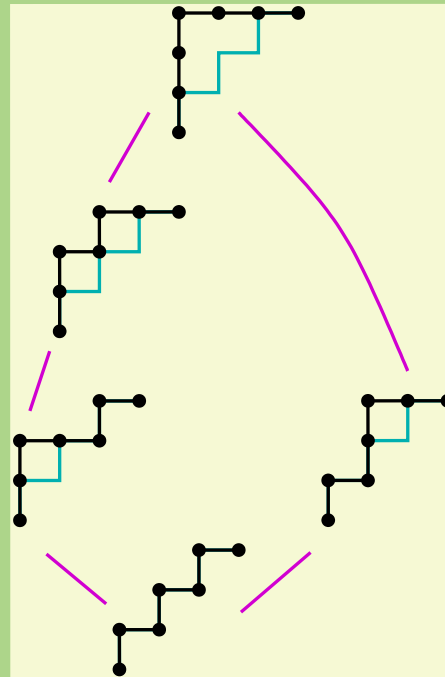
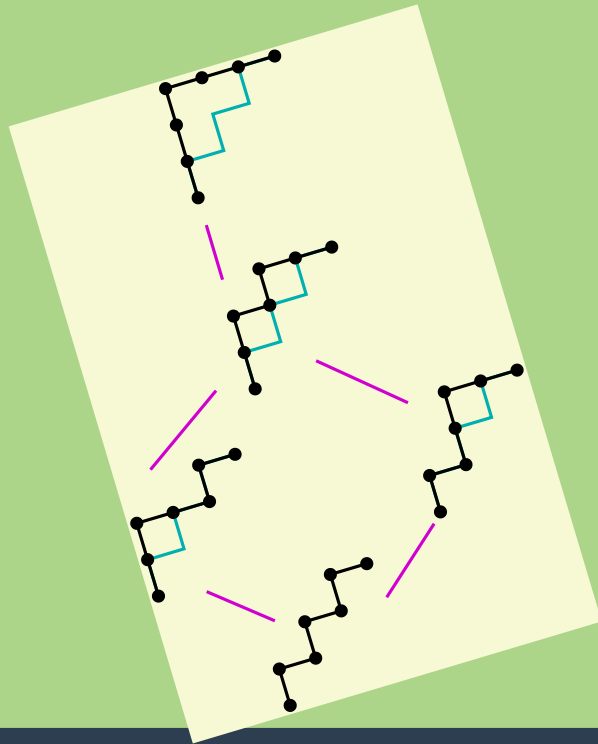


# Posets on Dyck paths and their intervals

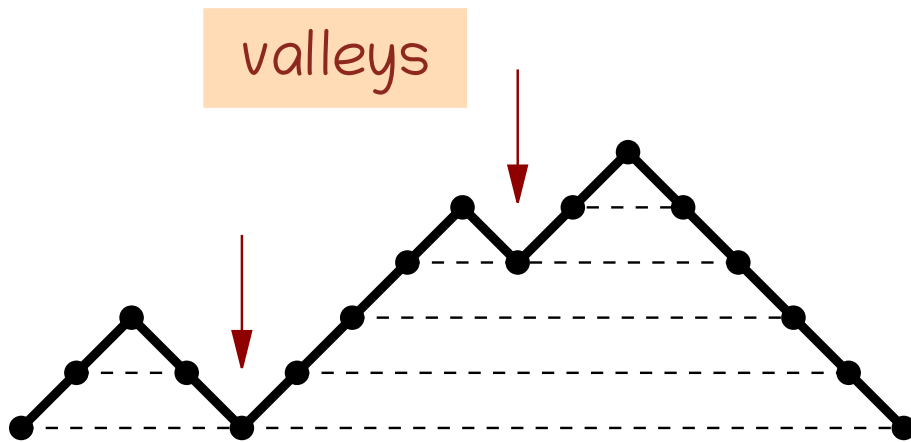


Dov Tamari

Mireille Bousquet-Mélou  
CNRS, Université de Bordeaux, France

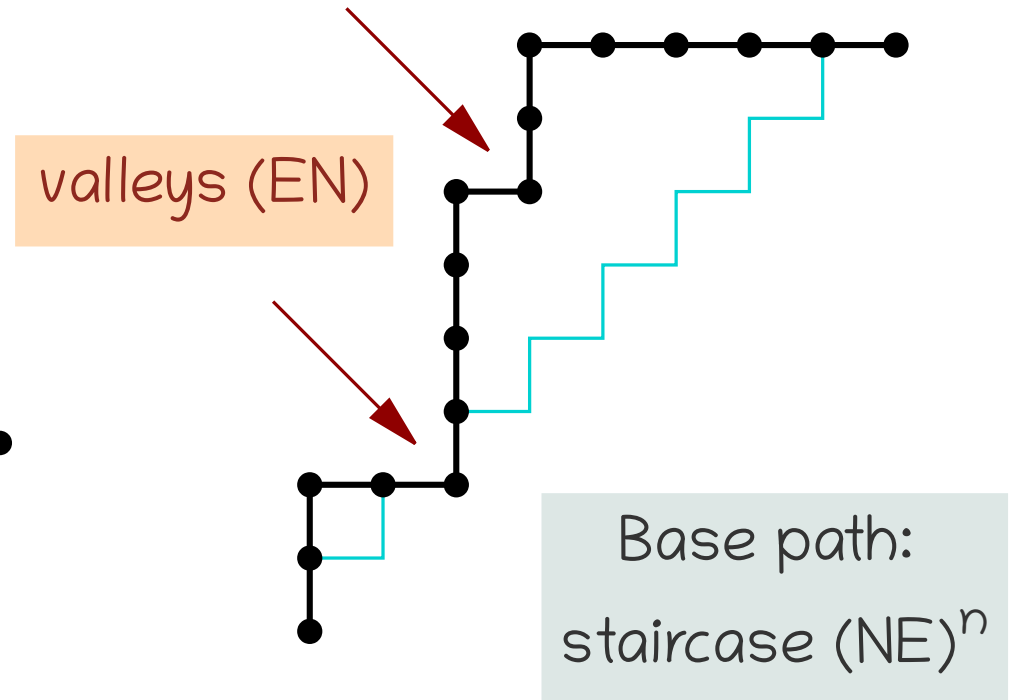
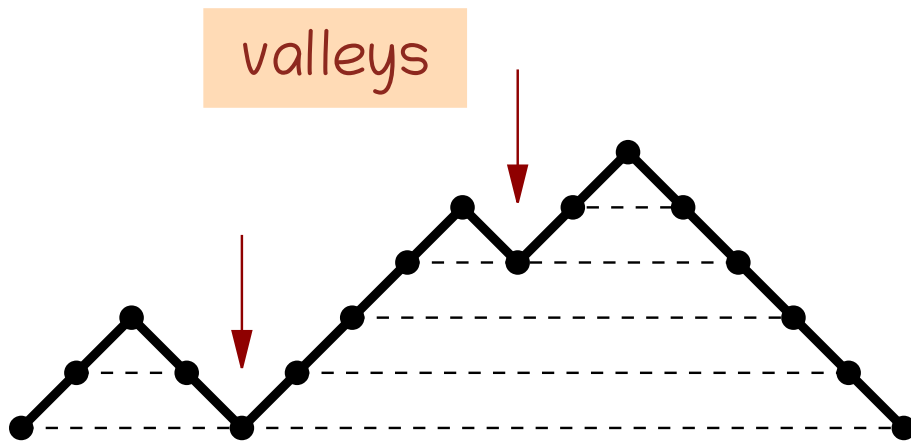
# Dyck paths

- A Dyck path of **size**  $n=8$  (size=number of up steps)



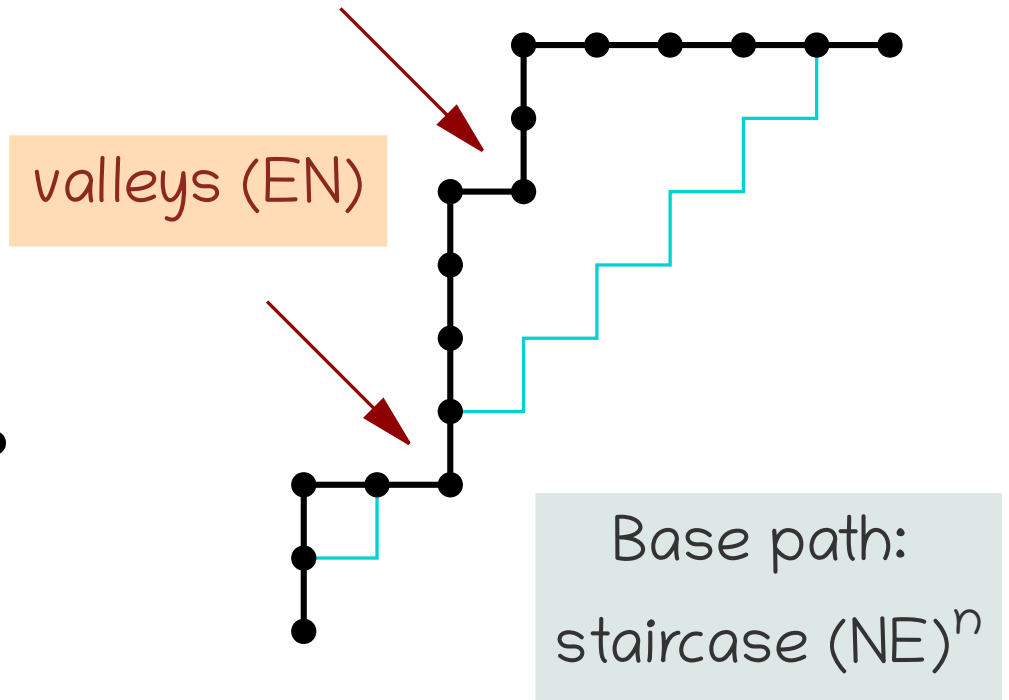
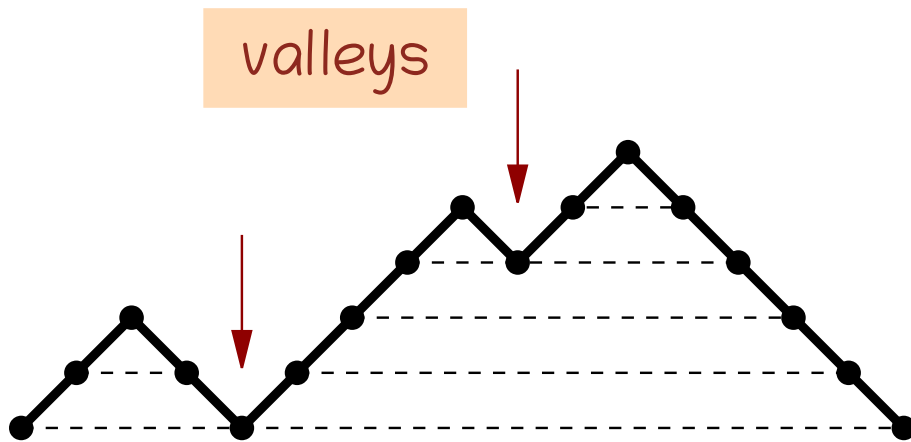
# Dyck paths

