

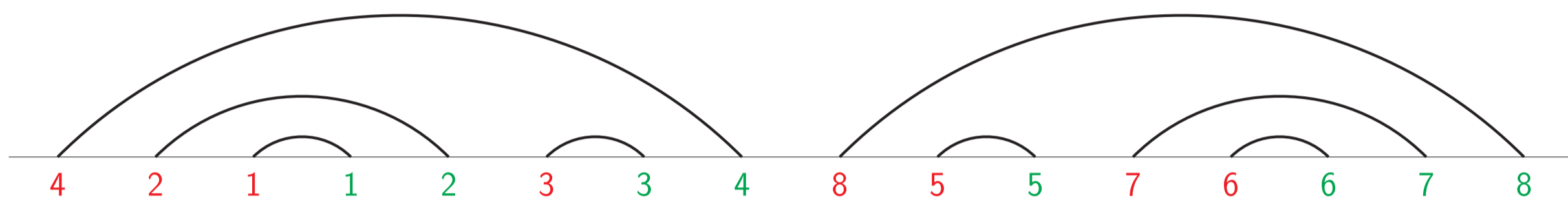
Sorting with a stack

Can we characterise, recognise, and enumerate those permutations that can be sorted using a single stack?



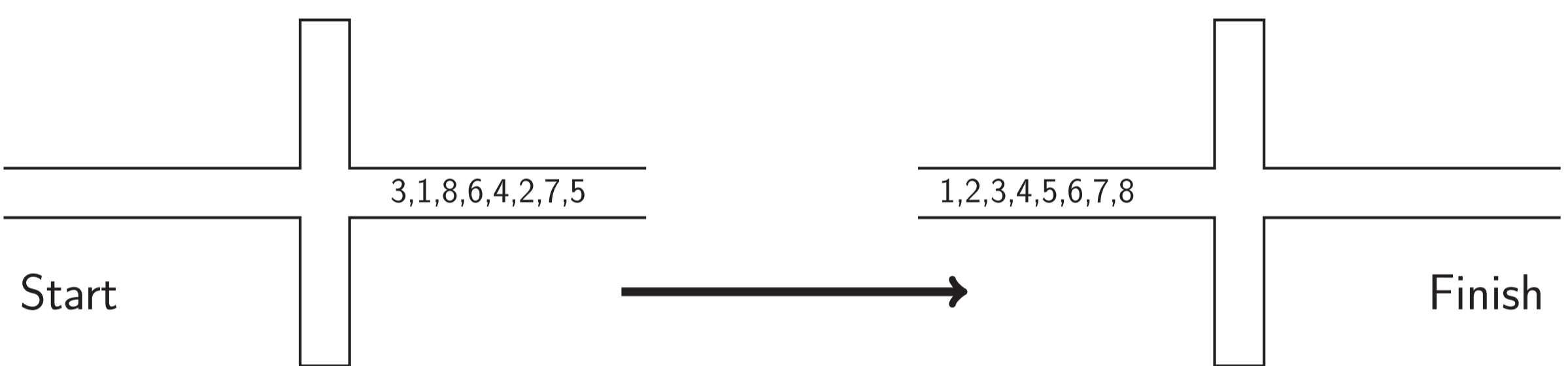
Answers (Knuth, 1968)

- Sortable permutations do not contain the pattern 231.
- They can be recognised in linear time by trying to sort them.
- There are $\frac{\binom{2n}{n}}{n+1}$ of them (i.e. Catalan) of size n .
- They are in one to one correspondence with:
 - ▶ walks on \mathbb{N} from 0 to 0 of semi length n ,
 - ▶ arch systems having n arches.



Sorting with two stacks

What if we have two stacks available in parallel? What's the story then?



