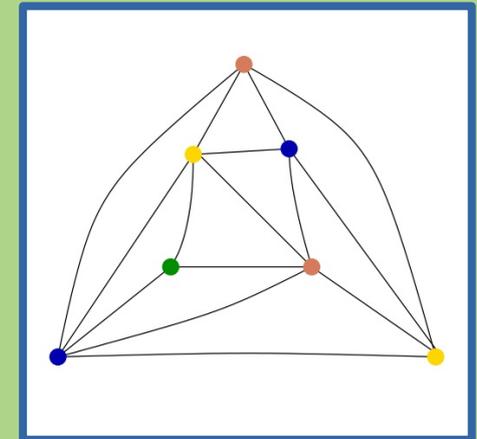
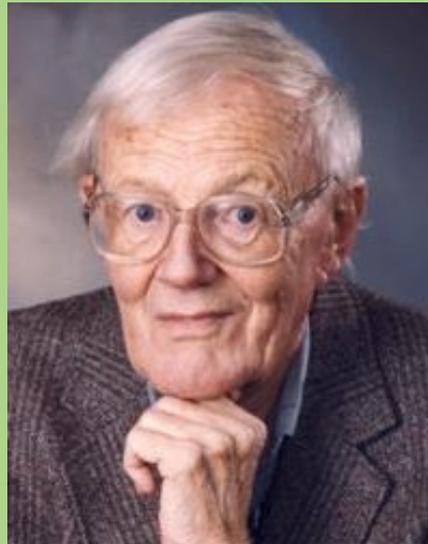
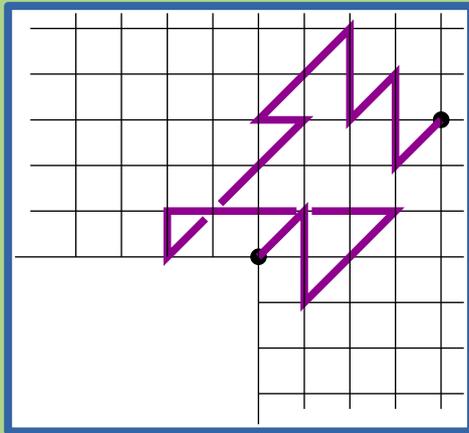


# Tutte's invariants



Keywords: enumerative combinatorics, functional equations, algebraic series

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# A “Tutte invariant” ?

The chromatic polynomial of a graph ?

Its generalization, **the Tutte polynomial** ?

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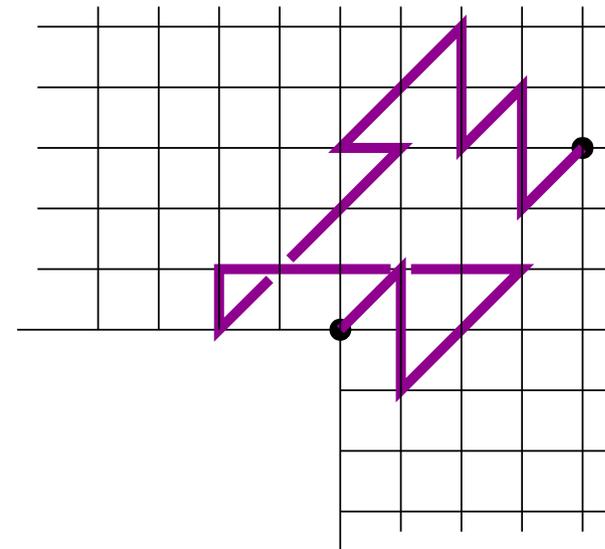
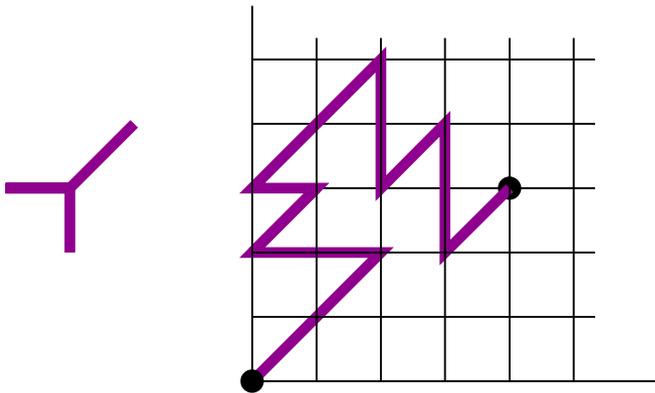
No graph invariants!

Instead, we'll see many

**functional equations for combinatorial enumeration.**

# In this talk

- I. An equation
- II. More equations with two catalytic variables
- III. Catalytic variables:  $0 < 1 < 2 < \dots$
- IV. Tutte's invariants
- V. Quadrant walks: the whole picture
- VI. Three-quadrant walks: a partial picture



# **I. An equation**

# An equation

Series  $Q(t;x,y) \equiv Q(x,y)$  :

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$$Q(x,y) = 1 + txy + t \frac{1-1}{y} + t \frac{1-1}{x} + \mathcal{O}(t^2)$$

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Tautological equation at  $x=0$  and  $y=0$

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# An equation for walks in a quadrant

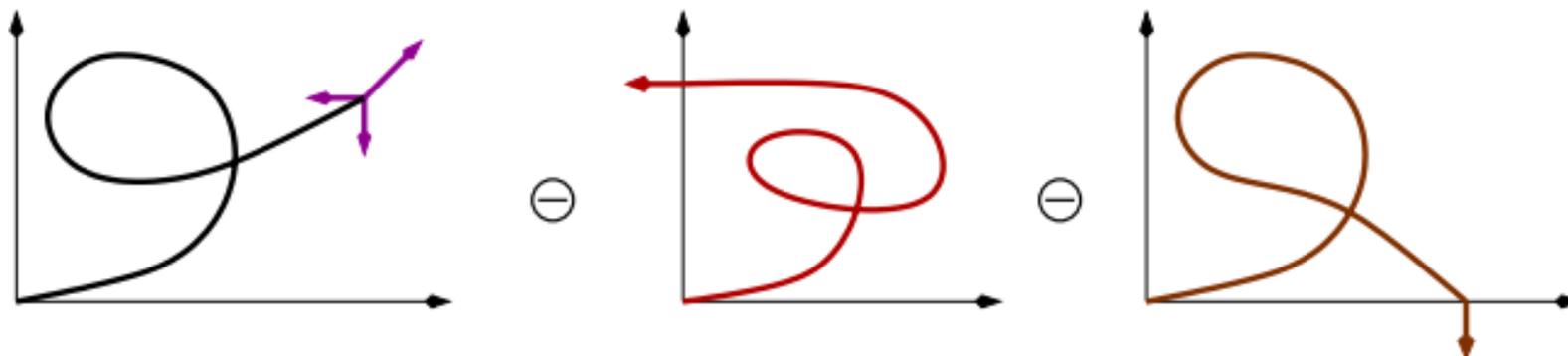
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$$Q(x,y) = \sum_{n \geq 0} \left( \sum_{i,j \geq 0} q_{i,j}(n) x^i y^j \right) t^n.$$

- Then  $q_{i,j}(n)$  is the number of walks with  $n$  steps NE, W, S going from  $(0,0)$  to  $(i,j)$  in the first quadrant.



# An equation for walks in a quadrant

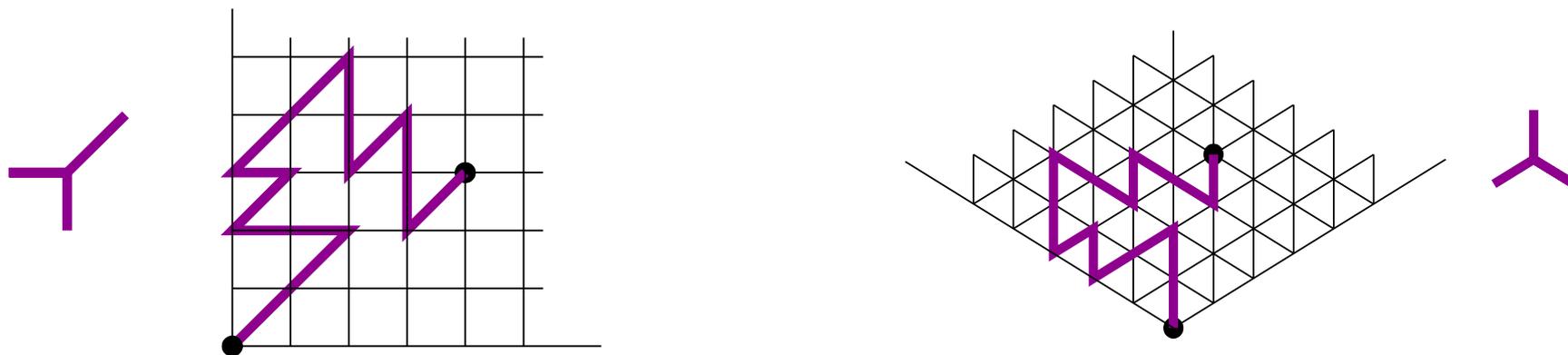
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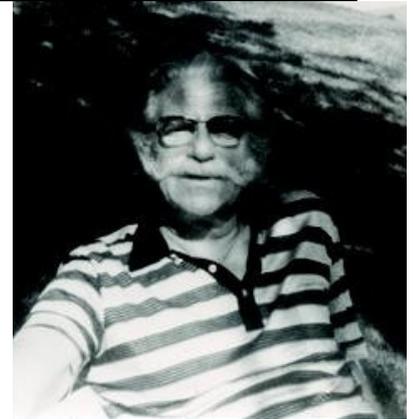
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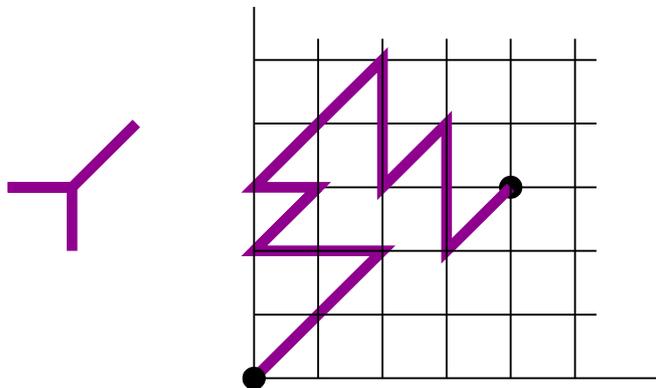
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$$q_{i,0}(3n+2i) = \frac{4^n (2i+1)}{(n+i+1)(2n+2i+1)} \binom{2i}{i} \binom{3n+2i}{n}$$



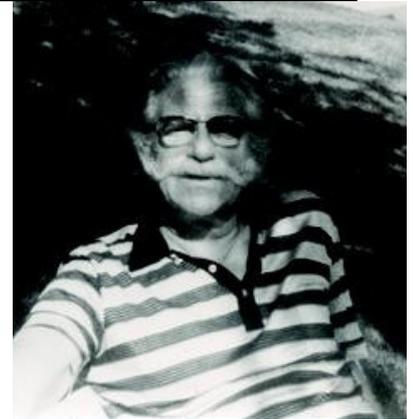
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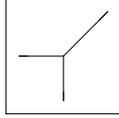
- The series  $Q(t;x,y)$  is **algebraic** ! [Gessel 86]: there exists a non-zero polynomial  $P(u,x,y,t)$  such that

$$P(Q(t;x,y), t, x, y) = 0.$$

## **II. More equations in two catalytic variables**

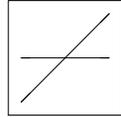
# Quadrant walks with different steps

- Kreweras' walks



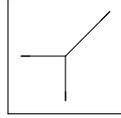
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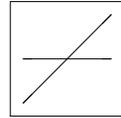
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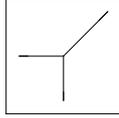
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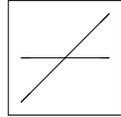
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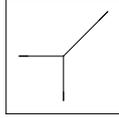


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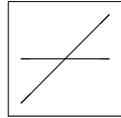
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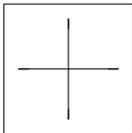
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- The **kernel** describes the new steps
- Coefficients on the r.h.s. have changed as well
- This series is **algebraic** again ! [Bostan, Kauers 10]
- Some nice coefficients [Kauers, Koutschan, Zeilberger 09]

$$q_{0,0}(2n) = 16^n \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n}$$

with  $(a)_n = a(a+1) \cdots (a+n-1)$ .