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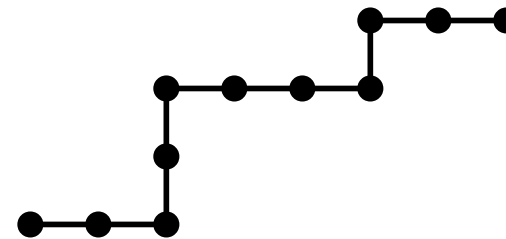


# Dyck paths

- A Dyck path of **size**  $n=8$  (size=number of up steps)

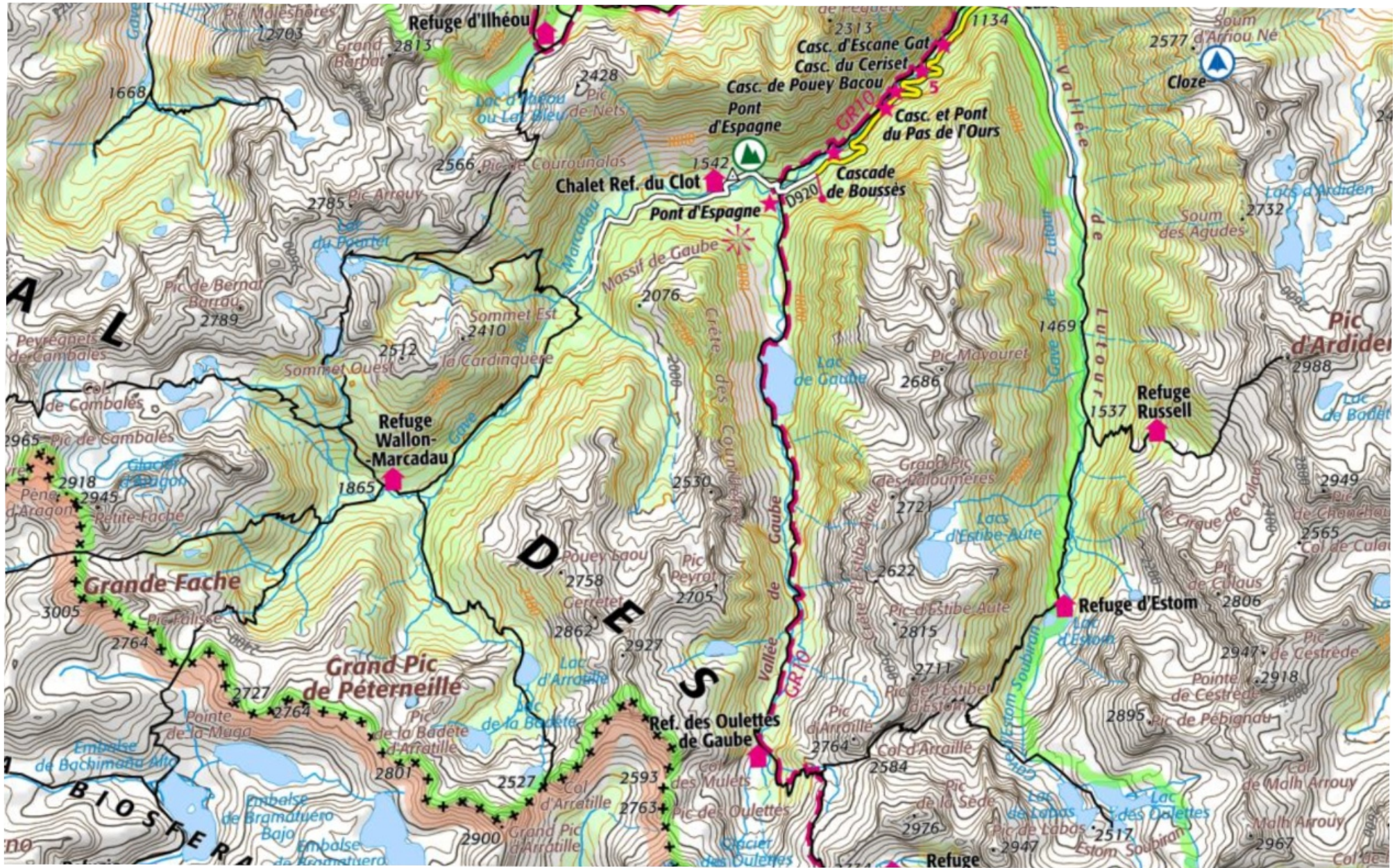


- Also: arbitrary NE paths/walks



# Paths, walks and maps: natural connections

# Paths, walks and maps: natural connections

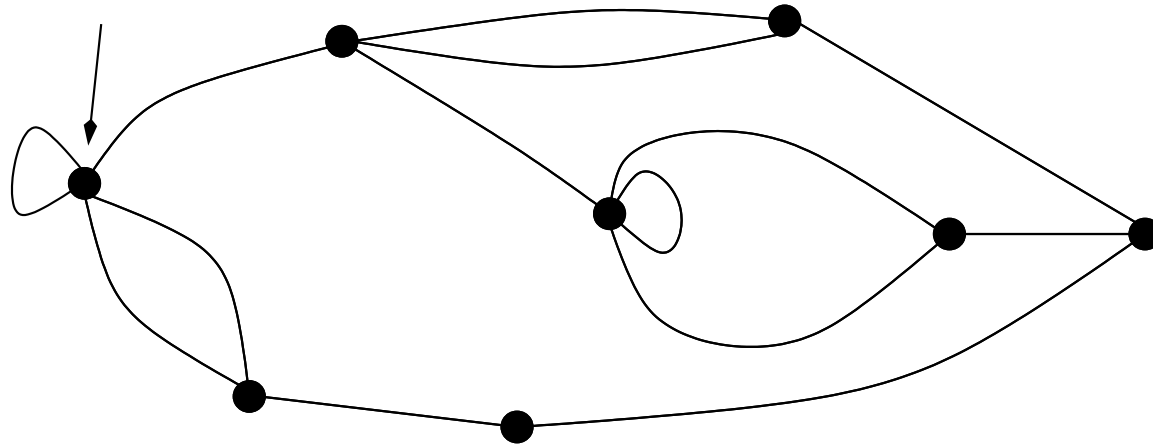


# Planar maps

**Def.** A connected planar (multi)graph, given with an embedding in the plane, taken up to continuous deformation.

## Components:

- vertices
- edges
- faces

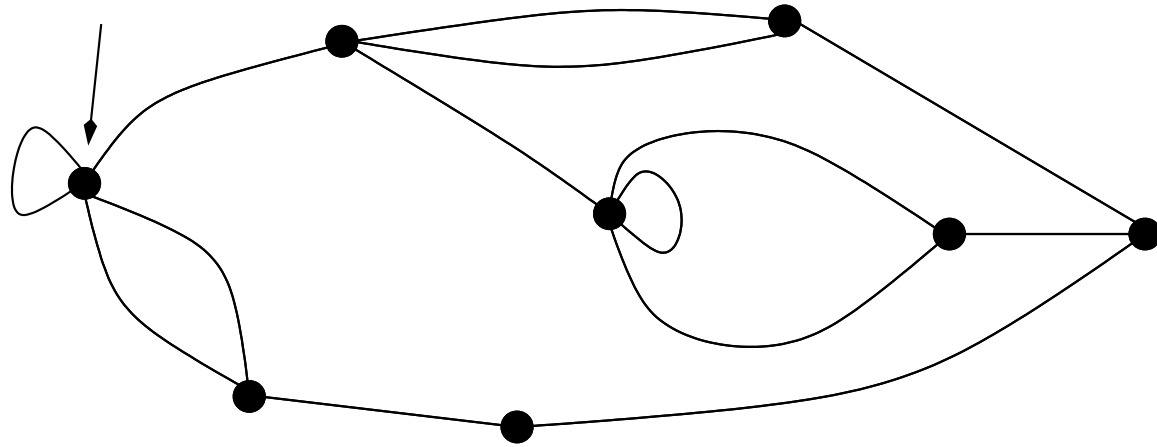


# Planar maps

**Def.** A connected planar (multi)graph, given with an embedding in the plane, taken up to continuous deformation.

## Components:

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**Rooted map:** a distinguished corner in the outer face

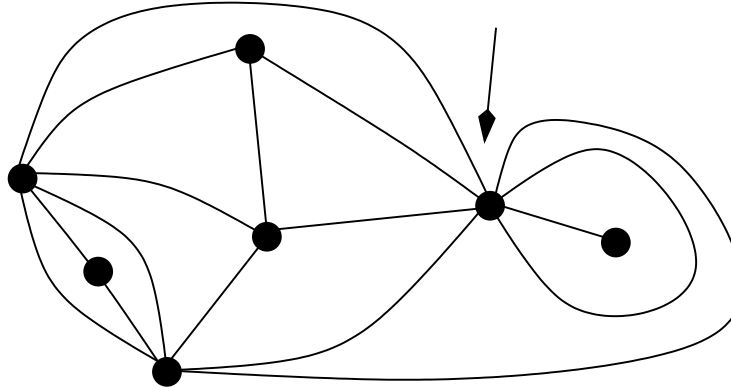


# Planar maps

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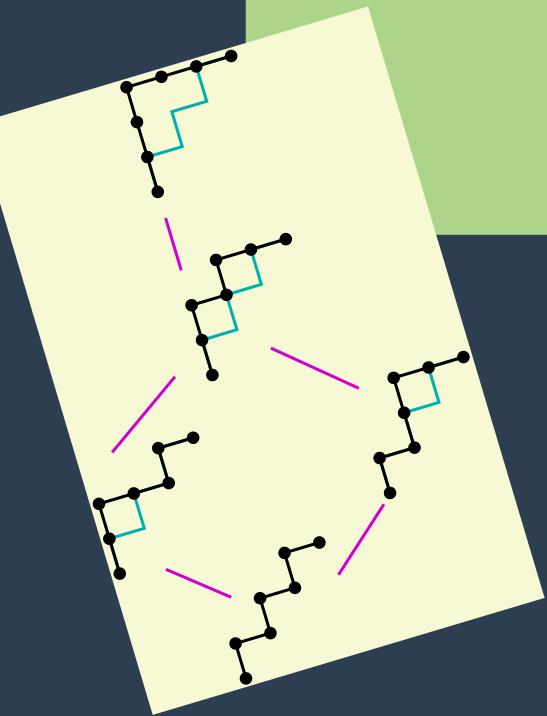
**Rooted map:** a distinguished corner in the outer face

**Triangulation:** all faces have **degree 3**

# Outline

- Stanley lattices
- Tamari lattices
- Recursive constructions and functional equations
- Greedy Tamari posets
- Final comments

