n^3},$$

and this asymptotic behaviour prevents the corresponding generating function from being algebraic. The above formula is easily proven by looking at the projections of the walk onto the horizontal and vertical axes. A bijective proof is given in [21].

4.2. Walks on the slit plane

Let $\mathcal{S}_0$ be the set of words u on the alphabet $\{a,b,c\}$ satisfying the following two conditions:

(i) $|u|_a = 1 + |u|_b = 1 + |u|_c$,
(ii) for every non-empty prefix v of u, if $|v|_b = |v|_c$ then $|v|_a > |v|_b$.

These words encode certain walks on the plane: these walks start at $(0,0)$, end at $(2,0)$, are made of three types of steps, $a = (2,0)$, $b = (-1,1)$ and $c = (-1,-1)$, and never hit the non-positive x-axis once they have left their starting point. We call such walks "walks on the slit plane" (Figure 6). The pumping lemma, applied to the word $b^n a^{n+1} c^n$, shows that the language $\mathcal{S}_0$ is not context-free. However, its generating function is algebraic – and even $\mathbb{N}$-algebraic. Indeed, denoting by $s_{2,0}(3n+1)$ the number of words of $\mathcal{S}_0$ of length $3n+1$, one has

$$s_{2,0}(3n+1) = \frac{4^n}{n+1}\binom{3n}{n}.$$

In other words, this number is 4^n times the number of ternary trees with n nodes, which we encountered in Example 2.

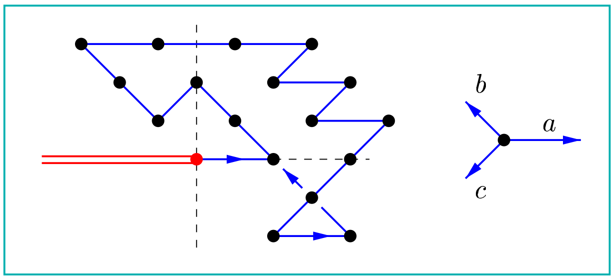


FIGURE 6. A walk on the slit plane ending at $(2,0)$.

In contrast to the case of walks in the quarter plane, the algebraicity of walks on the slit plane is a robust property: that is, it is rather resistant to changes in the set of allowed steps. Take any finite set of steps $S \subset \mathbb{Z} \times \{-1, 0, 1\}$ (we say that these steps have *small height variations*). Let $s_{i,j}(n)$ be the number of walks of length n that start from the origin, never return to the non-positive horizontal axis, consist of steps of S and end at (i, j). Let $S(u, v; t)$ be the associated generating function:

$$S(u, v; t) = \sum_{i,j \in \mathbb{Z}, n \geq 0} s_{i,j}(n) u^i v^j t^n.$$

Then this series is always algebraic, as well as the series $S_{i,j}(t) := \sum_n s_{i,j}(n) t^n$ that counts walks ending at (i, j) [9, 17].

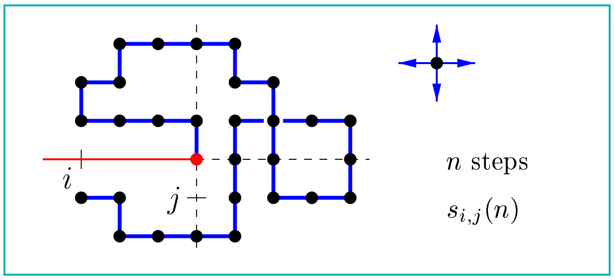


FIGURE 7. A walk on the slit plane, with steps in $\{N, S, E, W\}$.

Let us consider for instance the case where S is formed of the usual square lattice steps (North, South, West and East). Then

$$S(u, v; t) = \frac{\left(1 - 2t(1 + \bar{u}) + \sqrt{1 - 4t}\right)^{1/2} \left(1 + 2t(1 - \bar{u}) + \sqrt{1 + 4t}\right)^{1/2}}{1 - t(u + \bar{u} + v + \bar{v})}$$

with $\bar{u} = 1/u$ and $\bar{v} = 1/v$. Moreover, the number of walks ending at certain specific points is remarkably simple. For instance:

$$s_{1,0}(2n + 1) = C_{2n+1}, \quad s_{0,1}(2n + 1) = 4^n C_n, \quad s_{-1,1}(2n) = C_{2n}.$$

where $C_n = \binom{2n}{n}/(n + 1)$ is the celebrated nth Catalan number, which counts binary trees, Dyck words, and numerous other combinatorial objects [47, Ch. 6]. The first of these three identities has been proved combinatorially by Barcucci et al. [3]. The others still defeat our understanding.