# Quadrant walks with different steps

- The simple walk 

$$(1 - t(x + \bar{x} + y + \bar{y}))xyQ(x, y) = xy - txQ(x, 0) - tyQ(0, y)$$

- The kernel describes the new steps.
- Coefficients on the r.h.s. have changed as well
- This series is **not algebraic**, but still **D-finite**.
- Nice coefficients:

$$q_{i,j}(n) = \frac{(i+1)(j+1)}{(n+1)(n+2)} \binom{n+2}{\frac{n-i-j}{2}} \binom{n+2}{\frac{n+i-j}{2} + 1}$$

# A hierarchy of formal power series

- Rational

$$A(t) = \frac{1-t}{1-t-t^2}$$

- Algebraic

$$1 - A(t) + tA(t)^2 = 0$$

- D-finite

$$t(1 - 16t)A''(t) + (1 - 32t)A'(t) - 4A(t) = 0$$

- D-algebraic

$$(2t + 5A(t) - 3tA'(t))A''(t) = 48t$$



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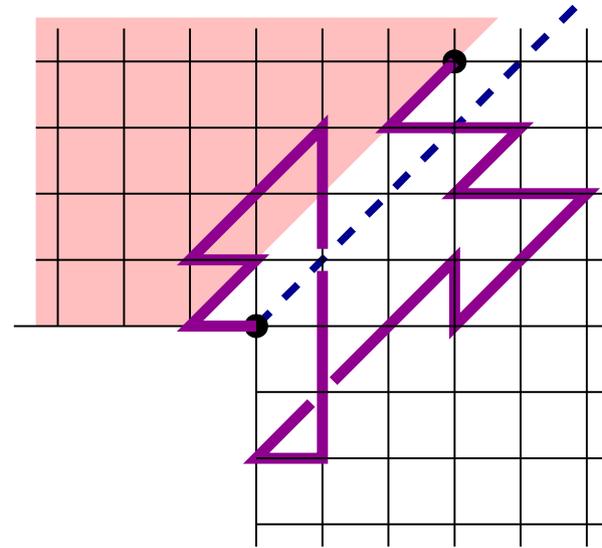
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# Kreweras' walks in three quadrants

- Walks in a three-quadrant cone, ending above the diagonal:

$$2(1 - t(\bar{x}\bar{y} + x + y))xyU(x, y) = y - 2tU(x, 0) + (ty + 2tx - 1)yD(y)$$



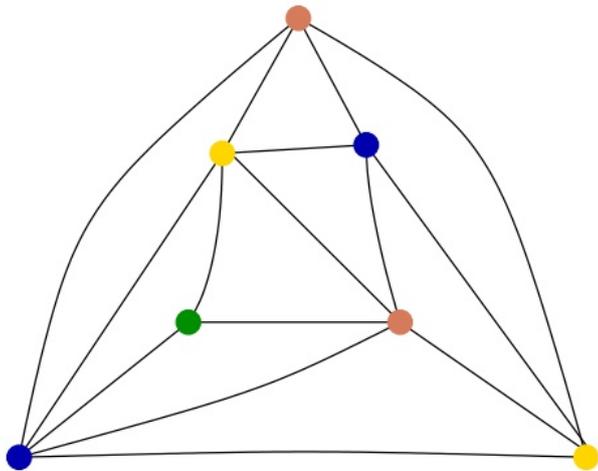
$$-2tU(0, y) = y - 2tU(0, 0) + (ty - 1)yD(y)$$

# Coloured triangulations: a historical example

Properly  $q$ -coloured triangulations : series  $T(t, q; x, y) \equiv T(x, y)$ :

$$T(x, y) = x(q - 1) + txyT(1, y)T(x, y)$$

$$+ tx \frac{T(x, y) - T(x, 0)}{y} - tx^2y \frac{T(x, y) - T(1, y)}{x - 1}$$



- Known :  $T(1, 0)$  is **D-algebraic**, and **algebraic** for some  $q$  ( $q=2, q=3...$ )

[Tutte, 1973-1984]

The birth of invariants

# Three-stack sortable permutations

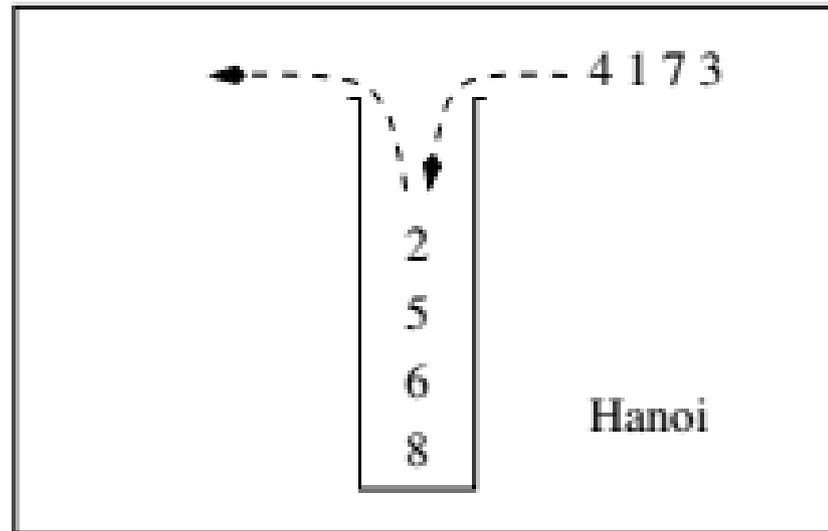
- A non-linear example [Defant, Elvey Price, Guttman 21]

$$P(x, y) = t(x + 1)^2(y + 1)^2 + ty(1 + x)P(x, y) \\ + t(1 + x) \frac{P(x, y) - P(x, 0)}{y} \left( (1 + y)^2 + y \frac{P(x, y) - P(0, y)}{x} \right)$$

Stack-sorting [Knuth 68]

Not D-finite?

$S(\sigma)$



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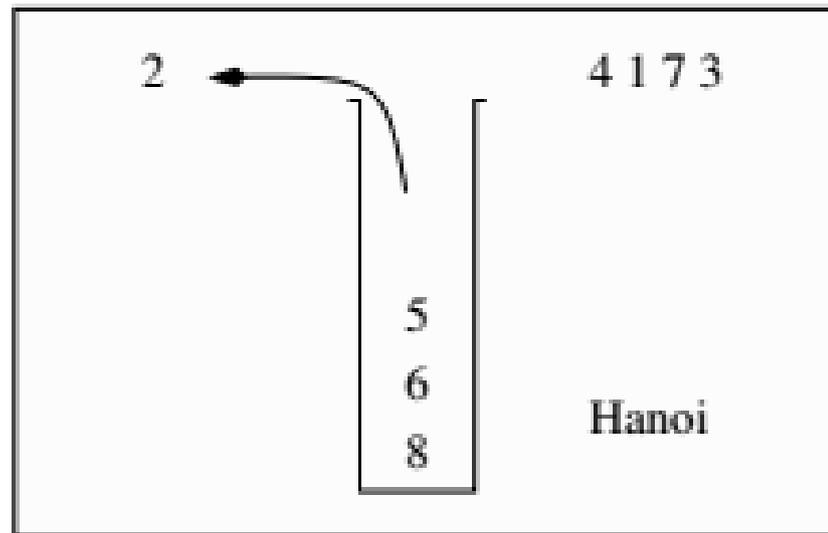
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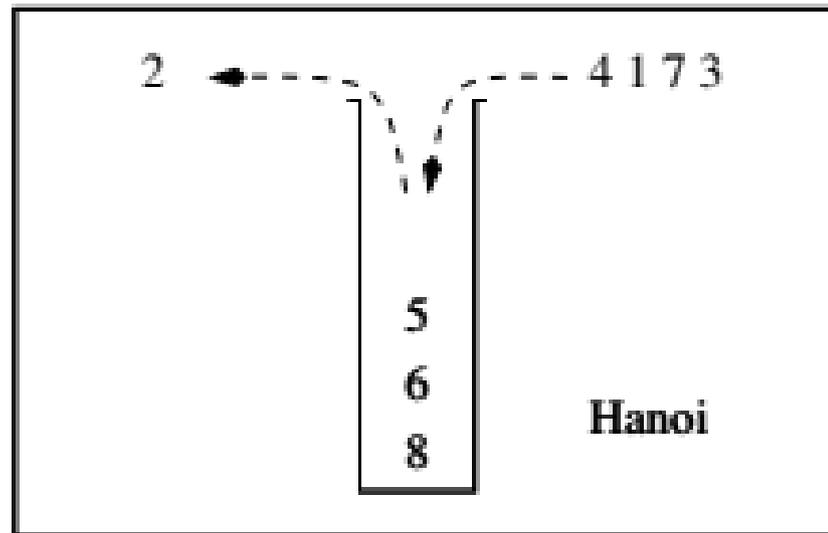
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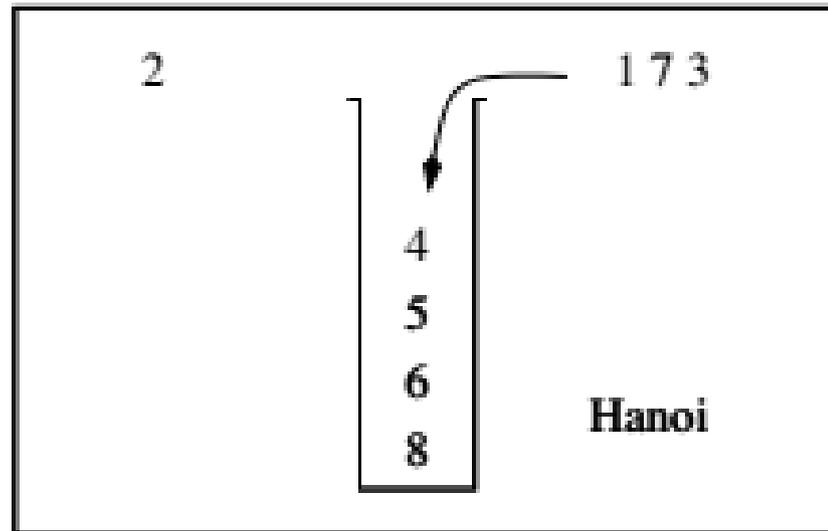
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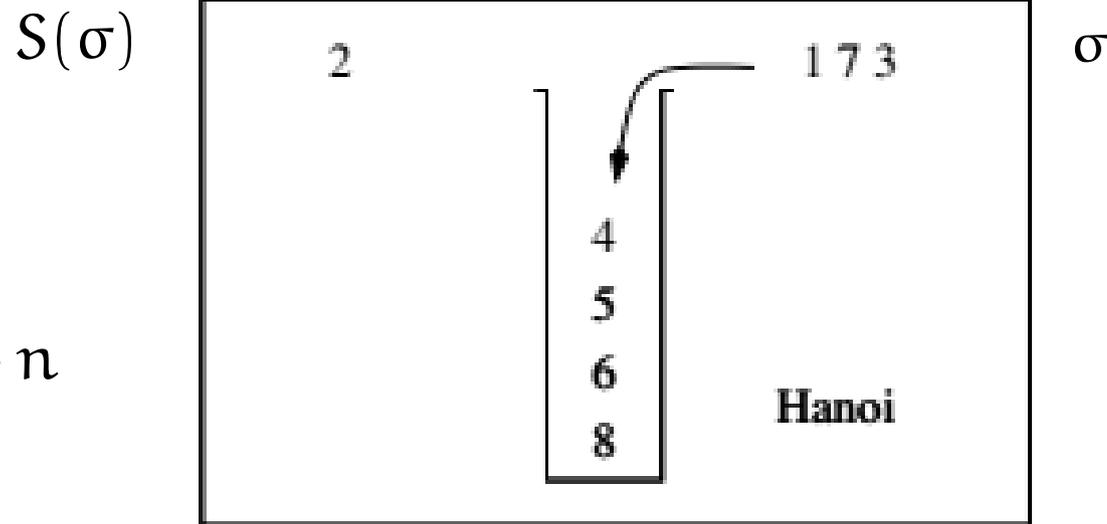
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Stack-sorting [Knuth 68]