# I. Stanley's lattice

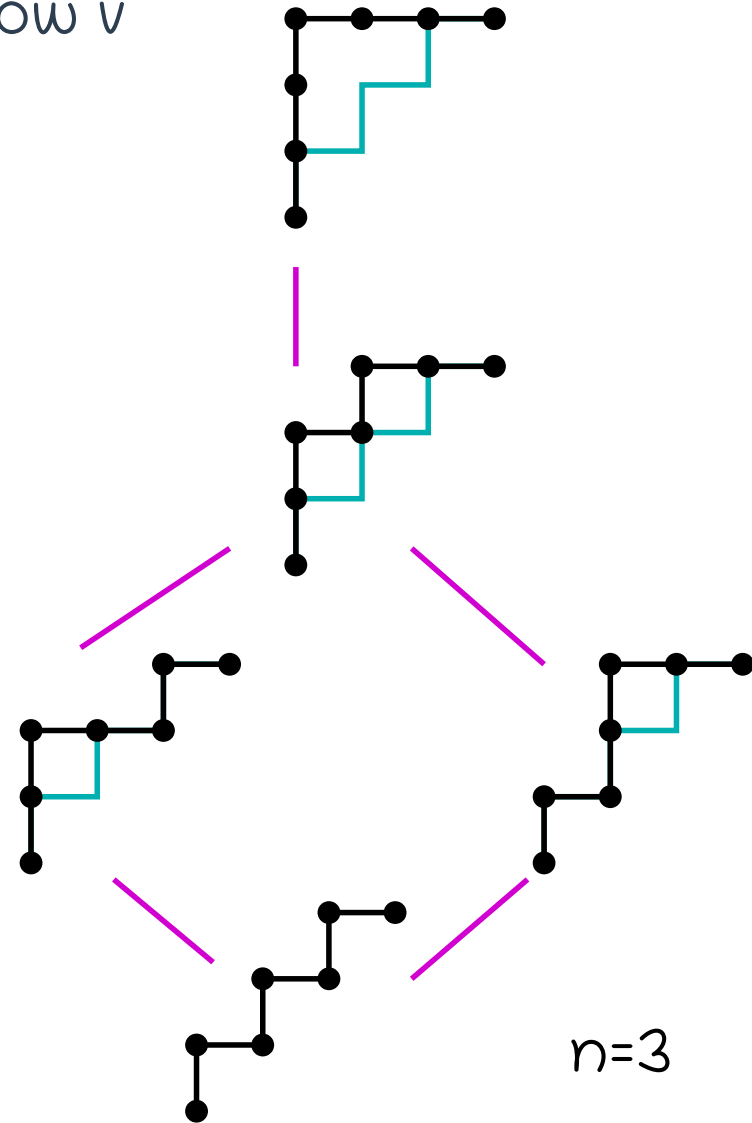


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# Stanley's lattice $S_n$

A poset on Dyck paths of size  $n$

**Def.** The path  $u$  is smaller than  $v$  if  $u$  lies below  $v$   
(or:  $u$  is contained in  $v$ ).

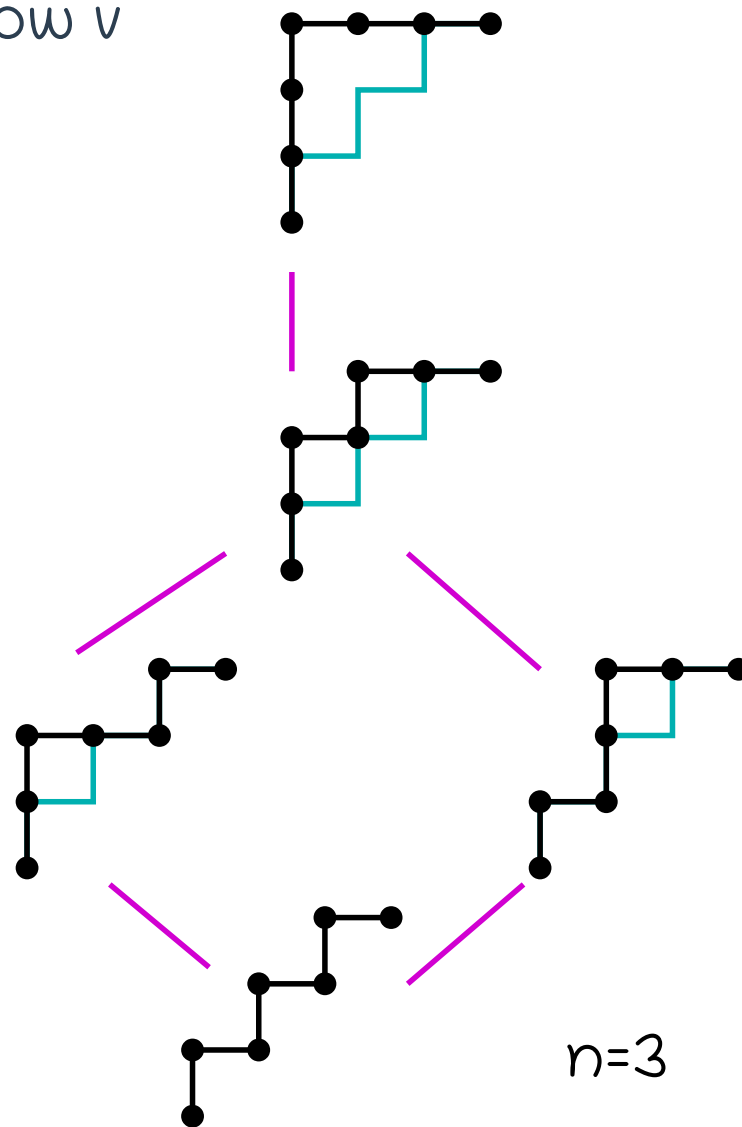


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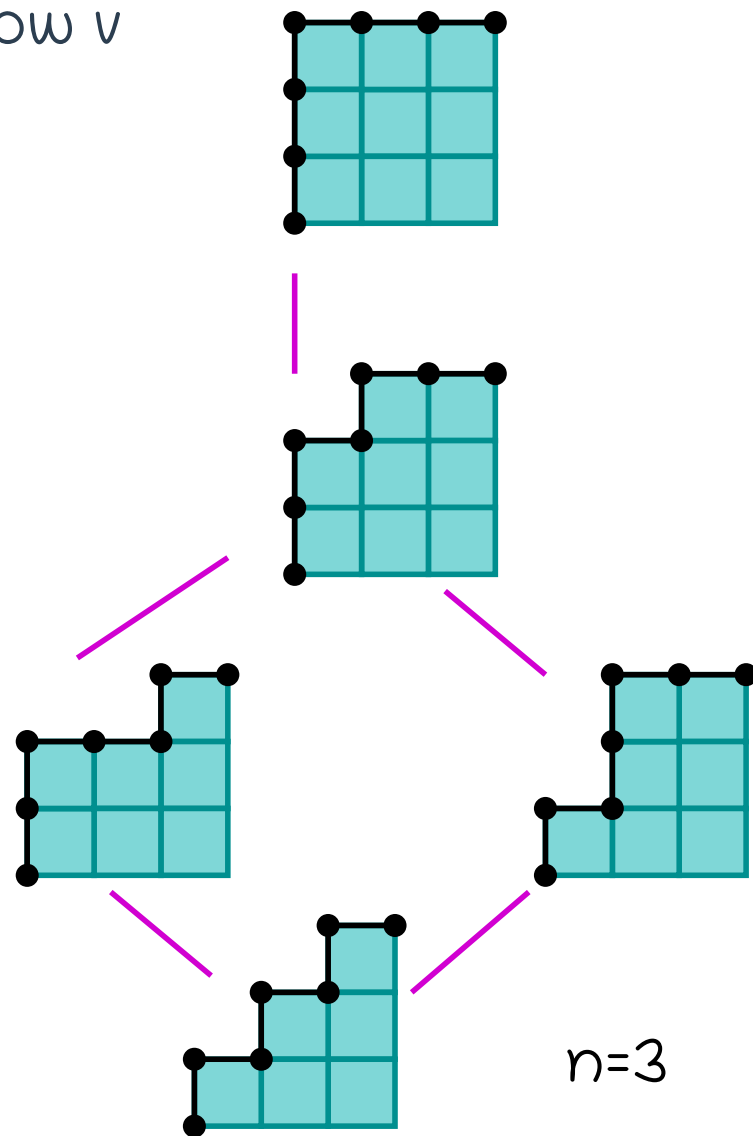


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- Covering relations are given by  $EN \rightarrow NE$  (that is, valley  $\rightarrow$  peak)
- A sub-lattice of the Young lattice

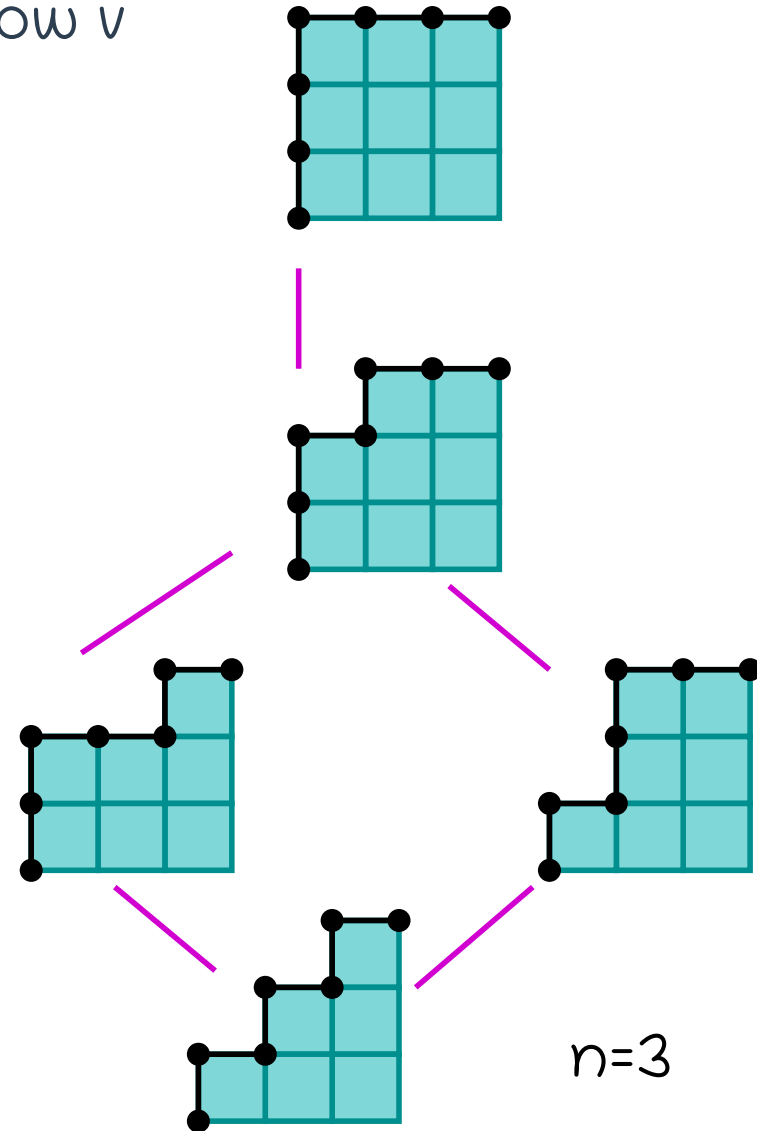


# Stanley's lattice $S_n$

A poset on Dyck paths of size  $n$

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- Covering relations are given by  $EN \rightarrow NE$  (that is, valley  $\rightarrow$  peak)
- A sub-lattice of the Young lattice
- Graded (by area)