4.3. **Embedded binary trees**

We consider here the classical binary trees, defined in much the same way as the ternary trees of Example 2: every vertex has either two ordered children (in which case it is called a node), or no child at all (in which case it is called a leaf). Let us associate with each node of a binary tree a label, equal to the difference between the number of right steps and the number of left steps one does when going from the root to the node. In other words, the label of the node is its abscissa in the natural integer embedding of the tree (Figure 8).
Let $S_j \equiv S_j(t, u)$ be the generating function of binary trees counted by the number of nodes (variable t) and the number of nodes at abscissa j (variable u). The standard decomposition of binary trees gives

$$\begin{array}{rcll} S_0 & = & 1 + tuS_{-1}S_1, & \\ S_j & = & 1 + tS_{j-1}S_{j+1} & \text{for } j \neq 0. \end{array}$$

It has be shown that for all $j \in \mathbb{Z}$, the series $S_j(t, u)$ is algebraic of degree (at most) 8 (while $S_j(t, 1)$ is quadratic) [8].

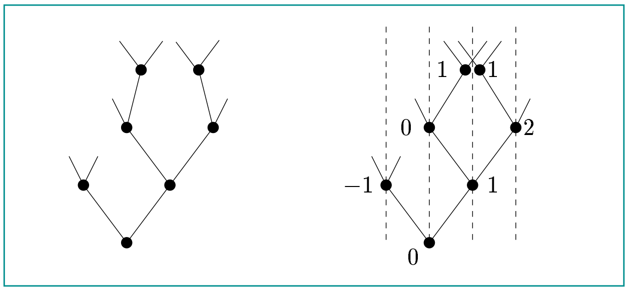


FIGURE 8. The integer embedding of a binary tree.

Let $j \geq 1$. Setting $u = 0$ in $S_j(t, u)$, we obtain the generating function $T_{j-1}(t)$ that counts binary trees in which all nodes lie at abscissa at most $j - 1$. Of course, the series T_j are algebraic too. Their degree is (at most) 2, and they admit a simple expression in terms of the series $T \equiv T(t)$ and $Z \equiv Z(t)$ defined as follows:

$$T = 1 + tT^2 \quad \text{and} \quad Z = t\,\frac{\left(1+Z^2\right)^2}{(1-Z+Z^2)}.$$

For $j \geq 0$,

$$T_j = T\,\frac{(1-Z^{j+2})(1-Z^{j+7})}{(1-Z^{j+4})(1-Z^{j+5})}.$$

Why is that so? From this, one can derive some limit results on the distribution of the number of nodes at abscissa $\lfloor \lambda n^{1/4} \rfloor$ in a random tree with n nodes [8]. These results may tell us something about the law of a "universal" random mass distribution called the integrated super-Brownian excursion [1, 37].

5. Proving the algebraicity of a generating function

The first (and best) strategy for proving the algebraicity of a generating function was described and illustrated at length in Section 3. It consists of finding a recursive description of the objects based on concatenation: this gives directly a polynomial equation (or a set of polynomial equations) for their generating function. This is illustrated by Example 2 and Figure 1 (ternary trees).

A variant of this strategy consists in describing a bijection with other objects, which admit a clear "algebraic" decomposition. We have already mentioned in Section 3 the rather recent example of the enumeration of planar maps by Schaeffer via balanced blossoming trees [43, 44]. Recall that the associated decomposition involves a "minus" sign. In the past few years, this type of construction has been extended to many families of planar maps, thus providing a satisfactory explanation for the algebraicity of their generating functions [15, 16, 40].

This approach, however, has not (yet) been successful to prove any of the results stated in Section 4. Then, how did one prove these results? What can be done if one cannot discover an "algebraic structure" in the objects one is trying to count? Well, the natural strategy is to discover *any* (recursive) structure, to translate it into a functional equation for the generating function, and finally to prove that the solution of this equation *is* algebraic.