Not D-finite?

$$S^3(\sigma) = 12 \cdots n$$



### **III. Equations with catalytic variables :**

$$0 < 1 < 2 < \dots$$





# One catalytic variable $x$

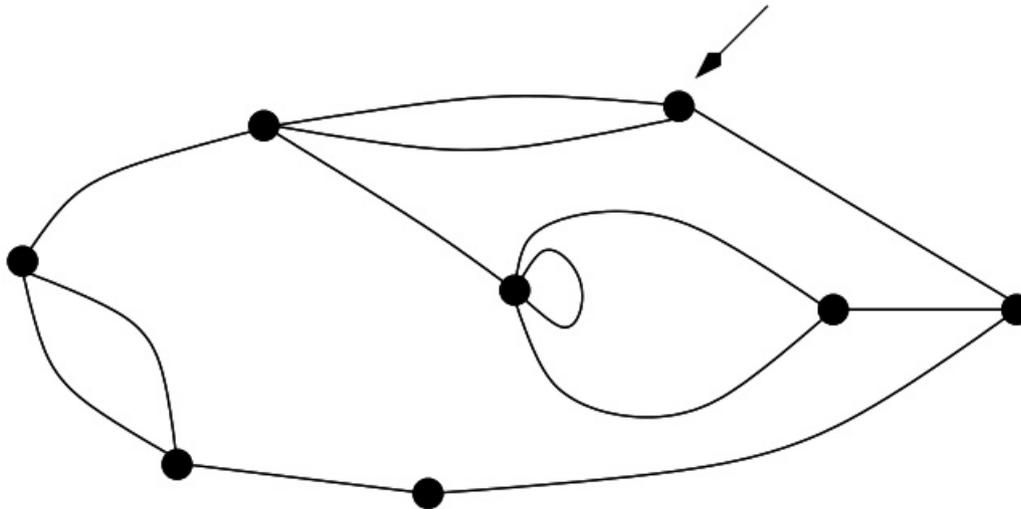
- Planar maps [Tutte 68]

$$M(x) = 1 + tx^2M(x)^2 + tx \frac{xM(x) - M(1)}{x-1}$$

or, with  $A(x)=M(x+1)$ :

$$A(x) = 1 + t(x+1)^2A(x)^2 + t(x+1) \frac{(x+1)A(x) - A(0)}{x}$$

- Many families of (uncoloured) maps



# One catalytic variable $x$

- Two-stack sortable permutations [Zeilberger 92]

$$A(x) = \frac{1}{1-tx} + xt \frac{x A(x) - A(1)}{x-1} \cdot \frac{A(x) - A(1)}{x-1}$$

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## Theorem [MBM-Jehanne 06]

Let  $P(A(x), A_1, A_2, \dots, A_k, t, x)$  be a polynomial equation in one catalytic variable  $x$ . Under natural assumptions, the series  $A(x)$  and the  $A_i$ 's are algebraic.



Algebraic series



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[Popescu 85, Swan 98]



Algebraic series



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$$A(x) = \frac{1}{1-tx} + xt \frac{x A(x) - A(1)}{x-1} \cdot \frac{A(x) - A(1)}{x-1}$$

**Theorem** [MBM-Jehanne 06]

Let  $P(A(x), A_1, A_2, \dots, A_k, t, x)$  be a polynomial equation in one catalytic variable  $x$ . Under natural assumptions, the series  $A(x)$  and the  $A_i$ 's are algebraic.

[Popescu 85, Swan 98]

effective



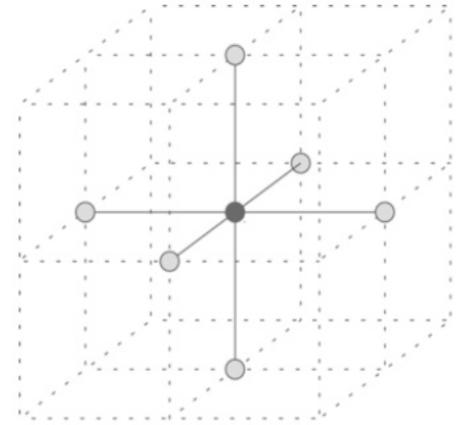
Algebraic series



# Arbitrarily many catalytic variables

- Walks in  $\mathbb{N}^d$  with unit steps  $(0, \dots, 0, \pm 1, 0, \dots, 0)$ :

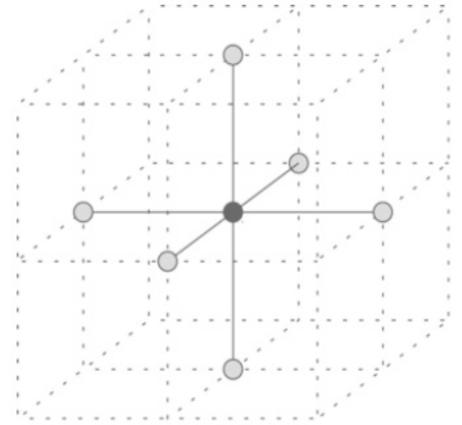
D-finite



# Arbitrarily many catalytic variables

- Walks in  $\mathbb{N}^d$  with unit steps  $(0, \dots, 0, \pm 1, 0, \dots, 0)$  :

D-finite



- Permutations with no ascending sub-sequence of length  $(d+2)$

[mbm 11]

D-finite [Gessel 90]

- Permutations sortable by  $(d+1)$  stacks ?

**IV. Back to two catalytic  
variables :  
Tutte's invariants**

A tool for proving (D)-algebraicity

# Framework

An equation in two catalytic variables, defining a series  $A(t;x,y)$ :

- **linear** in  $A(t;x,y) \equiv A(x,y)$ ,
- with **two divided differences** at  $x=0$  and  $y=0$ , of first order.

Typical form:

$$K(x,y)xyA(x,y) = R(t,x,y,A(x,0),A(0,y))$$

where the **kernel**  $K(x,y)$  satisfies:

$$K(x,y) = 1 - \bar{x}\bar{y}tS(t,x,y,A(x,0),A(0,y))$$

for **polynomials**  $R$  and  $S$ .

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for polynomials  $R$  and  $S$ .

**Example 1:** quadrant walks with Kreweras' steps

$$(1 - t(xy + \bar{x} + \bar{y}))xyQ(x, y) = xy - txQ(x, 0) - tyQ(0, y)$$

Here,

$$K(x, y) = 1 - t\bar{x}\bar{y}(x^2y^2 + y + x).$$

# Framework

Typical form:

$$K(x, y)xyA(x, y) = R(t, x, y, A(x, 0), A(0, y))$$

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for polynomials  $R$  and  $S$ .

**Example 2:** Tutte's coloured triangulations

$$T(u, y) = ux(q - 1) + tuyT(1, y)T(u, y) + tu\frac{T(u, y) - T(u, 0)}{y} - tu^2y\frac{T(u, y) - T(1, y)}{u - 1}$$

# Framework

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Or, with  $A(x, y) = T(x+1, y)$  and  $u = x+1$ ,

$$(1 - t\bar{x}\bar{y}(ux - u^2y^2 + uxy^2A(0, y)))xyA(x, y) = xuy(q - 1) - txuyA(x, 0) + tu^2y^2A(0, y)$$

# Framework

- In our equations,

$$\begin{aligned}\frac{1}{K(x, y)} &= \frac{1}{1 - t\bar{x}\bar{y}S(t, x, y, A(x, 0), A(0, y))} \\ &= \sum_{k \geq 0} t^k \bar{x}^k \bar{y}^k S(t, x, y, A(x, 0), A(0, y))^k\end{aligned}$$

has poles of unbounded order at  $x=0$  and  $y=0$ .

**Example:** for simple walks in the quadrant,

$$\frac{1}{K(x, y)} = \frac{1}{1 - t(x + \bar{x} + y + \bar{y})} = \sum_{n \geq 0} (x + \bar{x} + y + \bar{y})^n t^n$$

and the  $n$ -th coefficient has a pole of order  $n$  at  $x=0$  (and at  $y=0$ ).

# Invariants for a kernel $K(x,y)$

**Def.** A pair of series  $(I(x), J(y))$ , with rational coefficients in  $x$  (resp.  $y$ ) is a pair of invariants if

$$(I(x) - J(y)) / K(x, y)$$

has **poles of bounded order** (p.b.o.) at  $x=0$  and  $y=0$ .