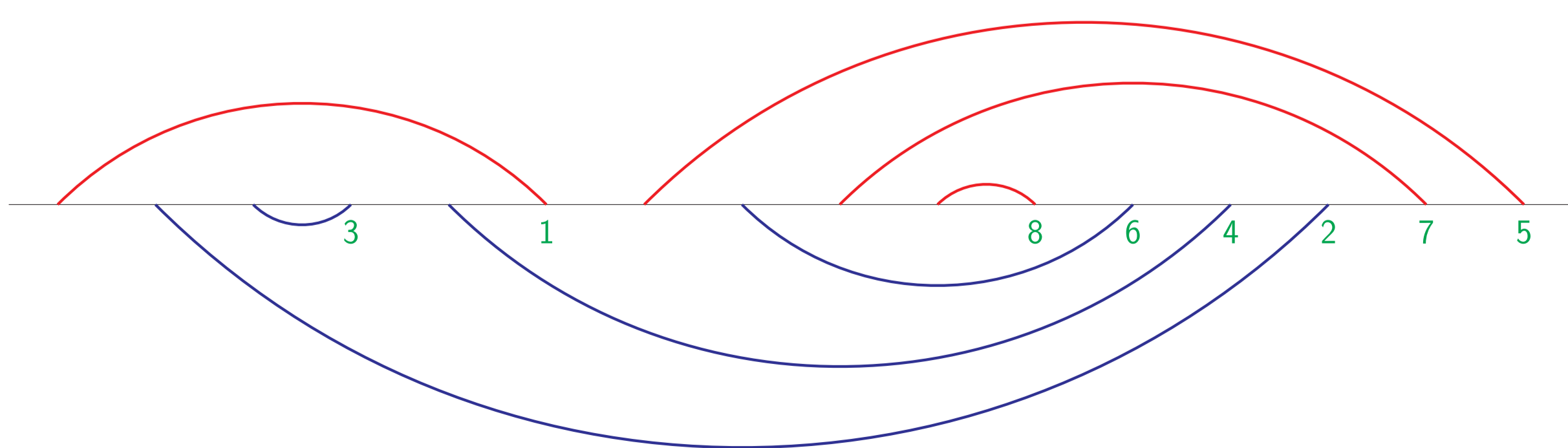
Answers

- Sortable permutations do not contain one of an infinite list of patterns (Pratt, 1973).
- They can be recognised in polynomial time (Even & Itai, 1971).
- They can be enumerated in polynomial time (A & M Bousquet-Mélou, 2013).
- They are *not* in one to one correspondence with:
 - ▶ walks in $\mathbb{N} \times \mathbb{N}$ of semi-length n from $(0, 0)$ to $(0, 0)$, or
 - ▶ two-sided arch systems (above and below the line) having n arches, because multiple legal operation sequences can sort the same permutation.

A slight change of perspective

- We can sort σ exactly if we can generate σ^{-1} from $1, 2, \dots, n$.
- So, given input in order $1, 2, \dots, n$ what permutations can be generated by arbitrary operation of the two stacks?
- Can we identify a normal form among operation sequences that generate the same permutation?

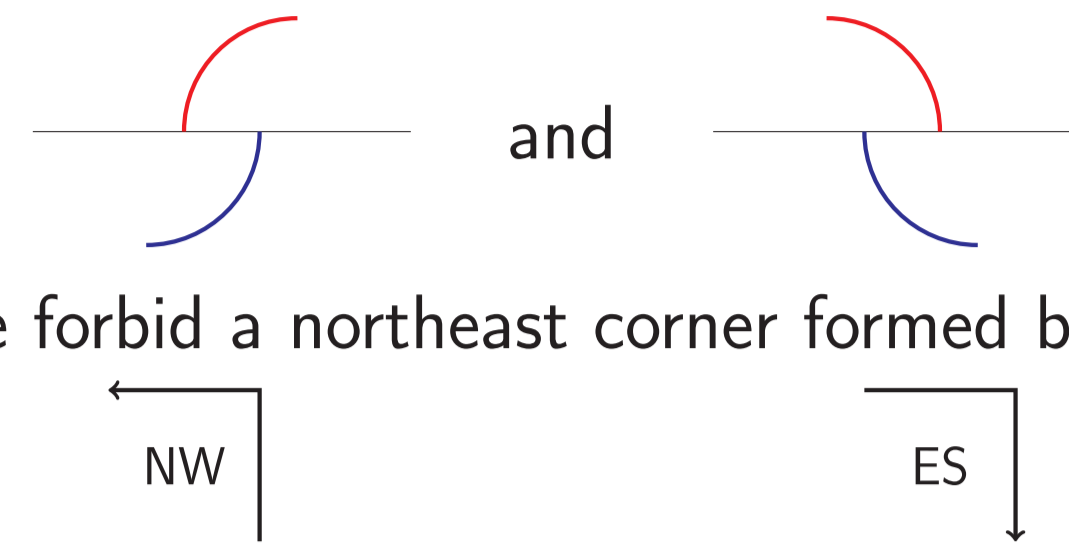
Producing 31864275



Eager output

If the next token needed is already in the stacks we might as well output it immediately.

- In arch systems we disallow the two pictures:



- In quarter plane walks we forbid a northeast corner formed by a NW or ES factor.