Let us examine again the "historical" example of planar maps. Tutte established the equations of Figure 4 without giving a combinatorial explanation for them. Yet, he gave a very simple decomposition of maps (based on the deletion of the root edge, see Figure 9). But, in order to exploit this decomposition, he had to *take into account an additional parameter* in the enumeration, namely the degree of the infinite face (also called outer-degree) [48]. This forced him to introduce refined numbers $m_{n,k}$ (counting maps with n edges and outer-degree

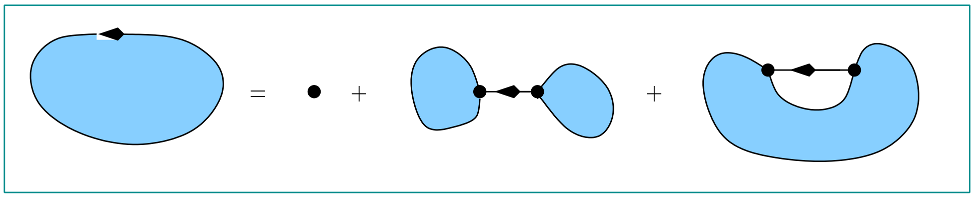


FIGURE 9. Tutte's decomposition of rooted planar maps.

k), and a *bivariate* generating function $M(u,t) = \sum_{n,k\geq 0} m_{n,k} t^n u^k$. At this cost, it became very easy to write a functional equation defining $M(u,t)$:

$$M(u,t) = 1 + tu^2 M(u,t)^2 + t\, \frac{uM(u,t) - M(1,t)}{u-1}.$$

Tutte then proved that the series $M(1,t)$ was algebraic. This implies that $M(u,t)$ is algebraic too. He actually guessed the value of $M_1(t) := M(t,1)$, then showed the existence of a series $M(u,t)$ that fits with $M_1(t)$ when $u = 1$, and satisfies the above equation.

The above example is actually rather typical: it very often happens that it becomes much much easier to find and exploit a decomposition of the objects (even a very naïve one) when taking into account one or several additional parameters. In the case of planar maps, one has to add a single parameter. For walks in the quarter plane (Section 4.1), writing an equation becomes almost trivial if one agrees to count walks ending at any point (i,j) of the quadrant, and to take into account, in the enumeration, the coordinates of this endpoint. At this cost, it becomes possible to exploit the most naïve decomposition of walks one can dream of, based on the deletion of the final step. This gives, for the series $L(u,v;t)$ defined in Section 4.1:

$$L(u,v;t) = 1 + tuvL(u,v;t) + \frac{t}{u}\left(L(u,v;t) - L(0,v;t)\right) + \frac{t}{v}\left(L(u,v;t) - L(u,0;t)\right) \qquad (4)$$

and this equation completely defines the series $L(u,v;t)$ – but does not tell us very clearly why it is algebraic. Similarly, for the walks on the slit plane of Figure 7,

$$S(u,v;t) = 1 + t(u + v + \frac{1}{u} + \frac{1}{v})S(u,v;t) - B(1/u;t)$$

where $B(1/u;t)$ is a series in t with polynomial coefficients in $1/u$ that corresponds to the "forbidden moves", and can be described explicitly in terms of the coefficients of $S(u,v;t)$ (this description is actually not necessary to solve the equation, see [9]).

Following a terminology introduced by Zeilberger [52], we say that the variables u and v are *catalytic variables*. The above examples suggest that we have to learn which equations with catalytic variables have an algebraic solution, and how to obtain an algebraic system defining this solution. More generally, one would like to be able to *solve* such equations, whether their solution is algebraic or not...

5.1. Polynomial equations with one catalytic variable

The case of one catalytic variable has now been completely clarified. Consider an equation of the form

$$P(F(u), F_1, \ldots, F_k, t, u) = 0 \qquad (5)$$

and assume it defines uniquely a $(k+1)$-tuple $(F(u), F_1, \ldots, F_k)$ of formal power series in t. Typically, $F(u) \equiv F(t,u)$ has *polynomial* coefficients in u, and $F_i \equiv F_i(t)$ is the coefficient of u^{i-1} in $F(t,u)$. In a recent work with A. Jehanne, we proved that *the solution of "every" such equation is algebraic* [18]. Moreover, a practical strategy allows one to solve specific examples (that is, to derive from (5) an algebraic equation for $F(u)$, or $F_1, \ldots, F_k$). Our method extends what had been done before for linear and quadratic equations (the *kernel method* [27, 2, 13] and the *quadratic method* [20, 33, Section 2.9]). See [18] for more references and details.