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**Poles of bounded order ?**

$$\frac{1}{K(x, y)} = \frac{1}{1 - t(x + \bar{x} + y + \bar{y})} = \sum_{n \geq 0} (x + \bar{x} + y + \bar{y})^n t^n \quad \text{no}$$

$$\sum_{n \geq 0} \frac{x^n}{(1 - x^n) y^n} t^n \quad \text{no}$$

$$\sum_{n \geq 0} \frac{1 + xy}{x^2 y^5 (1 + x)^{2n} (1 - 5y)^n} t^n \quad \text{yes}$$

# Explicit kernels, explicit invariants

**Def.** A pair of series  $(I(x), J(y))$  is a pair of invariants if

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**Example 1.** Simple walks in the quadrant :

$$K(x, y) = 1 - t(x + \bar{x} + y + \bar{y}) = (1 - t(x + \bar{x})) - t(y + \bar{y})$$

Then

$$I(x) := 1 - t(x + \bar{x}) \quad \text{and} \quad J(y) := t(y + \bar{y})$$

form a pair of invariants since

$$\frac{I(x) - J(y)}{K(x, y)} = 1.$$

# Explicit kernels, explicit invariants

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has poles of bounded order at  $x=0$  and  $y=0$ .

**Example 2.** Kreweras' walks in the quadrant :

$$K(x, y) = 1 - t(xy + \bar{x} + \bar{y})$$

Then

$$I_0(x) := \bar{x}^2 - \bar{x}/t - x, \quad J_0(y) = I_0(y)$$

form a pair of invariants since

$$\frac{I_0(x) - J_0(y)}{K(x, y)} = \frac{x - y}{x^2 y^2} \cdot \frac{1}{t}$$

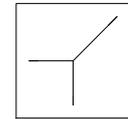
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**Def.** A pair of series  $(I(x), J(y))$  is a pair of invariants if

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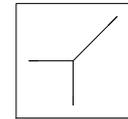
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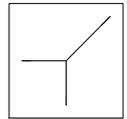
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Since

$$xy = \frac{1}{t} - \bar{x} - \bar{y} - \frac{K(x, y)}{t},$$

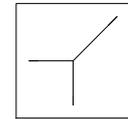
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$$K(x, y)xyQ(x, y) = xy - txQ(x, 0) - tyQ(0, y)$$

Since

$$xy = \frac{1}{t} - \bar{x} - \bar{y} - \frac{K(x, y)}{t},$$

we have

$$K(x, y) \left( xyQ(x, y) + \frac{1}{t} \right) = I_1(x) - J_1(y)$$

and a second pair of invariants:

$$I_1(x) = \frac{1}{2t} - \bar{x} - txQ(x, 0), \quad J_1(y) = -I_1(y).$$

# The invariant lemma

**Lemma.** Let  $(I(x), J(y))$  be a pair of invariants such that the series

$$(I(x) - J(y)) / K(x, y)$$

not only has p.b.o. at  $x=0$  and  $y=0$ , but in fact **vanishes** at  $x=0$  and  $y=0$  (“strict” invariants). Then  $I(x)$  and  $J(y)$  are **trivial**:

$$I(x) = J(y) \text{ is independent of } x \text{ (and } y).$$

**Lemma.** The componentwise sum and product of two pairs of invariants  $(I_0(x), J_0(y)), (I_1(x), J_1(y))$  is another pair of invariants.

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$I(x) = J(y)$  is independent of  $x$  (and  $y$ ).

$$\frac{I(x) - J(y)}{K(x, y)} = \sum_n \frac{xy p_n(x, y)}{d_n(x) d'_n(y)} t^n$$

**Lemma.** The componentwise sum and product of two pairs of invariants  $(I_0(x), J_0(y)), (I_1(x), J_1(y))$  is another pair of invariants.

# Application: Kreweras' walks in the quadrant

- Two pairs of invariants:

$$I_0(x) := \bar{x}^2 - \bar{x}/t - x = J_0(x)$$

$$\frac{I_0(x) - J_0(y)}{K(x, y)} = \frac{x - y}{x^2 y^2} \cdot \frac{1}{t}$$

$$I_1(x) = \frac{1}{2t} - \bar{x} - txQ(x, 0) = -J_1(x)$$

$$\frac{I_1(x) - J_1(y)}{K(x, y)} = xyQ(x, y) + \frac{1}{t}$$

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- **Two pairs of invariants:**

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- **Observation:** the following pair of invariants has **no pole** at  $x=0$  or  $y=0$ :

$$I(x) := I_1(x)^2 - I_0(x), \quad J(y) := J_1(y)^2 - J_0(y)$$

# Application: Kreweras' walks in the quadrant

- **Two pairs of invariants:**

$$I_0(x) := \bar{x}^2 - \bar{x}/t - x = J_0(x) \quad \left| \quad I_1(x) = \frac{1}{2t} - \bar{x} - txQ(x, 0) = -J_1(x)\right.$$
$$\frac{I_0(x) - J_0(y)}{K(x, y)} = \frac{x - y}{x^2 y^2} \cdot \frac{1}{t} \quad \left| \quad \frac{I_1(x) - J_1(y)}{K(x, y)} = xyQ(x, y) + \frac{1}{t}\right.$$

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Moreover,  $(I(x) - J(y))/K(x, y)$  **vanishes** at  $x=0$  and  $y=0$ !

# Application: Kreweras' walks in the quadrant

- **Two pairs of invariants:**

$$I_0(x) := \bar{x}^2 - \bar{x}/t - x = J_0(x) \quad \left| \quad I_1(x) = \frac{1}{2t} - \bar{x} - txQ(x, 0) = -J_1(x)\right.$$
$$\frac{I_0(x) - J_0(y)}{K(x, y)} = \frac{x - y}{x^2 y^2} \cdot \frac{1}{t} \quad \left| \quad \frac{I_1(x) - J_1(y)}{K(x, y)} = xyQ(x, y) + \frac{1}{t}\right.$$

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Moreover,  $(I(x) - J(y))/K(x, y)$  **vanishes** at  $x=0$  and  $y=0$ !

By the invariant lemma,  $I(x)=I(0)$ , that is,

$$(txQ(x, 0) - 1/(2t))^2 + 2tQ(x, 0) + x = -1/(2t)^2 + 2tQ(0, 0)$$

# Application: Kreweras' walks in the quadrant

**Summary:** starting from an equation in two catalytic variables, involving  $Q(x,y)$ ,  $Q(x,0)$  and  $Q(0,y)$ , we have derived an equation in **only one catalytic variable**, involving  $Q(x,0)$  and  $Q(0,0)$  only

⇒ algebraicity

$$(txQ(x,0) - 1/(2t))^2 + 2tQ(x,0) + x = -1/(2t)^2 + 2tQ(0,0)$$

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**Summary:** starting from an equation in two catalytic variables, involving  $Q(x,y)$ ,  $Q(x,0)$  and  $Q(0,y)$ , we have derived an equation in **only one catalytic variable**, involving  $Q(x,0)$  and  $Q(0,0)$  only  
**⇒ algebraicity**

$$(txQ(x,0) - 1/(2t))^2 + 2tQ(x,0) + x = -1/(2t)^2 + 2tQ(0,0)$$

**Theorem** [Kreweras 65, Gessel 86, mbm 05...]: let  $Z(t) \equiv Z$  be the only series in  $t$  such that  $Z=t(2+Z^3)$ . Then

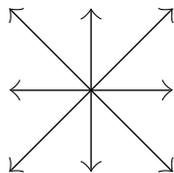
$$Q(x,0) = \frac{1}{tx} \left( \frac{1}{2t} - \frac{1}{x} - \left( \frac{1}{Z} - \frac{1}{x} \right) \sqrt{1 - xZ^2} \right).$$

**V. Walks in a quadrant:  
the whole picture**

# About twenty years ago...

- Systematic study of quadrant walks

Set of steps (“model”) in



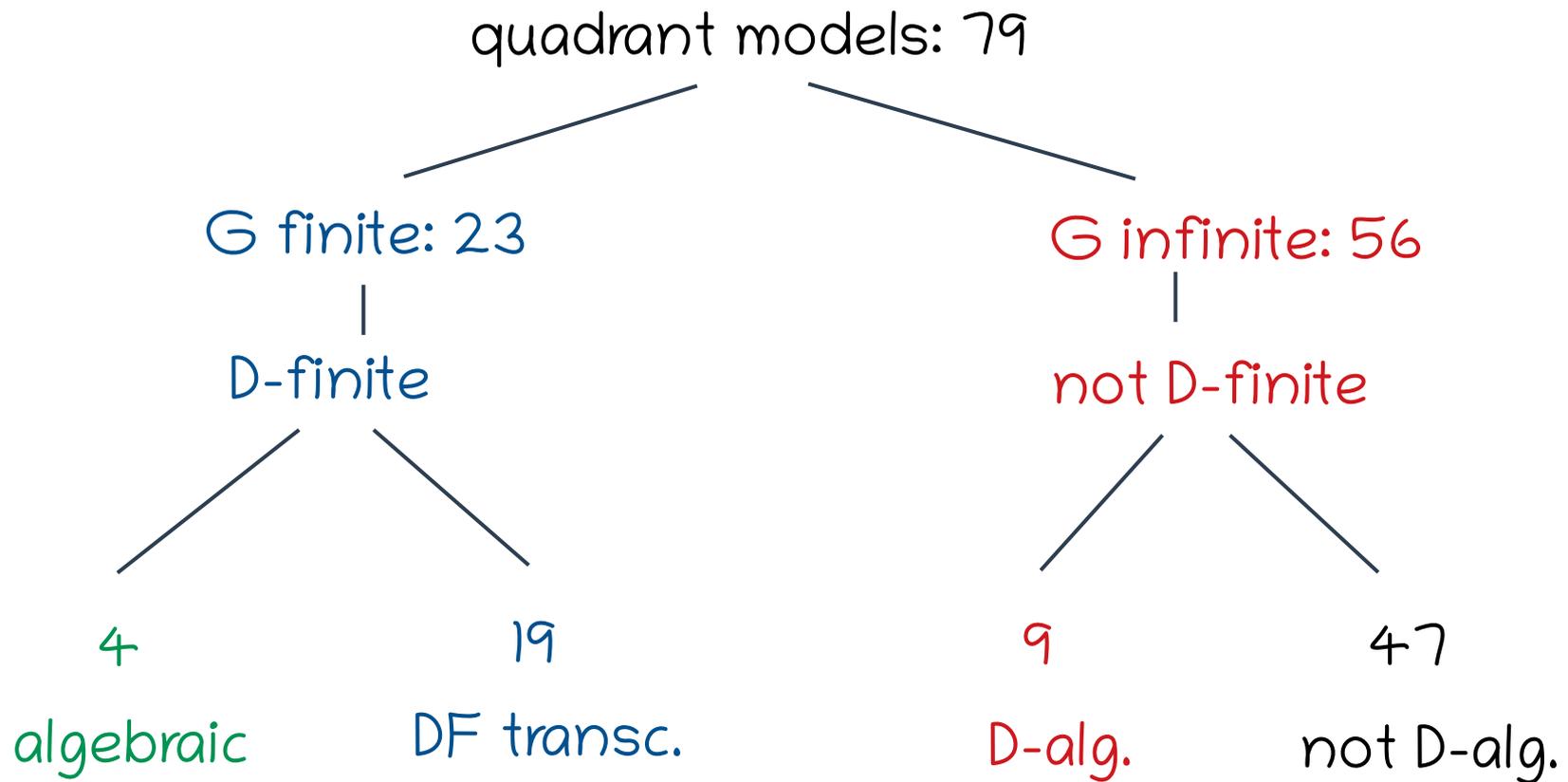
- Some models are trivial, or equivalent to a half plane problem