# Stanley's lattice: interesting numbers

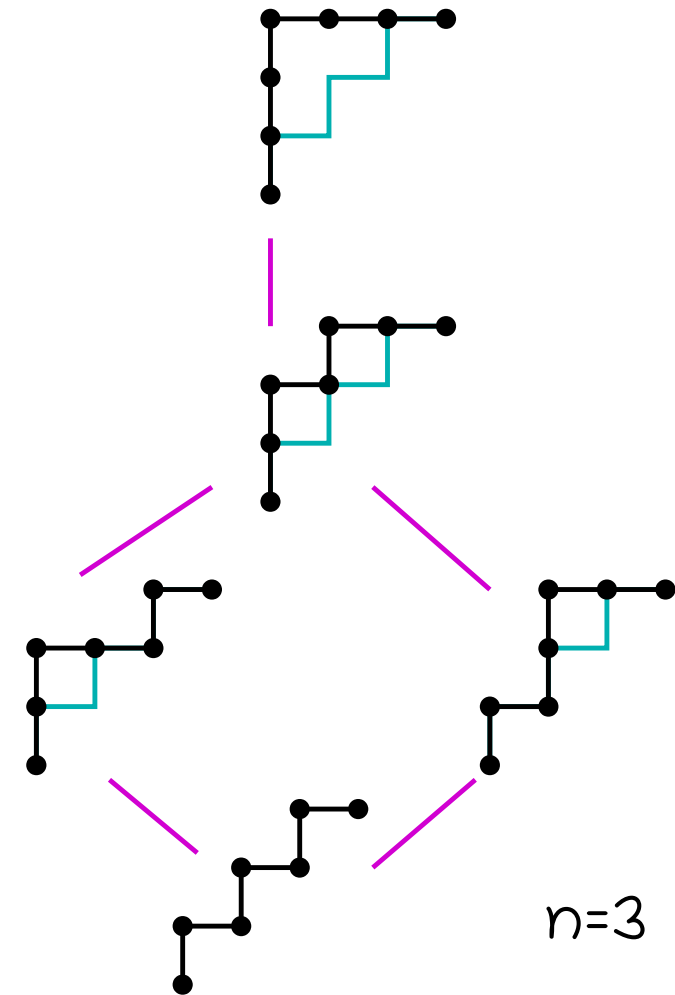
Number of:

- elements

$$C_n = \frac{1}{n+1} \binom{2n}{n}$$

- covering relations

$$\binom{2n-1}{n-2}$$





# Stanley's lattice: interesting numbers

Number of:

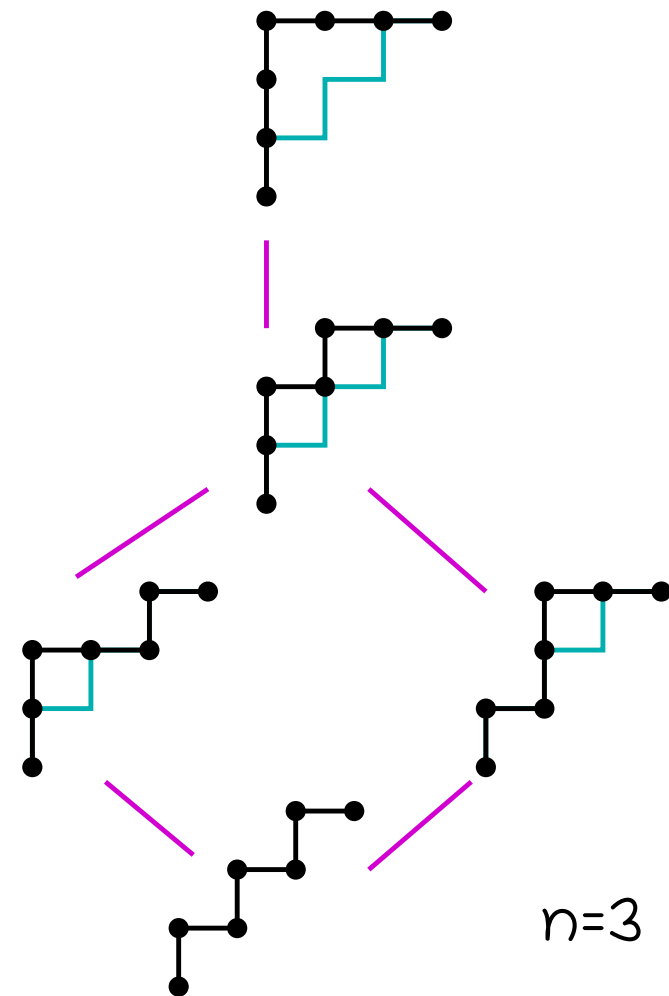
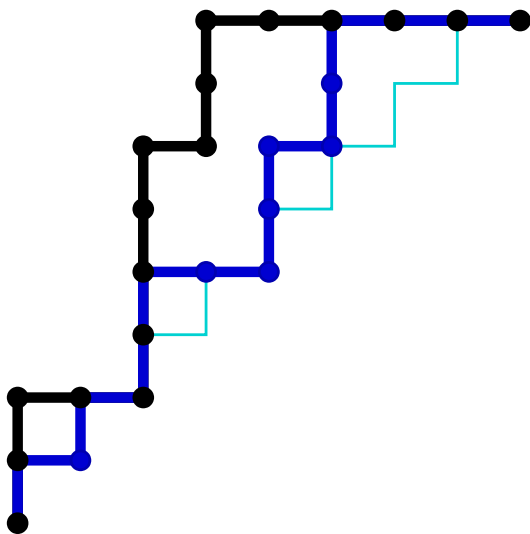
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- intervals: pairs of comparable Dyck paths



# Stanley's lattice: interesting numbers

Number of:

- elements

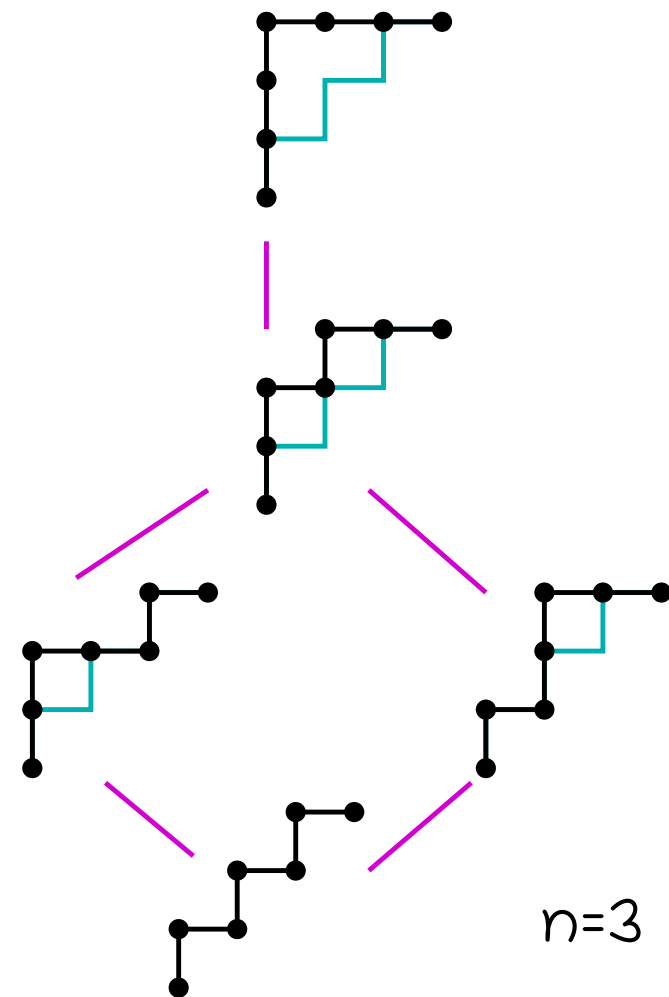
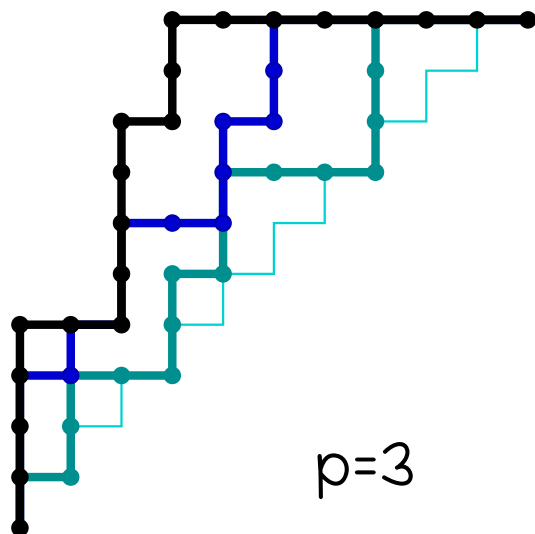
$$C_n = \frac{1}{n+1} \binom{2n}{n}$$

- covering relations

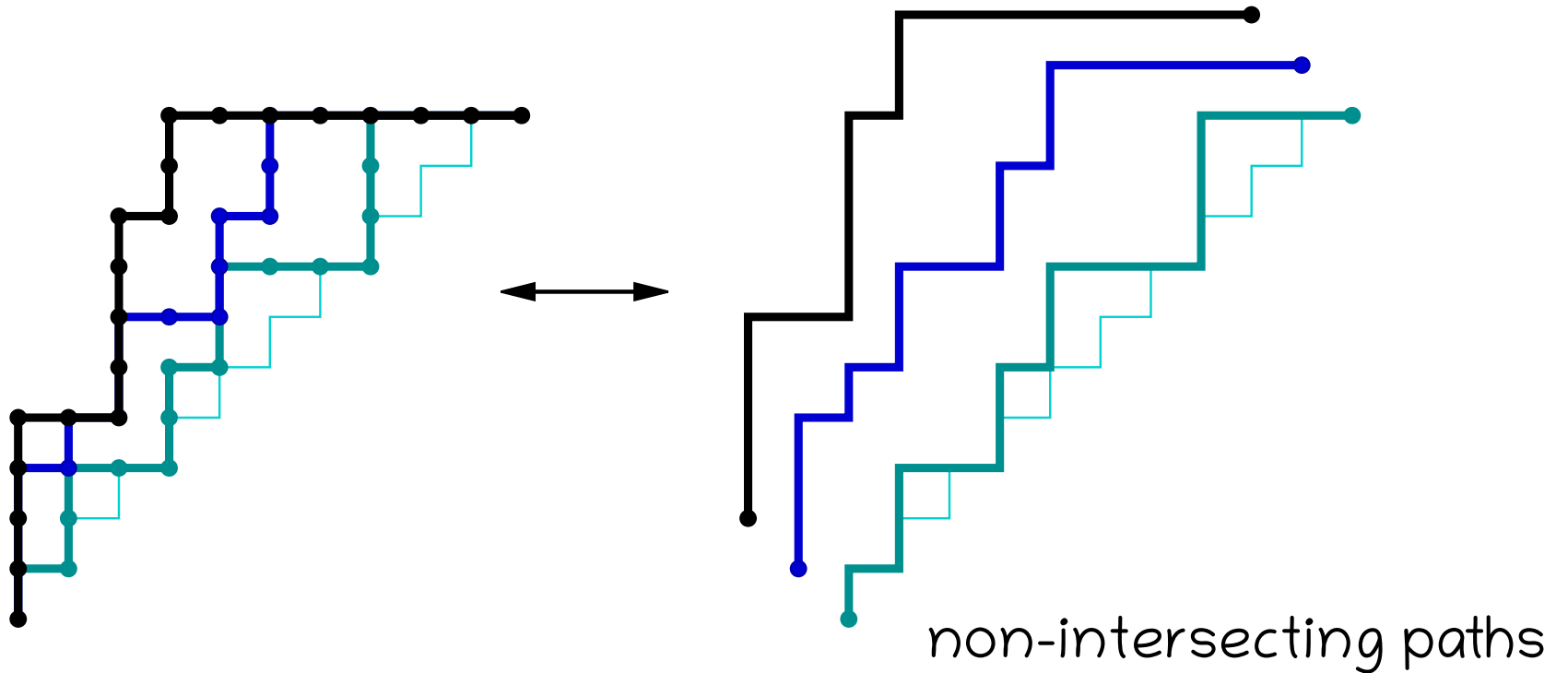
$$\binom{2n-1}{n-2}$$

- intervals: pairs of comparable Dyck paths

- p-chains



# p-Chains in Stanley's lattice



## Tools

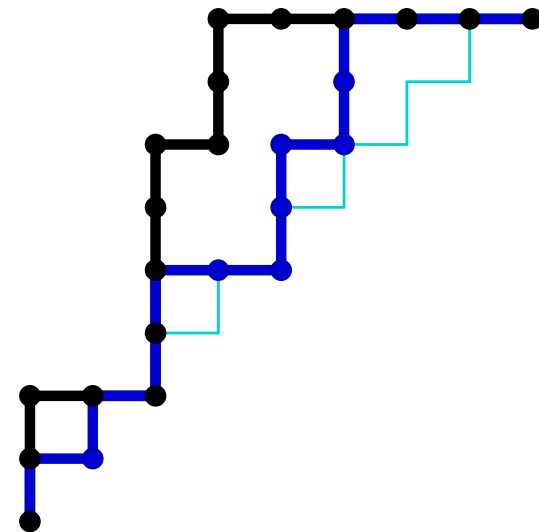
- Lindström-Gessel-Viennot determinants
- [Krattenthaler 99] "Advanced determinant calculus"