These results provide a second general strategy for proving the algebraicity of the generating function of a class of objects: it suffices to establish a polynomial equation with one catalytic variable. Recent applications include the solution of a model of hard-particles on planar maps [18], and the enumeration of triangulations with high vertex degrees [4].

5.2. Linear equations with two catalytic variables

In contrast to the case of one catalytic variable, there is no hope that *all* linear equations with two catalytic variables have an algebraic solution. This is shown by the enumeration of square lattice walks constrained to stay in the first quadrant. Their three-variable generating function satisfies

$$Q(u,v;t) = 1 + t(u+v)Q(u,v;t) + \frac{t}{u}\left(Q(u,v;t) - Q(0,v;t)\right) + \frac{t}{v}\left(Q(u,v;t) - Q(u,0;t)\right)$$

but, as mentioned at the end of Section 4.1, the series $Q(0,0;t)$ is transcendental, which prevents the complete series $Q(u,v;t)$ from being algebraic.

To our knowledge, there is, at the moment, no way to solve systematically a linear equation with two catalytic variables. However, some principles are beginning to emerge, and have proved successful for several instances of such equations. In particular, it is shown in [10, 12] how to derive the algebraicity of the generating function of Kreweras' walks in the quarter plane from the functional equation (4). See also [10, 14] for other results on walks confined to the quarter plane, and [11] for other occurrences of such equations in the enumeration of pattern avoiding permutations. Some of the key ideas in the treatment of these equations were inspired by the book of Fayolle, Iasnogorodski and Malyshev, in which related equations are solved in a more analytic context [28].

Let us finally underline the word "linear" in the title of this subsection: we only know of *one* example of a non-linear (but polynomial) equation with two catalytic variables that has been solved. It is related to the enumeration of planar triangulations, weighted by their chromatic polynomial. It took Tutte ten years and ten papers to solve the equation he had established in 1973. The solution turned out to be *differentially algebraic*, meaning that it satisfies a polynomial differential equation

$$P(A(t), A'(t), A''(t), t) = 0.$$

See [49] for a summary of this *tour de force.*

6. Concluding remarks and questions

Some of the questions below may have been solved already, or be simple to solve.

Regarding $\mathbb{N}$-algebraic series

It is still an open question to know whether all (combinatorial) generating functions that are rational (resp. algebraic) are actually $\mathbb{N}$-rational (resp. $\mathbb{N}$-algebraic). It seems that the answer could be yes in the first case, and no in the second. The generating function for planar maps (Figure 4) is possibly not $\mathbb{N}$-rational.

How can one decide whether an algebraic series with positive coefficients is $\mathbb{N}$-algebraic, or not? What about the singularities of $\mathbb{N}$-algebraic series? What about the asymptotic behaviour of their coefficients? Can we find context-free, non-ambiguous languages $\mathcal{L}$ such that the number of words of length n in $\mathcal{L}$ grows like $\alpha\mu^n n^\gamma$, for any $\gamma \in \mathbb{Q} \setminus \{-1, -2, \ldots\}$?

Regarding D-finite series

Regular languages, and context-free languages, fit well with the first two steps of our hierarchy of formal power series, namely rational series and algebraic series. Is there somewhere a well-polished class of languages that would fit with the third step of our hierarchy, namely the class of D-finite series defined in Section 1? Recall that a series $A(t) = \sum_n a_n t^n$ is D-finite if and only if its coefficients satisfy a linear recurrence relation with polynomial coefficients:

$$P_0(n)a_n + P_1(n)a_{n-1} + \cdots + P_k(n)a_{n-k} = 0$$

for n large enough.

Acknowledgements. I am very grateful to Frédérique Bassino and Mark Dukes, whose extremely valuable comments helped improving a preliminary version of this paper.

References

[1] D. Aldous. Tree-based models for random distribution of mass. *J. Statist. Phys.*, 73(3-4):625–641, 1993.

[2] C. Banderier, M. Bousquet-Mélou, A. Denise, P. Flajolet, D. Gardy, and D. Gouyou-Beauchamps. Generating functions for generating trees. *Discrete Math.*, 246(1-3):29–55, 2002.