⇒ 79 really interesting and distinct small step models [mbm-Mishna 10]

- Systematic approach via a functional equation

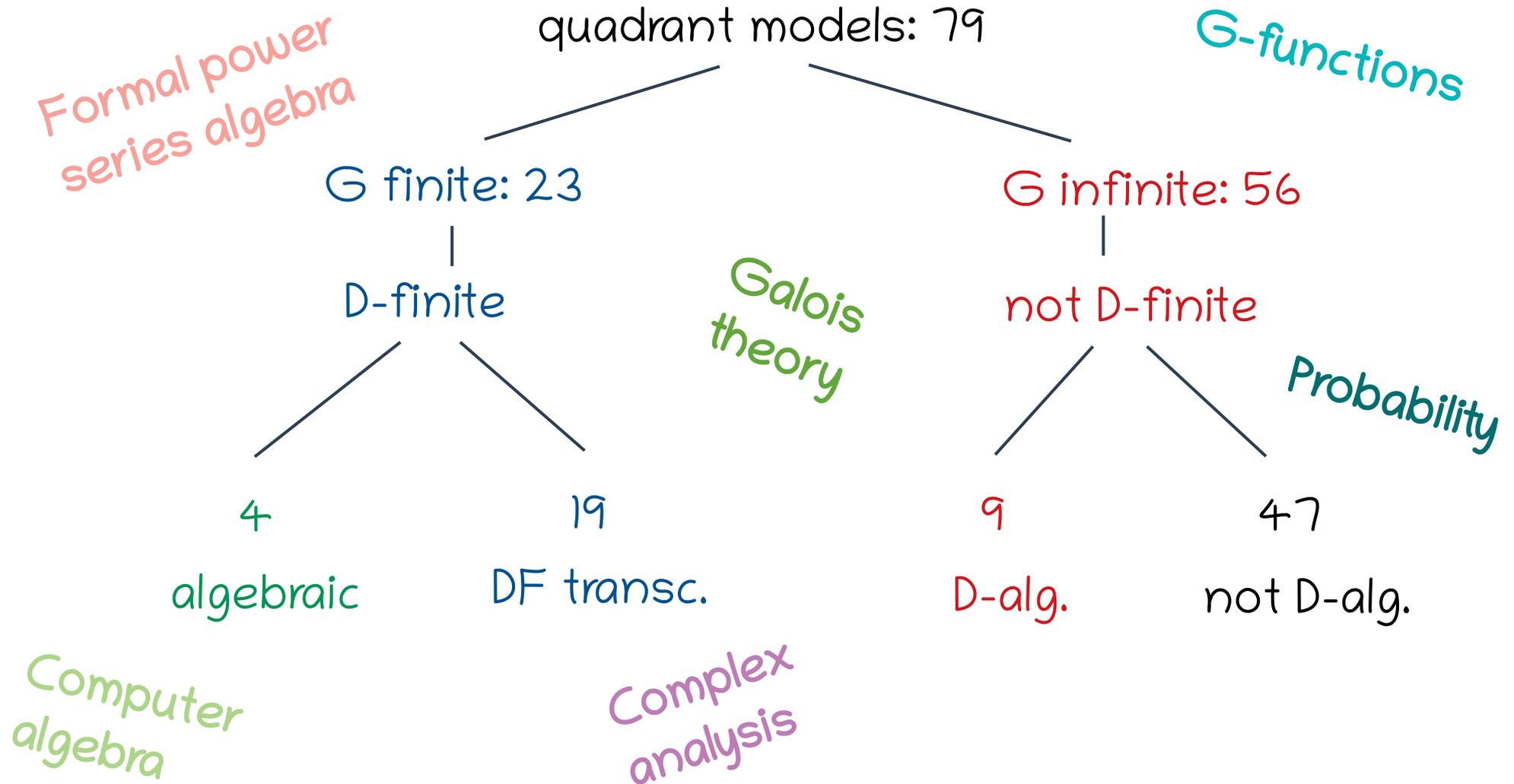
$$K(x, y)xyQ(x, y) = xy - txQ(x, 0) - tyQ(0, y)$$

# Twenty years later: classification of quadrant walks



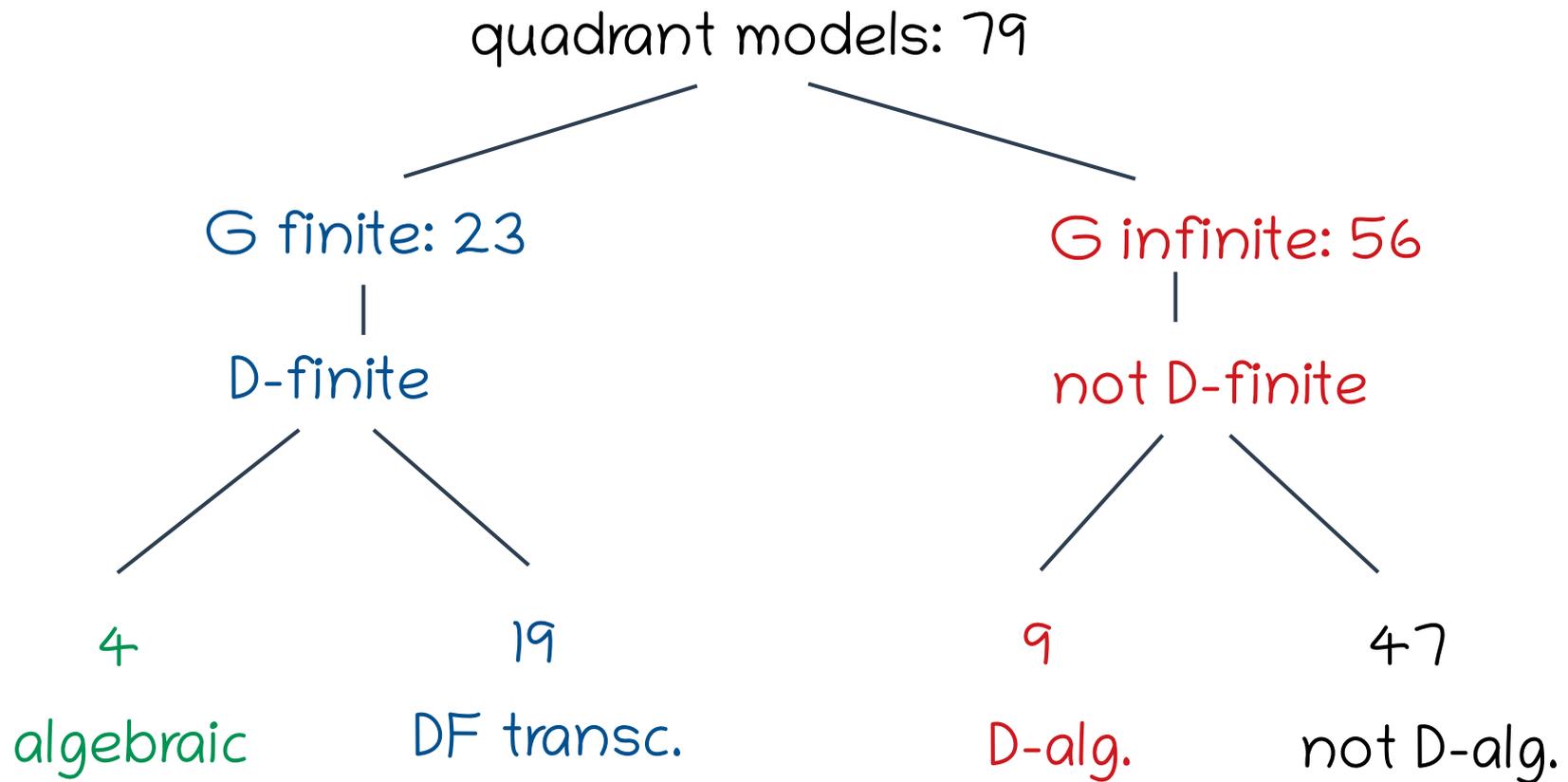
Bernardi, Bostan, mbm, Budd, Chyzak, Dreyfus, Elvey Price, Gessel, Hardouin, Kauers, Koutschan, Kurkova, Melczer, Mishna, Pech, Raschel, Rechnitzer, Roques, Salvy, Singer, van Hoeij, Zeilberger...