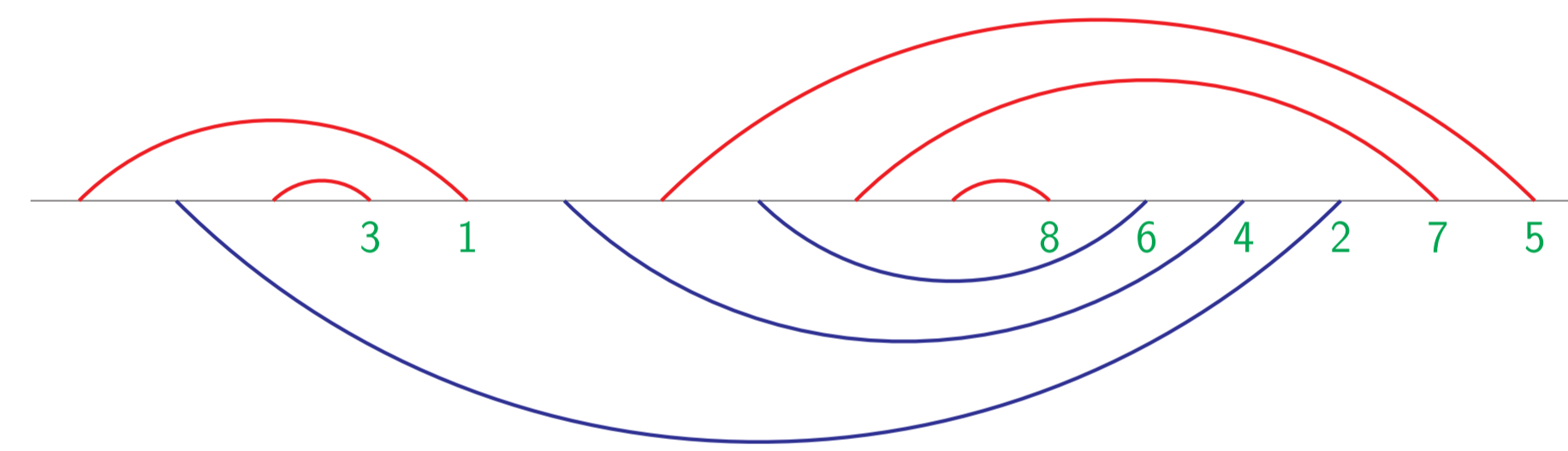
Connected components

For every cab pattern $(a < b < c)$, a and b must be handled by opposite stacks.

- Because both must still be in the stacks at the time c is input, and if they were in the same stack they would come out in reversed order.
- Defining a and b (or their corresponding arches) to be related if this occurs gives us a graph structure. Within a connected component of this graph the choice of stack for any one element forces that for all the others.

Normal form

If a permutation can be generated by two stacks in parallel, then there is exactly one operation sequence that generates it which outputs eagerly and in which each component begins with a red arch.



Show me the power series

Let $Q(a, u; x, y)$ be the generating function for walks in $\mathbb{N} \times \mathbb{N}$ starting from $(0, 0)$: u counts steps, a counts northeast corners, and x, y the coordinates of the endpoint:

$$Q(a, u; x, y) = 1 + (x + y)u + (2 + 2xy + x^2 + y^2)u^2 + ((a + 4)(x + y) + 3x^2y + 3xy^2 + x^3 + y^3)u^3 + O(u^4).$$

By decomposition on the last step or pair of steps we obtain:

$$(1 - u(x + \bar{x} + y + \bar{y}) - u^2(a - 1)(x\bar{y} + y\bar{x}))Q = 1 - u\bar{y}(1 + ux(a - 1))Q_{y=0} - u\bar{x}(1 + uy(a - 1))Q_{x=0}$$

where $\bar{x} = 1/x, \bar{y} = 1/y$.

Defining $Q(a, u) = Q(a, \sqrt{u}; 0, 0)$ we get loops in the quarter plane, and can compute as many coefficients as needed.

More importantly if $S(t)$ is the generating function for permutations that can be generated by two stacks in parallel then

$$S(t) = 1 + t + 2t^2 + 6t^3 + 23t^4 + 103t^5 + O(t^6) = \frac{1}{1 - S^\bullet(t)},$$

where S^\bullet is the unique solution of:

$$Q\left(-S^\bullet, \frac{t}{(1 + S^\bullet)^2}\right) = \frac{1 + S^\bullet}{1 - S^\bullet}.$$

This arises from manipulations of the functional equations that connect Q with S through the normal form described above.

Can we find asymptotic information about the coefficients, s_n , of S ?