[3] E. Barcucci, E. Pergola, R. Pinzani, and S. Rinaldi. A bijection for some paths on the slit plane. *Adv. in Appl. Math.*, 26(2):89–96, 2001.

[4] O. Bernardi. On triangulations with high vertex degrees. Submitted, 2004.

[5] J. Berstel. Sur les pôles et le quotient de Hadamard de séries $\mathbb{N}$-rationnelles. *C. R. Acad. Sci. Paris Sér. A-B*, 272:A1079–A1081, 1971.

[6] J. Bétréma and J.-G. Penaud. Animaux et arbres guingois. *Theoret. Comput. Sci.*, 117(1-2):67–89, 1993.

[7] J. Bétréma and J.-G. Penaud. Modèles avec particules dures, animaux dirigés et séries en variables partiellement commutatives. Technical report, LaBRI, Université Bordeaux 1, 1993. arXiv:math.CO/0106210.

[8] M. Bousquet-Mélou. Limit results for embedded trees. Applications to the integrated super-Brownian excursion. In preparation.

[9] M. Bousquet-Mélou. Walks on the slit plane: other approaches. *Adv. in Appl. Math.*, 27(2-3):243–288, 2001.

[10] M. Bousquet-Mélou. Counting walks in the quarter plane. In *Mathematics and computer science* 2, *(Versailles,* 2002*)*, Trends Math., pages 49–67. Birkhäuser, Basel, 2002.

[11] M. Bousquet-Mélou. Four classes of pattern-avoiding permutations under one roof: generating trees with two labels. *Electronic J. Combinatorics*, 9(2):Research Paper 19, 2003.

[12] M. Bousquet-Mélou. Walks in the quarter plane: Kreweras' algebraic model. *Ann. Appl. Proba.*, to appear.

[13] M. Bousquet-Mélou and M. Petkovšek. Linear recurrences with constant coefficients: the multivariate case. *Discrete Math.*, 225(1-3):51–75, 2000.

[14] M. Bousquet-Mélou and M. Petkovšek. Walks confined in a quadrant are not always D-finite. *Theoret. Comput. Sci.*, 307:257–276, 2003.

[15] M. Bousquet-Mélou and G. Schaeffer. Enumeration of planar constellations. *Adv. in Appl. Math.*, 24(4):337–368, 2000.

[16] M. Bousquet-Mélou and G. Schaeffer. The degree distribution of bipartite planar maps: applications to the Ising model. ArXiv math.CO/0211070, 2002.

[17] M. Bousquet-Mélou and G. Schaeffer. Walks on the slit plane. *Probab. Theory Related Fields*, 124(3):305–344, 2002.

[18] M. Bousquet-Mélou and A. Jehanne. Planar maps and algebraic series: a generalization of the quadratic method. In preparation.

[19] J. Bouttier, P. Di Francesco, and E. Guitter. Census of planar maps: from the one-matrix model solution to a combinatorial proof. *Nuclear Phys. B*, 645(3):477–499, 2002.

[20] W. G. Brown. On the existence of square roots in certain rings of power series. *Math. Ann.*, 158:82–89, 1965.

[21] R. Cori, S. Dulucq, and G. Viennot. Shuffle of parenthesis systems and Baxter permutations. *J. Combin. Theory Ser. A*, 43(1):1–22, 1986.

[22] R. Cori and B. Vauquelin. Planar maps are well labeled trees. *Canad. J. Math.*, 33(5):1023–1042, 1981.

[23] M.-P. Delest. Generating functions for column-convex polyominoes. *J. Combin. Theory Ser. A*, 48(1):12–31, 1988.

[24] M.-P. Delest, D. Gouyou-Beauchamps, and B. Vauquelin. Enumeration of parallelogram polyominoes with given bond and site perimeter. *Graphs Combin.*, 3(4):325–339, 1987.

[25] M.-P. Delest and G. Viennot. Algebraic languages and polyominoes enumeration. *Theoret. Comput. Sci.*, 34(1-2):169–206, 1984.

[26] I. Dutour. *Grammaires d'objets : énumération, bijections et génération aléatoire.* PhD thesis, Université Bordeaux 1, France, 1996.

[27] G. Fayolle and R. Iasnogorodski. Two coupled processors: the reduction to a Riemann-Hilbert problem. *Z. Wahrsch. Verw. Gebiete*, 47(3):325–351, 1979.