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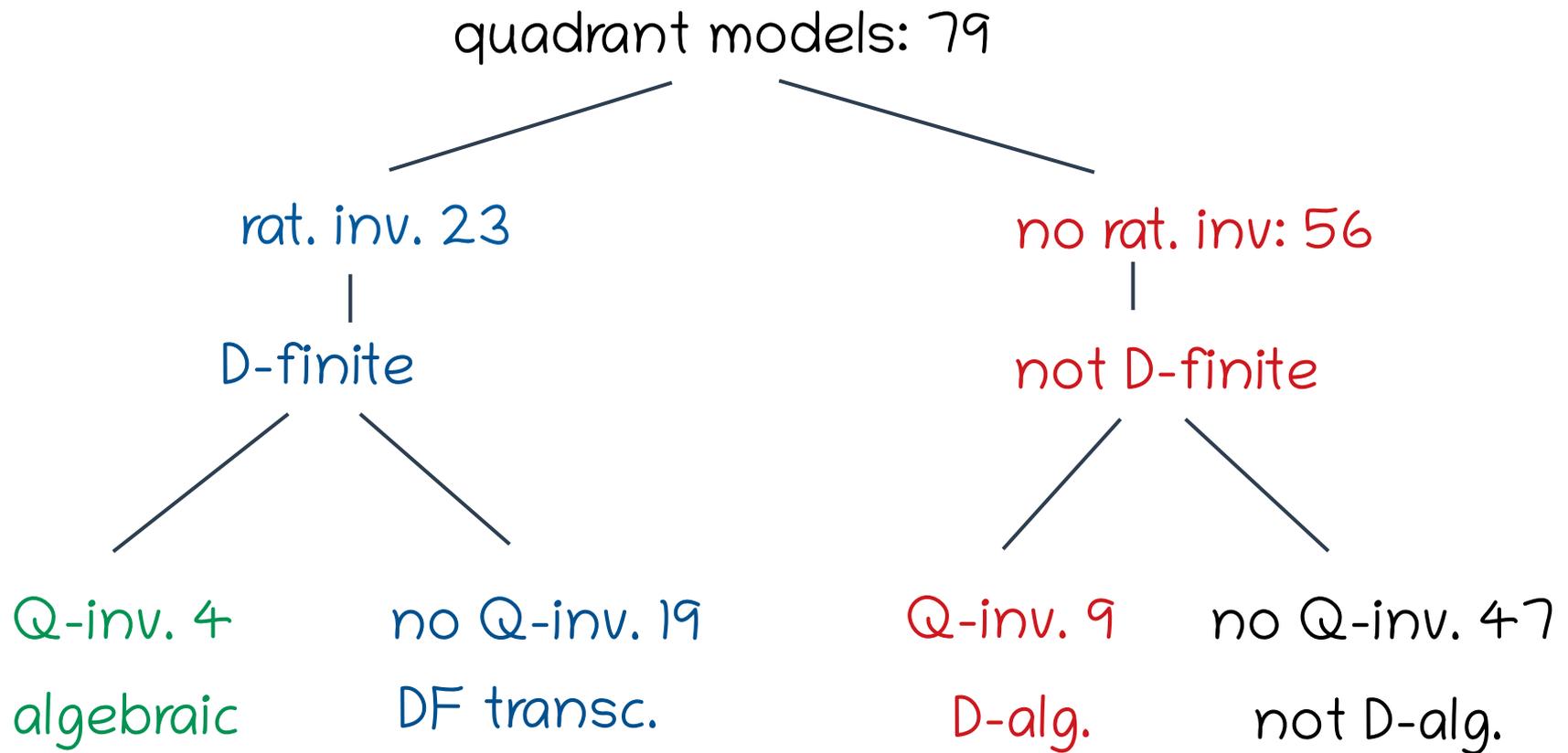
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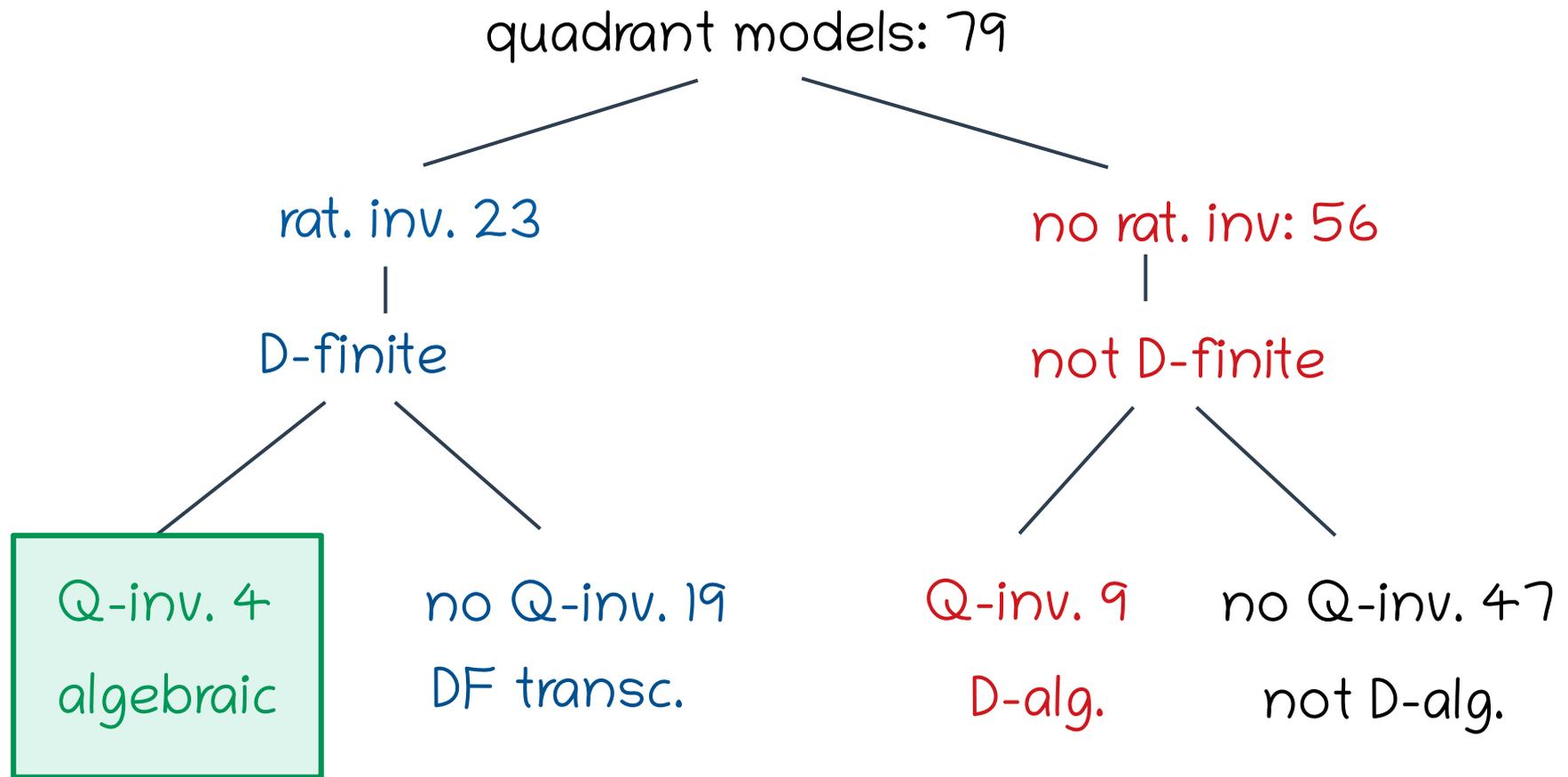
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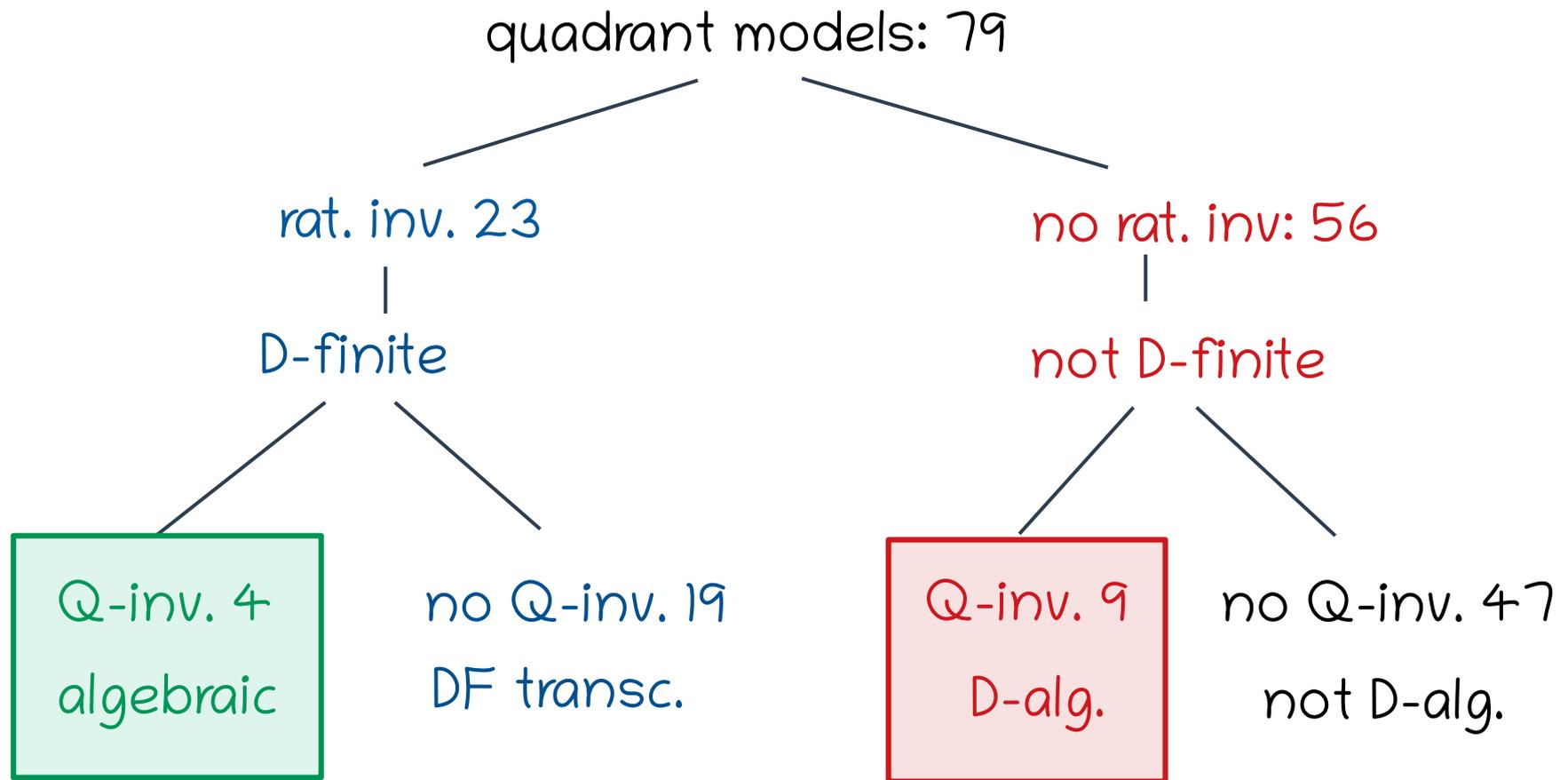
# Twenty years later: classification of quadrant walks



[Bernardi, mbm, Raschel 17(a)]

Bernardi, Bostan, mbm, Budd, Chyzak, Dreyfus, Elvey Price, Gessel, Hardouin, Kauers, Koutschan, Kurkova, Melczer, Mishna, Pech, Raschel, Rechnitzer, Roques, Salvy, Singer, van Hoeij, Zeilberger...