# Stanley's lattice: interesting numbers

## Number of

- **intervals:** pairs of comparable Dyck paths

$$C_{n-1}C_{n+1} - C_n^2 = \frac{6(2n)!(2n+2)!}{n!(n+1)!(n+2)!(n+3)!}$$



- **p-chains:** product formula  $\sim \kappa 4^{pn} n^{-p^2-p/2}$

Connection with plane partitions, and symplectic tableaux

[Proctor 86, 88] [Krattenthaler, Guttman, Viennot 00]

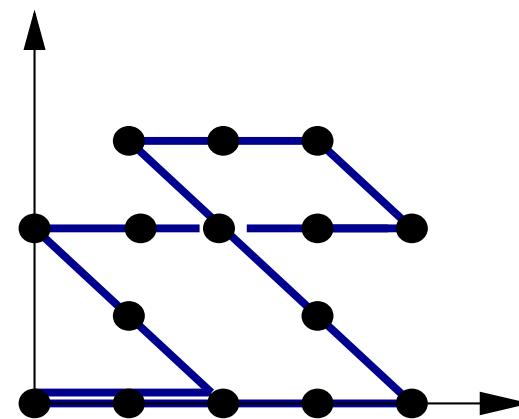
# Intervals in Stanley's lattice and planar maps

$$C_{n-1}C_{n+1} - C_n^2 = \frac{6(2n)!(2n+2)!}{n!(n+1)!(n+2)!(n+3)!} \sim \frac{24 \cdot 4^{2n}}{\pi n^5}$$

OEIS A005700

- Quadrant loops with steps E, W, NW, SE
- (Some) Young tableaux of height at most 4

[Gouyou-Beauchamps 89]



# Intervals in Stanley's lattice and planar maps

$$C_{n-1}C_{n+1} - C_n^2 = \frac{6(2n)!(2n+2)!}{n!(n+1)!(n+2)!(n+3)!} \sim \frac{24 \cdot 4^{2n}}{\pi n^5}$$

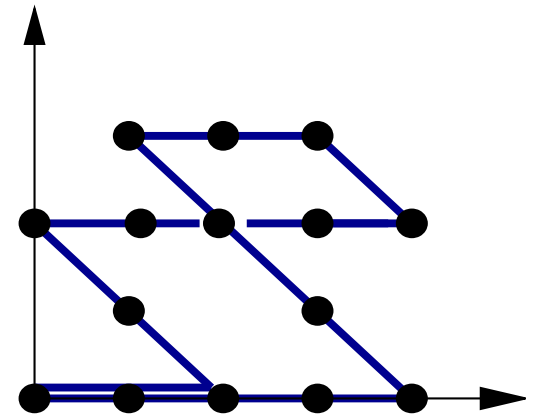
OEIS A005700

- Quadrant loops with steps E, W, NW, SE
- (Some) Young tableaux of height at most 4

[Gouyou-Beauchamps 89]

- **Some maps:** triangulations with a Schnyder wood

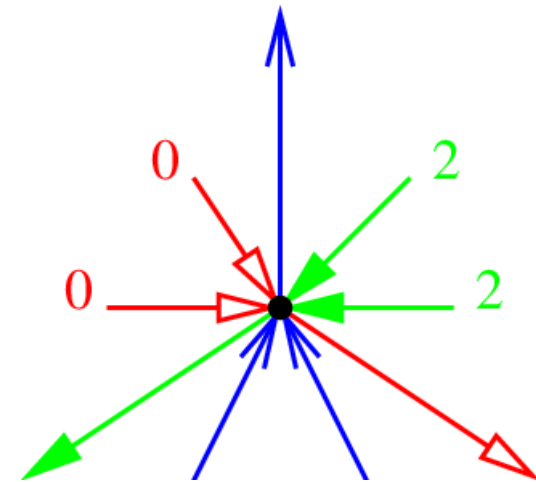
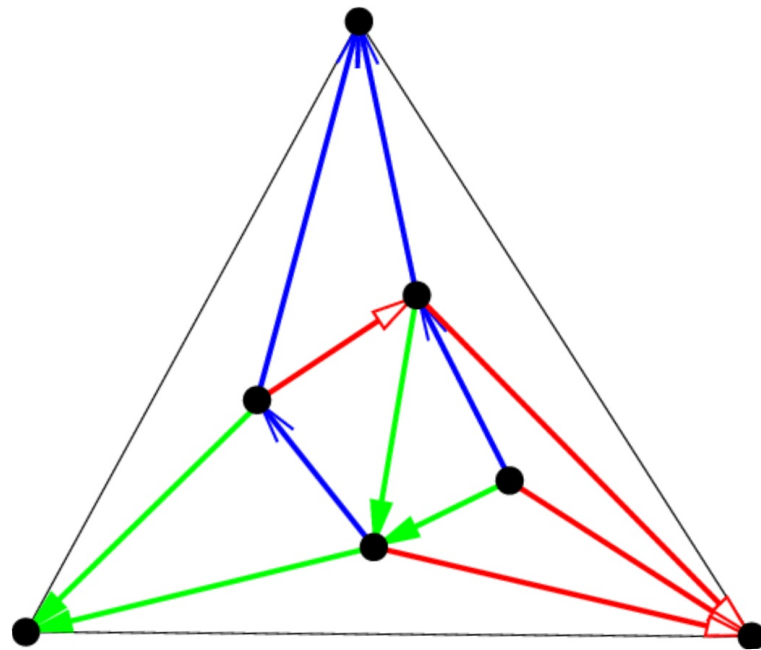
[Bonichon 05, Bernardi-Bonichon 09]



# Triangulations with a Schnyder wood

**Def.** A rooted triangulation where the inner edges are oriented and coloured following some rules [Schnyder 89, graph drawing]

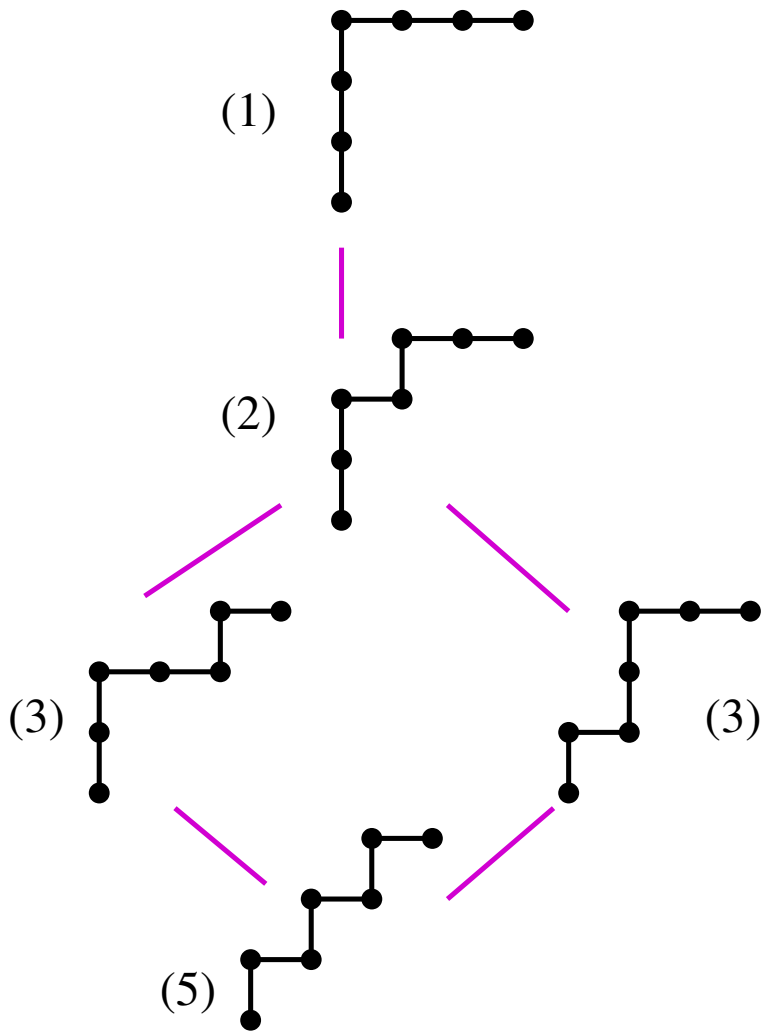
n vertices



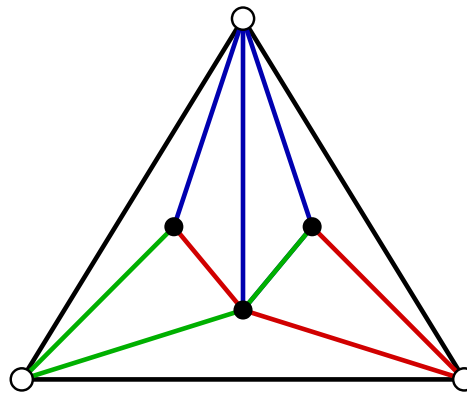
**Bijection** with pairs of non-intersecting Dyck paths of size n  
[Bonichon 05]

$$C_{n-1}C_{n+1} - C_n^2 = \frac{6(2n)!(2n+2)!}{n!(n+1)!(n+2)!(n+3)!}$$

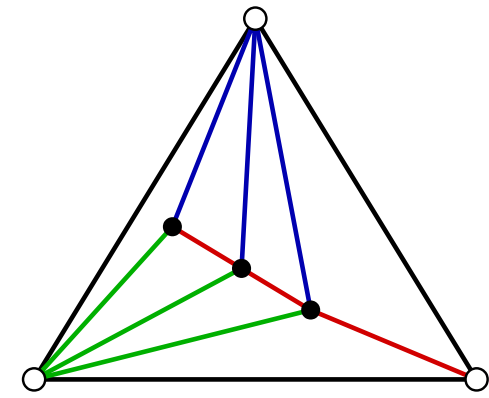
# 14 intervals for Dyck paths of size 3



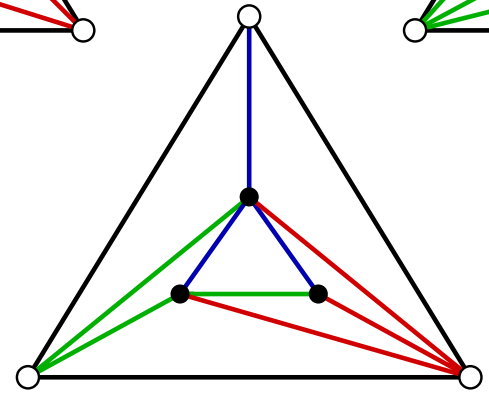
$n=3$



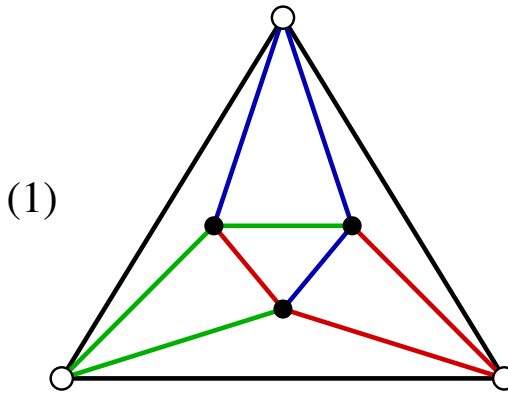
(3)