[28] G. Fayolle, R. Iasnogorodski, and V. Malyshev. *Random walks in the quarter-plane: Algebraic methods, boundary value problems and applications*, volume 40 of *Applications of Mathematics*. Springer-Verlag, Berlin, 1999.

[29] P. Flajolet. Analytic models and ambiguity of context-free languages. *Theoret. Comput. Sci.*, 49(2-3):283–309, 1987.

[30] R. Flajolet and R. Sedgewick. Analytic combinatorics: functional equations, rational, and algebraic functions. Technical Report RR4103, INRIA, 2001. A component of the book project "Analytic Combinatorics". Available at http://www.inria.fr/rrrt/rr-4103.html.

[31] I. Gessel. Rational functions with nonnegative power series coefficients. 50th Séminaire Lotharingien de Combinatoire, Ottrott, France, March 2003.

[32] I. M. Gessel. A probabilistic method for lattice path enumeration. *J. Statist. Plann. Inference*, 14(1):49–58, 1986.

[33] I. P. Goulden and D. M. Jackson. *Combinatorial enumeration*. John Wiley & Sons Inc., New York, 1983. Wiley-Interscience Series in Discrete Mathematics.

[34] J. E. Hopcroft and J. D Ullman. *Formal languages and their relation to automata*. Addison-Wesley, 1969.

[35] E. L. Ince. *Ordinary Differential Equations*. Dover Publications, New York, 1944.

[36] G. Kreweras. Sur une classe de problèmes liés au treillis des partitions d'entiers. *Cahiers du B.U.R.O.*, 6:5–105, 1965.

[37] J.-F. Marckert and S. Janson. Convergence of discrete snakes. Technical report, Université de Versailles-Saint-Quentin, 2003.

[38] H. Niederhausen. Sheffer polynomials in path enumeration. In *Proceedings of the West Coast Conference on Combinatorics, Graph Theory and Computing (Humboldt State Univ., Arcata, Calif.,* 1979*)*, Congress. Numer., XXVI, pages 281–294, 1980.

[39] H. Niederhausen. The ballot problem with three candidates. *European J. Combin.*, 4(2):161–167, 1983.

[40] D. Poulalhon and G. Schaeffer. A bijection for triangulations of a polygon with interior points and multiple edges. *Theoret. Comput. Sci.*, 307(2):385–401, 2003.

[41] A. Salomaa and M. Soittola. *Automata-theoretic aspects of formal power series*. Springer-Verlag, New York, 1978. Texts and Monographs in Computer Science.

[42] B. Salvy and P. Zimmermann. Gfun: a Maple package for the manipulation of generating and holonomic functions in one variable. *ACM Transactions on Mathematical Software*, 20(2):163–177, 1994.

[43] G. Schaeffer. Bijective census and random generation of Eulerian planar maps with prescribed vertex degrees. *Electron. J. Combin.*, 4(1):Research Paper 20, 14 pp. (electronic), 1997.

[44] G. Schaeffer. *Conjugaison d'arbres et cartes combinatoires aléatoires*. PhD thesis, Université Bordeaux 1, France, 1998.

[45] M. Soittola. Positive rational sequences. *Theoret. Comput. Sci.*, 2(3):317–322, 1976.

[46] R. P. Stanley. *Enumerative combinatorics. Vol. 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1997.

[47] R. P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999.

[48] W. T. Tutte. On the enumeration of planar maps. *Bull. Amer. Math. Soc.*, 74:64–74, 1968.

[49] W. T. Tutte. Chromatic sums revisited. *Aequationes Math.*, 50(1–2):95–134, 1995.

[50] G. Viennot. Enumerative combinatorics and algebraic languages. In *Fundamentals of computation theory (Cottbus, 1985)*, volume 199 of *Lecture Notes in Comput. Sci.*, pages 450–464. Springer, Berlin, 1985.

[51] J. Wimp and D. Zeilberger. Resurrecting the asymptotics of linear recurrences. *J. Math. Anal. Appl.*, 111(1):162–176, 1985.

[52] D. Zeilberger. The umbral transfer-matrix method: I. Foundations. *J. Comb. Theory, Ser. A*, 91:451–463, 2000.

CNRS, LaBRI, Université Bordeaux 1, 351 cours de la Libération, 33405 Talence Cedex, France

E-mail address: mireille.bousquet@labri.fr