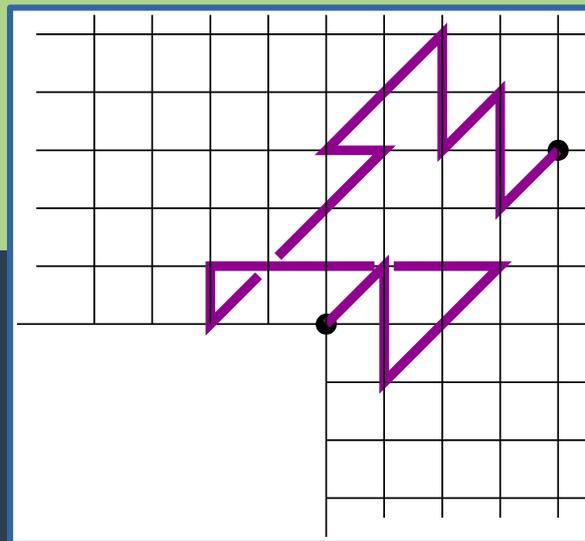
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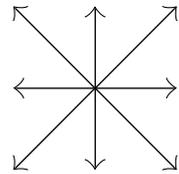
# VI. Walks in three quadrants: a partial picture



# Since 2015...

- Systematic study of **three-quadrant walks** with small steps

Set of steps in



- Some models are trivial, or equivalent to a half plane problem

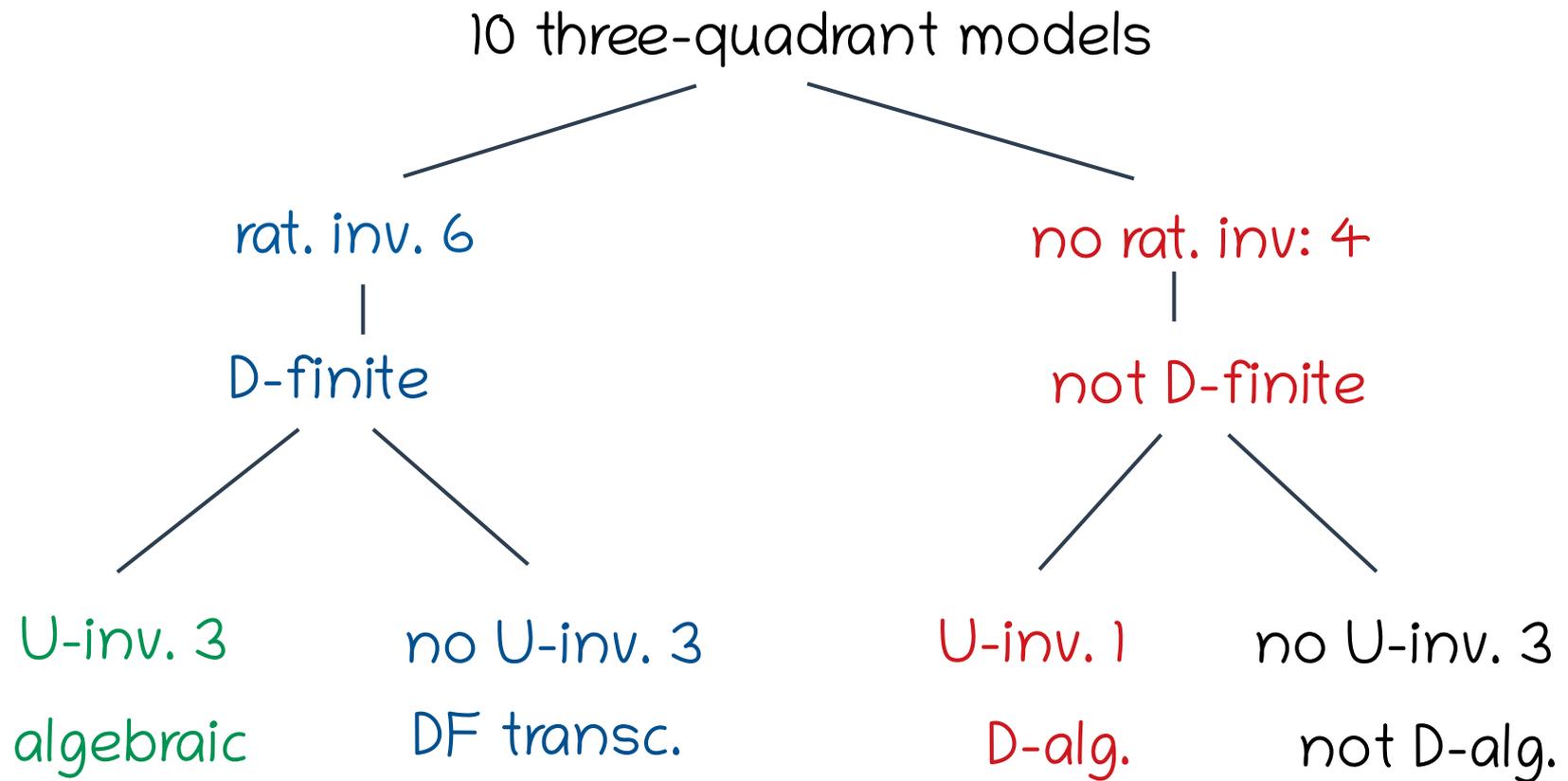
⇒ **74 really interesting and distinct models**

- For ten  $x/y$ -symmetric step sets\*, an equation reminiscent of quadrant equations:

$$2(1 - t(\bar{x}\bar{y} + x + y))xyU(x, y) = y - 2tU(x, 0) + (ty + 2tx - 1)yD(y)$$

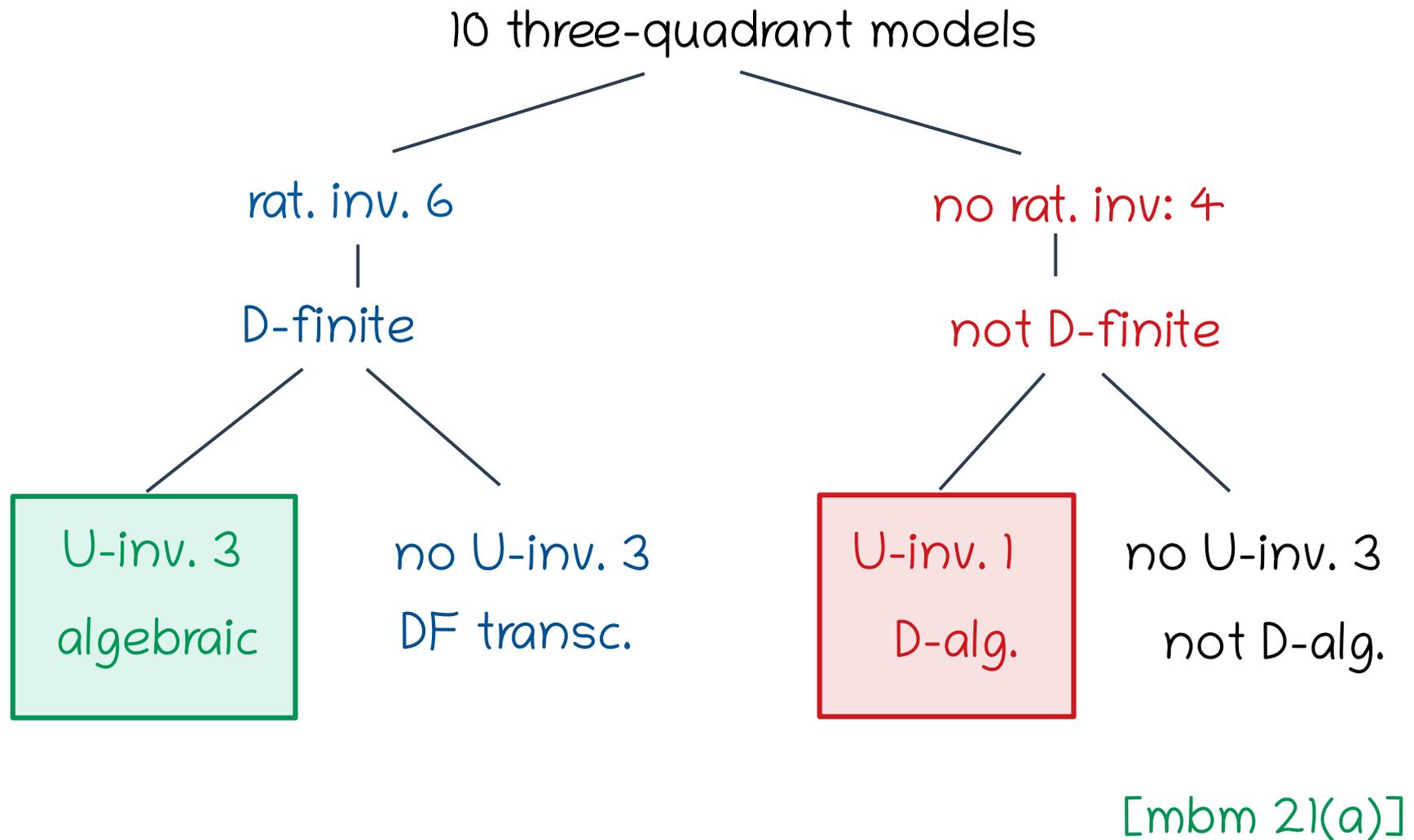
\* those with no NW nor SE step

# A partial classification of three-quadrant walks



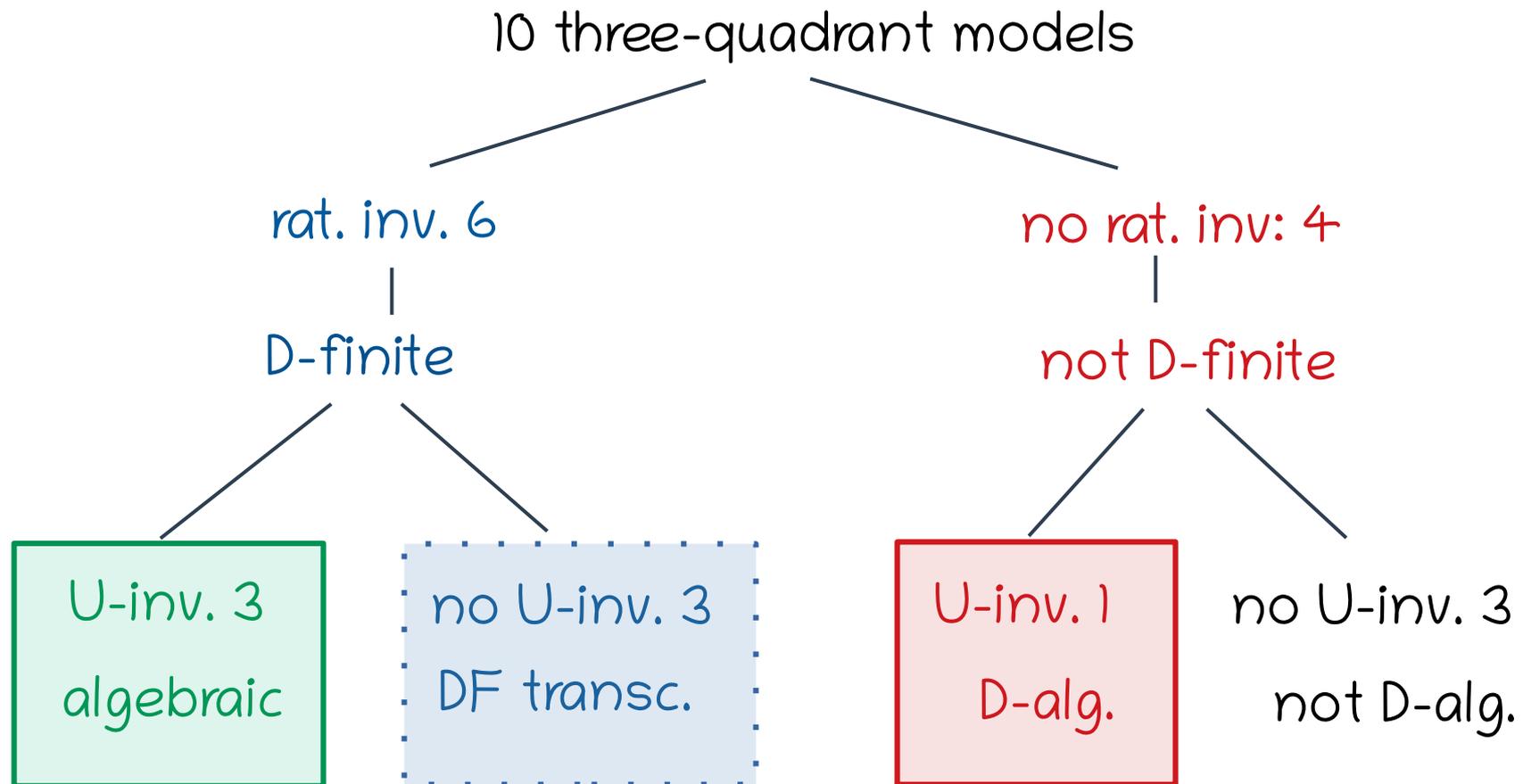
mbm, Budd, Dreyfus, Elvey Price, Mustapha, Raschel, Trotignon, Wallner...