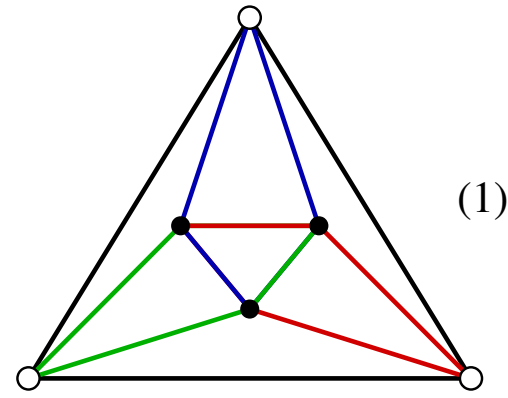
(3)



(6)



(1)



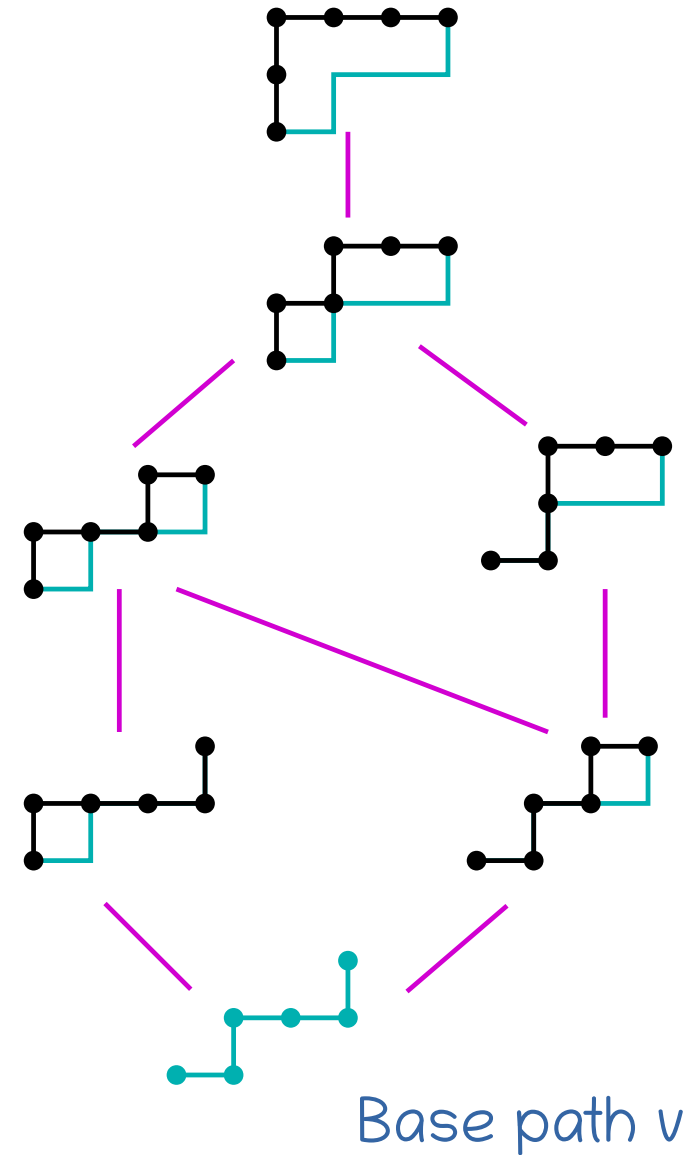
(1)



# The $v$ -Stanley lattice $S_v$

**Def.** For any **base path  $v$** , consider all NE paths above  $v$ .

Order them by containment: lattice  $S_v$



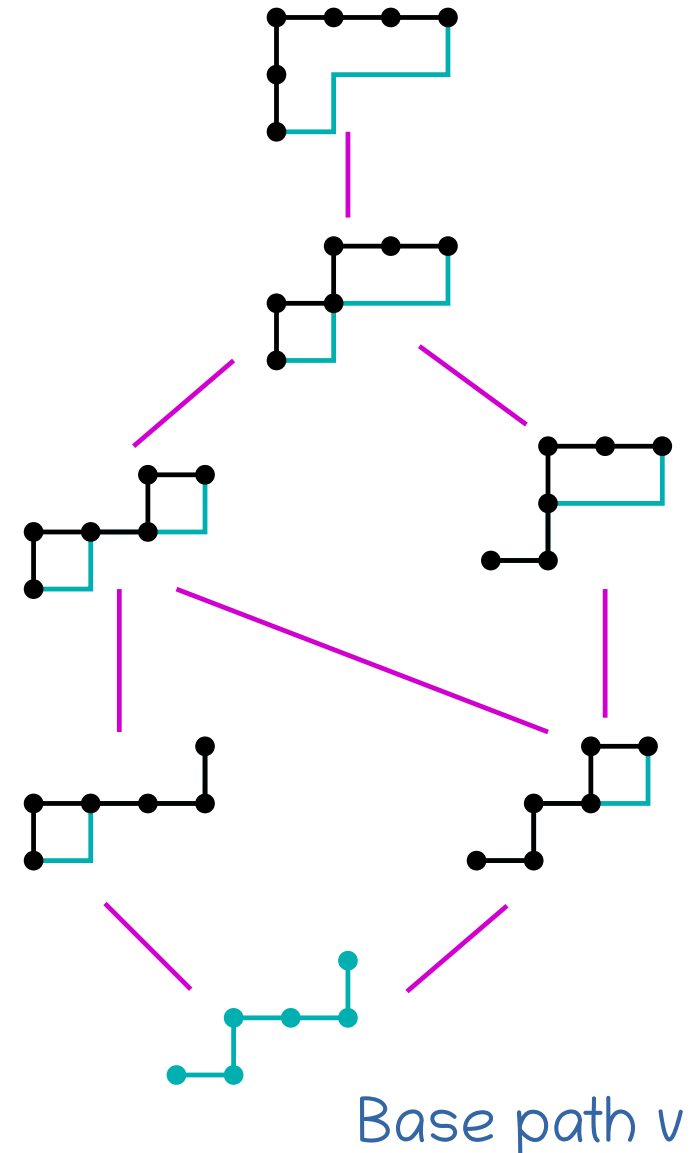
# The $v$ -Stanley lattice $S_v$

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**Interesting choices:**

- ◆  $v = E^i N^j$  : all paths ending at  $(i, j)$
- ◆  $v = (NE^m)^n$  :  $m$ -Dyck paths



# The $v$ -Stanley lattice $S_v$

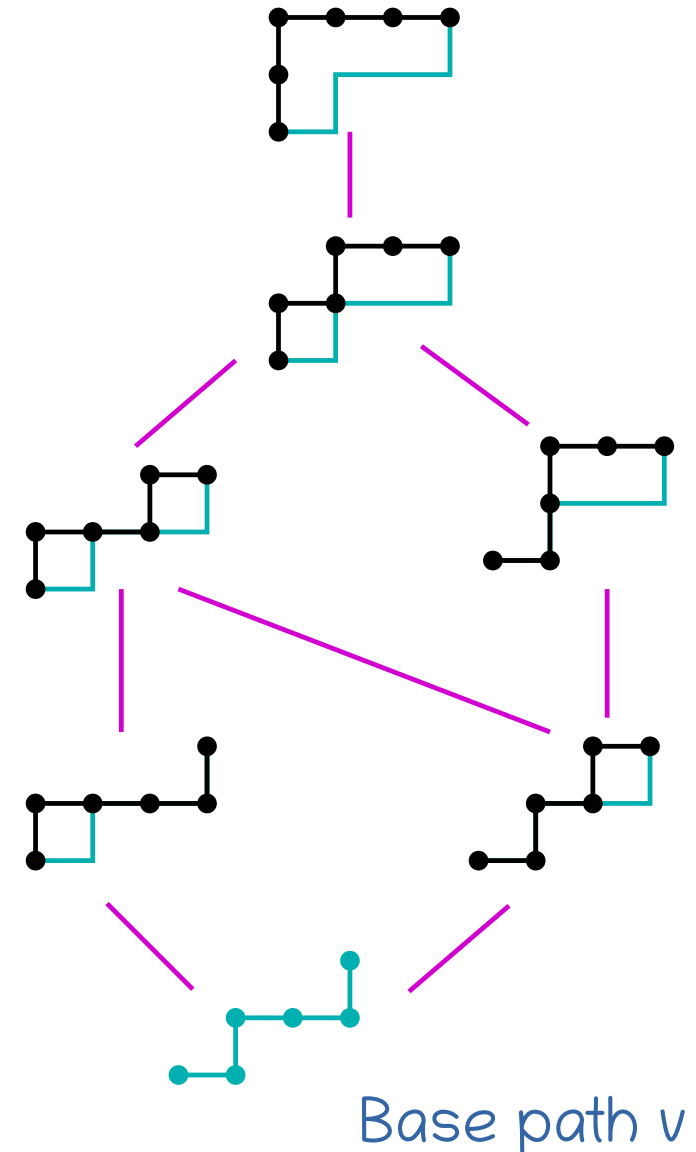
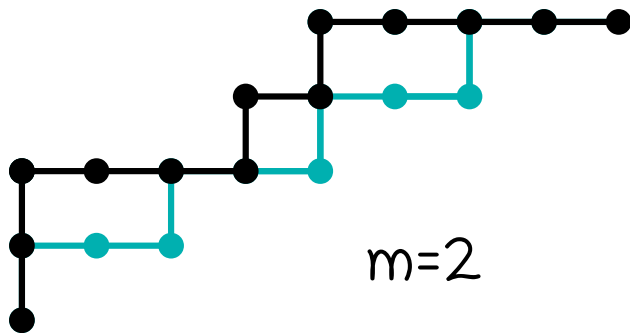
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A 2-Dyck path:



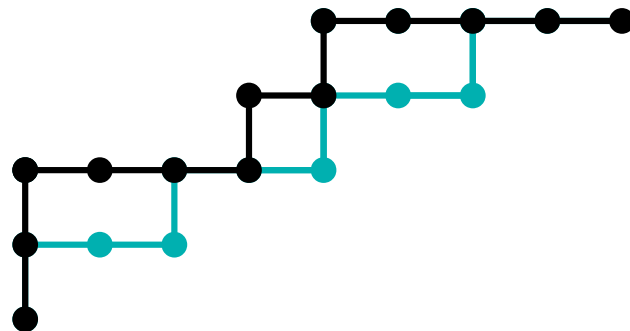
# The $v$ -Stanley lattices: number of elements

**Number of elements** in the lattice  $S_v$  (all walks above  $v$ ):

- arbitrary  $v$ : determinant [Kreweras 65]

- $v = E^i N^j$ : all paths ending at  $(i, j)$   $\binom{i+j}{i}$

- $v = (NE^m)^n$ :  $m$ -Dyck paths  $\frac{1}{mn+1} \binom{(m+1)n}{n}$



# The $v$ -Stanley lattices: number of intervals

For  $v=(NE^m)^n$ :

$$\text{Int}(S_v) = \frac{2(m+2)((m+1)n)!((m+1)(n+1))!}{n!(n+1)!(mn+2)!(m(n+2)+2)!}$$

Q. Are there  $m$ -Schnyder woods for  $m > 1$  ?

Q. Product formula for  $p$ -chains ?

# The $v$ -Stanley lattices: number of intervals

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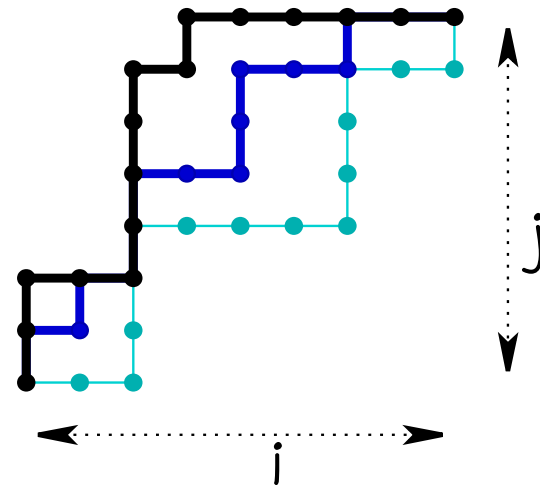
YES! (added after the talk)  
See Ex. 7.101(b) in Stanley's EC2

Thanks to Richard and Sam  
(Hopkins)

# The $v$ -Stanley lattices: number of intervals

Summing over some  $v$ : triples of non-intersecting paths

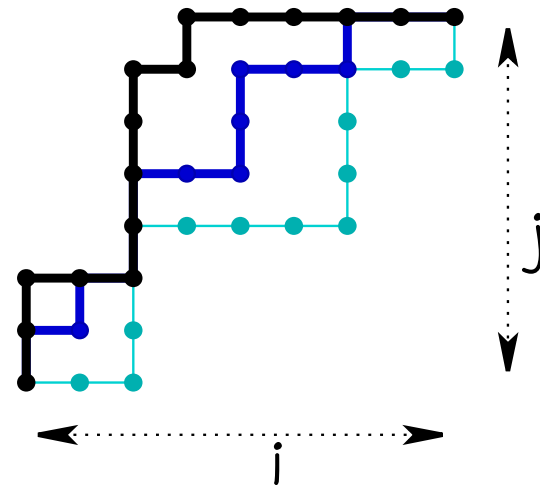
$$\sum_{v \rightsquigarrow (i,j)} \text{Int}(S_v) = 2 \frac{(i+j)!(i+j+1)!(i+j+2)!}{i!(i+1)!(i+2)!j!(j+1)!(j+2)!}$$



# The $v$ -Stanley lattices: number of intervals

Summing over some  $v$ : triples of non-intersecting paths

$$\sum_{v \rightsquigarrow (i,j)} \text{Int}(S_v) = 2 \frac{(i+j)!(i+j+1)!(i+j+2)!}{i!(i+1)!(i+2)!j!(j+1)!(j+2)!}$$



- Baxter permutations with  $i+1$  ascents and  $j+1$  descents

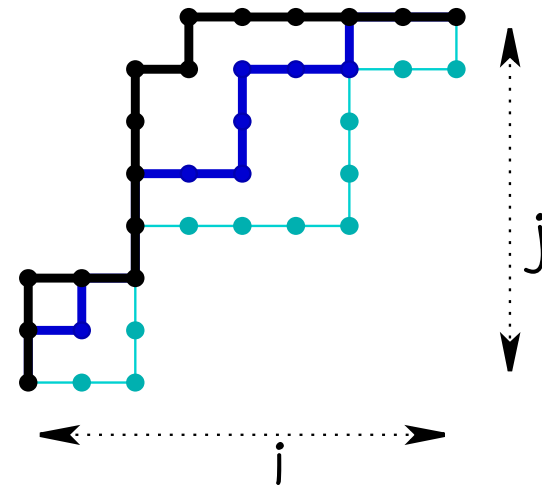
[Chung et al. 78, Mallows 79]



# The $v$ -Stanley lattices: number of intervals

Summing over some  $v$ : triples of non-intersecting paths

$$\sum_{v \rightsquigarrow (i,j)} \text{Int}(S_v) = 2 \frac{(i+j)!(i+j+1)!(i+j+2)!}{i!(i+1)!(i+2)!j!(j+1)!(j+2)!}$$



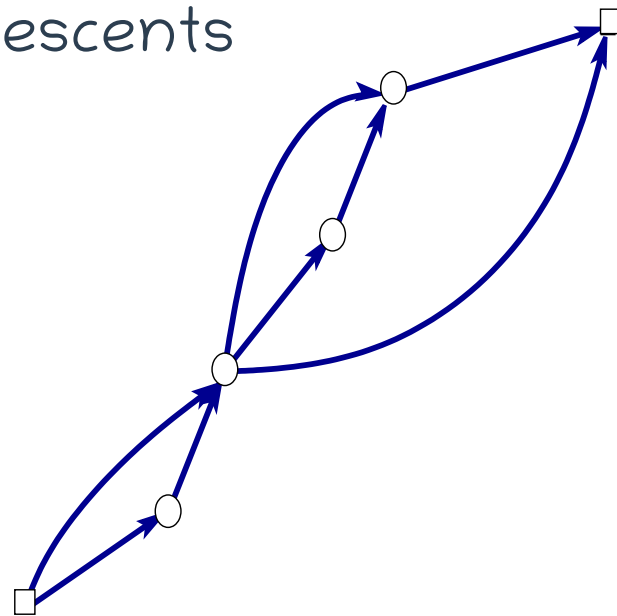
- Baxter permutations with  $i+1$  ascents and  $j+1$  descents

[Chung et al. 78, Mallows 79]

- **Maps** carrying a bipolar orientation

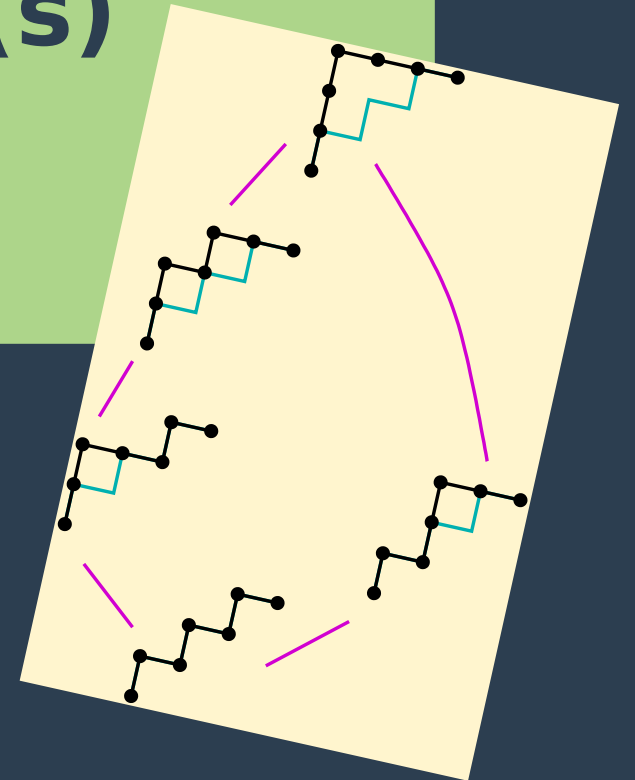
[Baxter 0] -- Bonichon, mbm, Felsner,

Fusy, Noy, Orden, Poulalhon...]



## II. Tamari lattice(s)

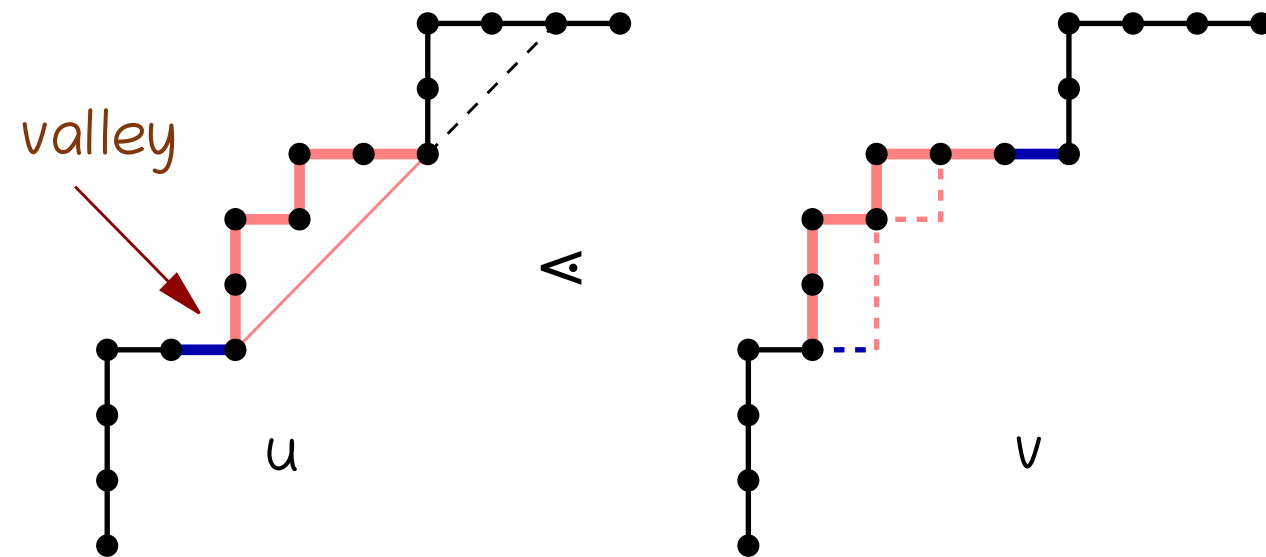
Tamari( $v$ )  $\subset$  Stanley( $v$ )



# The Tamari lattice

[Tamari 5]