And the algebraic combinatorics

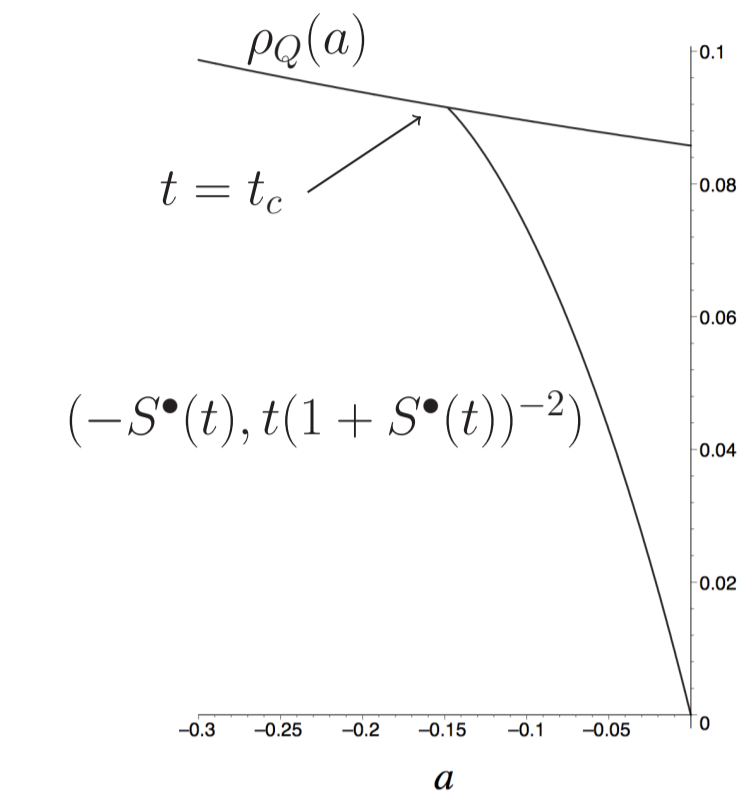
To find the radius of convergence of S^\bullet we must explore singularities of $Q(a, u)$ when $a < 0$.

Conjecture 1[†] The generating function of quarter plane loops is $(a + 1)$ -positive:

$$Q(a, u) = \sum_{n \geq 0} p_n(a + 1)u^n$$

where $p_n(b) \in \mathbb{N}[b]$.

Let $\rho_Q(a)$ denote the radius of convergence (in u) of $Q(a, u)$, and t_c the radius of convergence of $S^\bullet(t)$. We can plot $\rho_Q(a)$ and $(-S^\bullet(t), t(1 + S^\bullet(t))^{-2})$ in the (a, u) plane for $t \in [0, t_c]$. They must intersect at $t = t_c$.



Conjecture 2[†] The radius of convergence of $Q(a, \cdot)$ is

$$\rho_Q(a) = \begin{cases} \frac{1}{(2 + \sqrt{2 + 2a})^2} & \text{if } a \geq -1/2, \\ -\frac{a}{2(1 - a)^2} & \text{if } -1 \leq a \leq -1/2. \end{cases}$$

In combination with the preceding analysis, this allows us to estimate that $t_c \sim 0.12$ so that $s_n^{1/n} \rightarrow 8.2 \dots$ provided Conjectures 1 and 2 are true.

This is in very good agreement with numerical estimates from explicit computation of the actual coefficients.

Conclusions, open problems, etc.

- Prove Conjectures 1 and 2. In particular, Conjecture 2 would follow if one could show that $\rho_Q(a) = \rho_W(a)$ where W is defined similarly to Q but allowing walks in the full plane. Of note is that W is $(a + 1)$ -positive.
- There are many interesting identities in and around Q at the special values $a = \pm 1$. For instance:

$$Q(1, u) = \sum_{i, j \geq 0} \binom{2i + 2j}{2i} C_i C_j u^{i+j}$$

$$Q(-1, u) = \sum_{i, j \geq 0} \binom{i + j}{i} C_i C_j u^{i+j}$$

where C_n is the n^{th} Catalan number (these in fact reflect similar identities when we count horizontal and vertical steps separately). Combinatorial explanations for these at $a = -1$ might well help with Conjecture 1.

- What is the nature of the various series? Algebraic? D-finite? D-algebraic? Numerical data suggest that the exponent in the asymptotic behaviour of $q_n(a, u)$ varies continuously with a (and is not constant), which would rule out D-finiteness of $Q(a, u)$ for a generic value of a .
- What can be done with other data structures? Of particular interest is the *double ended queue*. There is a similar description in terms of arch systems, except on the cylinder with the top and bottom edges identified.