# A partial classification of three-quadrant walks



mbm, Budd, Dreyfus, Elvey Price, Mustapha, Raschel, Trotignon, Wallner...

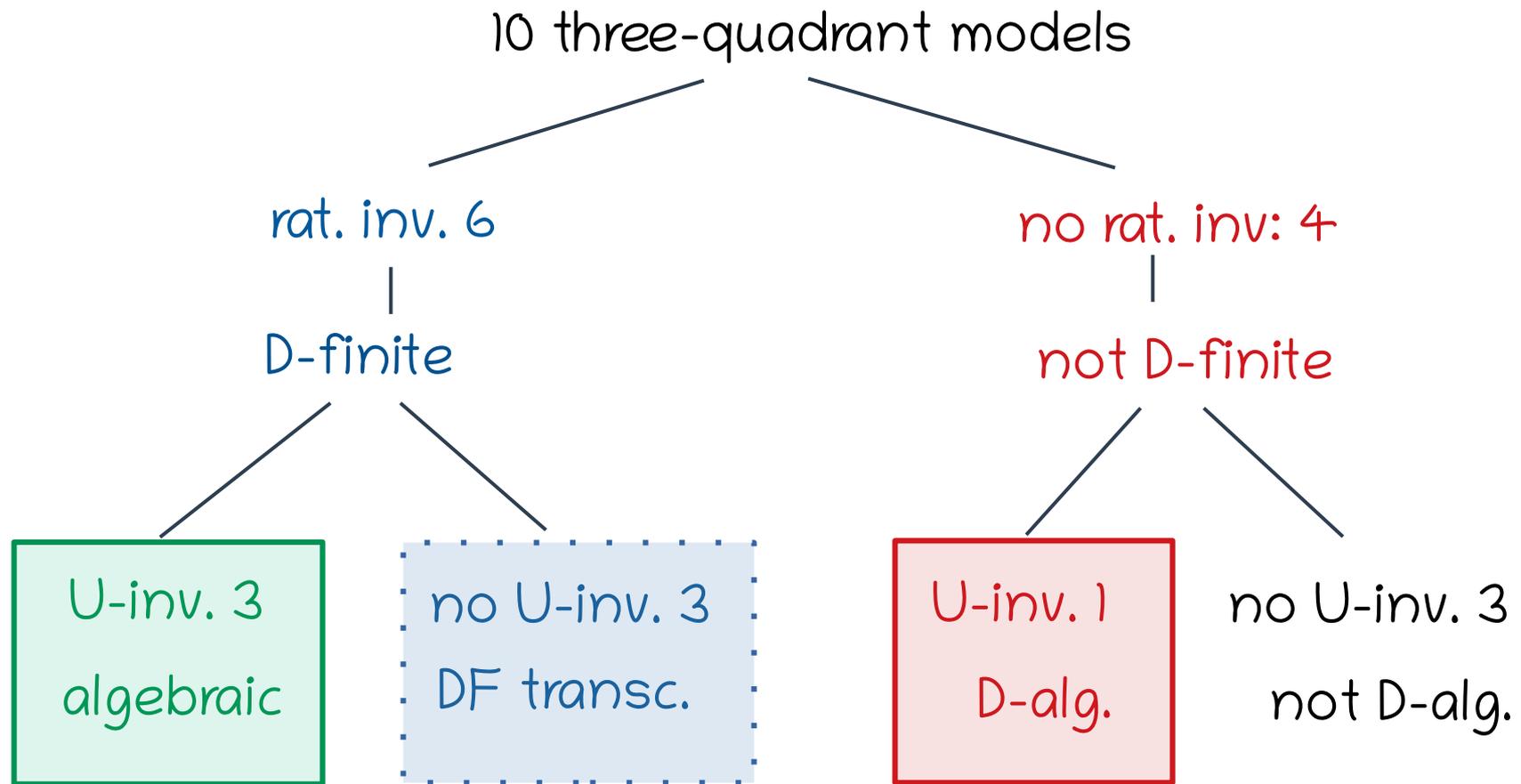
# A partial classification of three-quadrant walks



[mbm 2](a)]

mbm, Budd, Dreyfus, Elvey Price, Mustapha, Raschel, Trotignon, Wallner...

# A partial classification of three-quadrant walks



[mbm 21(a)]

**Theorem** [Elvey Price 22(a)] The Gfs of quadrant walks and three quadrant walks with the same (small) steps are of the same nature, at least w.r.t.  $x$  and  $y$ .

# Applications of Tutte's invariants

- Properly coloured triangulations [Tutte 73-84]
- General colourings of maps (= Potts model) [Bernardi-mbm 11-17]
- Quadrant walks [Bernardi, mbm, Raschel 17(a)]
- Three-quadrant walks [mbm 21(a)]
- **Continuous** walks in a cone [mbm, Elvey Price, Franceschi, Hardouin, Raschel...]



# Perspectives/work in progress

- (D)-algebraicity for more three-quadrant walks, e.g. Gessel's walks
- Quadrant walks with larger steps (P. Bonnet)
- 3-dimensional walks: from 3 to 2 catalytic variables?

# Perspectives/work in progress

- (D)-algebraicity for more three-quadrant walks, e.g. Gessel's walks
- Quadrant walks with larger steps (P. Bonnet)
- 3-dimensional walks: from 3 to 2 catalytic variables?

Constructing invariants,

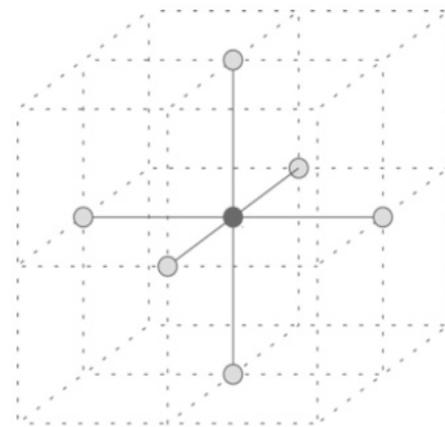
- from an explicit rational kernel
- or from a functional equation

should be **automatized** (if possible...).

[Buchacher, Kauers, Pogudin 20(a)]

# Arbitrarily many catalytic variables

- Walks in  $\mathbb{N}^d$  with unit steps  $(0, \dots, 0, \pm 1, 0, \dots, 0)$ :



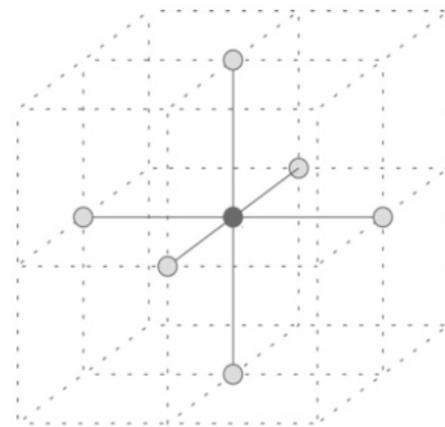
$$A(x_1, \dots, x_d) \equiv A(\mathbf{x}) = 1 + tA(\mathbf{x}) \sum_{i=1}^d (x_i + \bar{x}_i)$$

D-finite

$$- t \sum_{i=1}^d \bar{x}_i A(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_d)$$

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- Permutations with no ascending sub-sequence of length  $d+2$  :

$$A(\mathbf{x}) = x_{1,d} \left( 1 + tA(\mathbf{x}) + t \frac{A(\mathbf{x}) - x_1 A(1, x_2, \dots, x_d)}{x_1 - 1} \right. \\ \left. + t \sum_{j=2}^d x_{j,d} \frac{A(\mathbf{x}) - A(x_1, \dots, x_{j-2}, x_{j-1}x_j, 1, x_{j+1}, \dots, x_d)}{x_j - 1} \right)$$

with  $x_{j,k} = x_j x_{j+1} \cdots x_k$ .

D-finite [Gessel 90]

# Implicit kernels, implicit invariants

**Def.** A pair of series  $(I(x), J(y))$  is a pair of invariants if

$$(I(x) - J(y)) / K(x, y)$$

has poles of bounded order at  $x=0$  and  $y=0$ .

**Example 2.** Tutte's coloured triangulations (with  $u=x+1$ ):

$$(1 - t(u\bar{y} - \bar{x}u^2y + uyA(0, y)))xyA(x, y)$$

$$= xuy(q - 1) - txuyA(x, 0) + tu^2y^2A(0, y)$$

$$I(x) = \frac{ut^2}{x} - \frac{x}{u^2} + t^2A(x, 0)$$

$$J(y) = y^2t^2A(0, y)^2 + t(2t - y)A(0, y) + \frac{t^2}{y^2} - \frac{t}{y} + qty$$

form a pair of invariants since

$$\frac{I(x) - J(y)}{K(x, y)} = \frac{1}{x+1} (-tyA(x, y) + tyA(0, y) + t\bar{y} - 1) + \frac{1}{(x+1)^2}.$$

# Application: Kreweras' walks in the quadrant

- The kernel:

$$K(x, y) = 1 - t(xy + \bar{x} + \bar{y})$$

- **Explicit rational invariants** from the kernel:

$$I_0(x) := \bar{x}^2 - \bar{x}/t - x, \quad J_0(y) = I_0(y)$$

with

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$$K(x, y)xyQ(x, y) = xy - txQ(x, 0) - tyQ(0, y)$$

implies that the pair

$$I_1(x) = \frac{1}{2t} - \bar{x} - txQ(x, 0), \quad J_1(y) = -I_1(y)$$

satisfies

$$\frac{I_1(x) - J_1(y)}{K(x, y)} = -xyQ(x, y) - \frac{1}{t}.$$