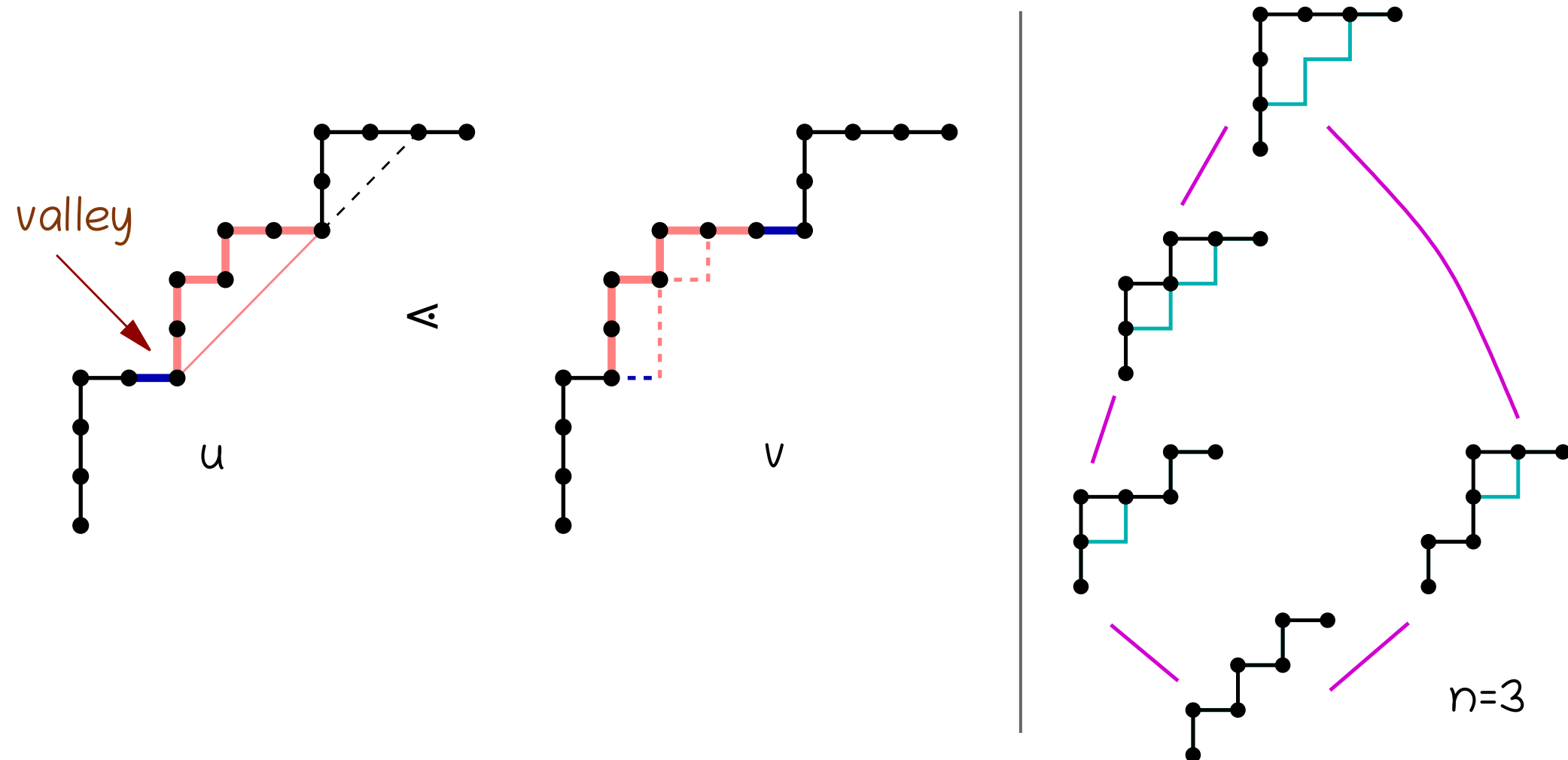
- A poset on Dyck paths with  $n$  up steps
- Covering relations: choose a valley in the path  $u$ .  
Swap the East step and the **shortest Dyck path** that follows



# The Tamari lattice

[Tamari 5]

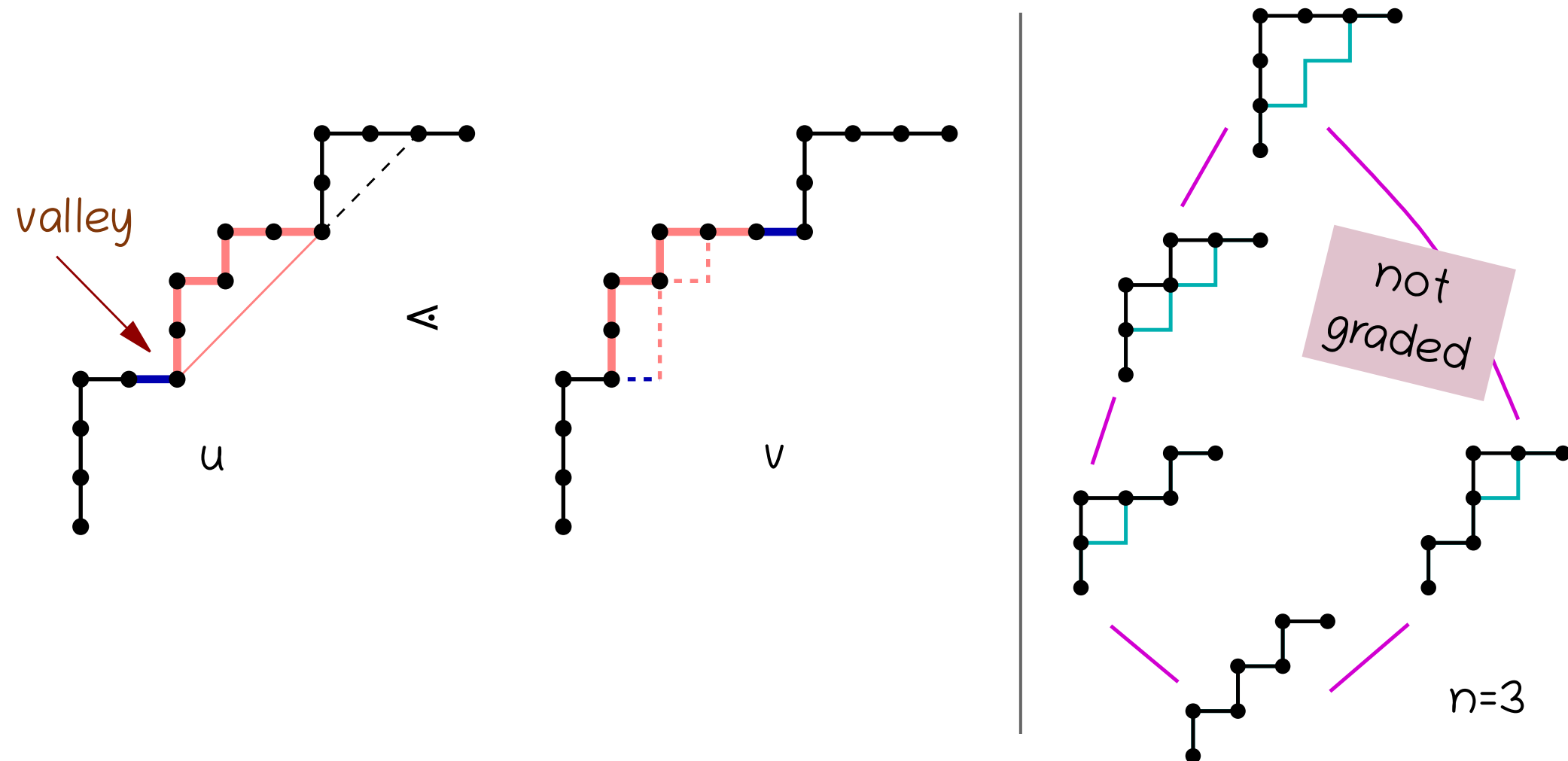
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# The Tamari lattice

[Tamari 5]

- A poset on Dyck paths with  $n$  up steps
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Swap the East step and the **shortest Dyck path** that follows

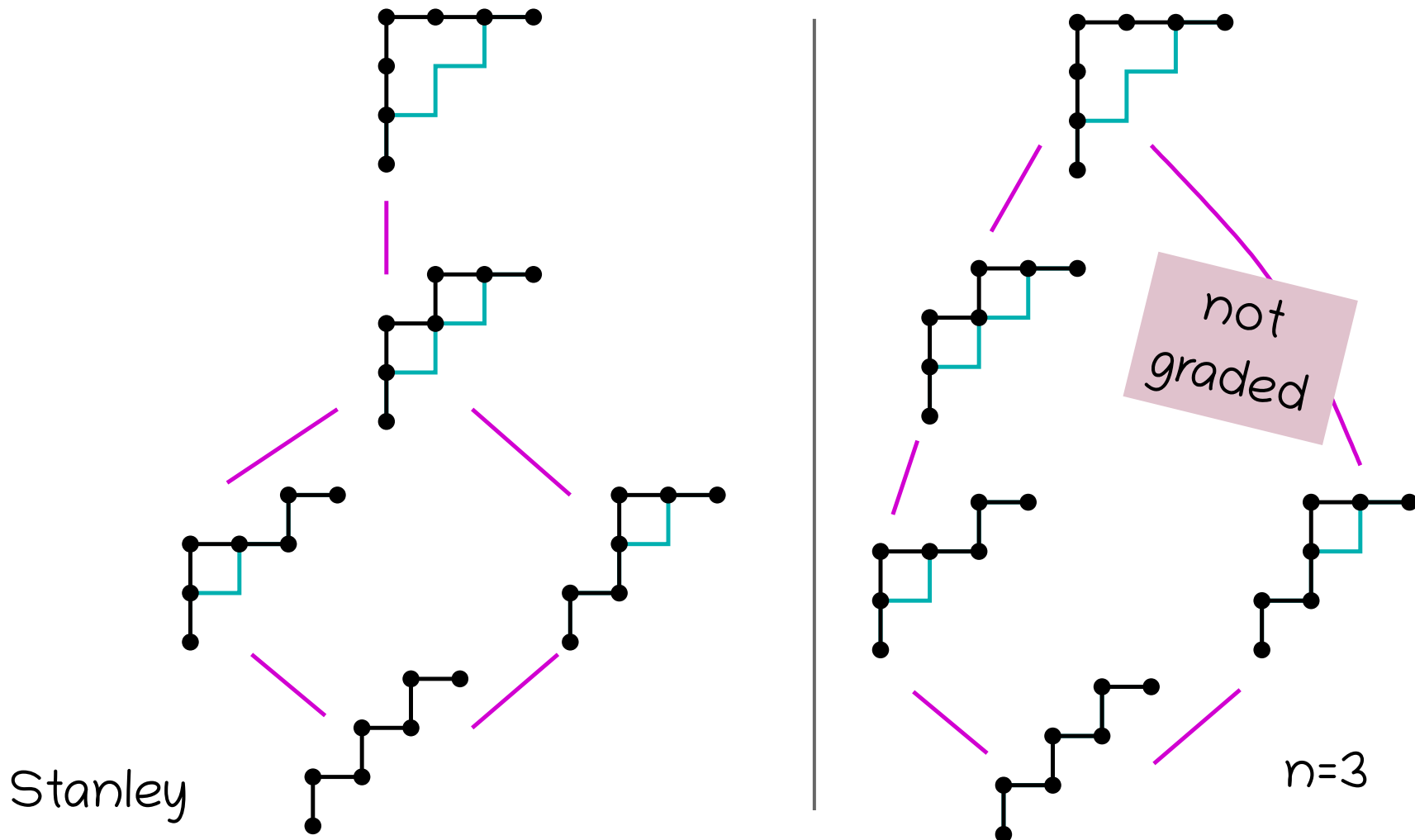


# The Tamari lattice

[Tamari 5]

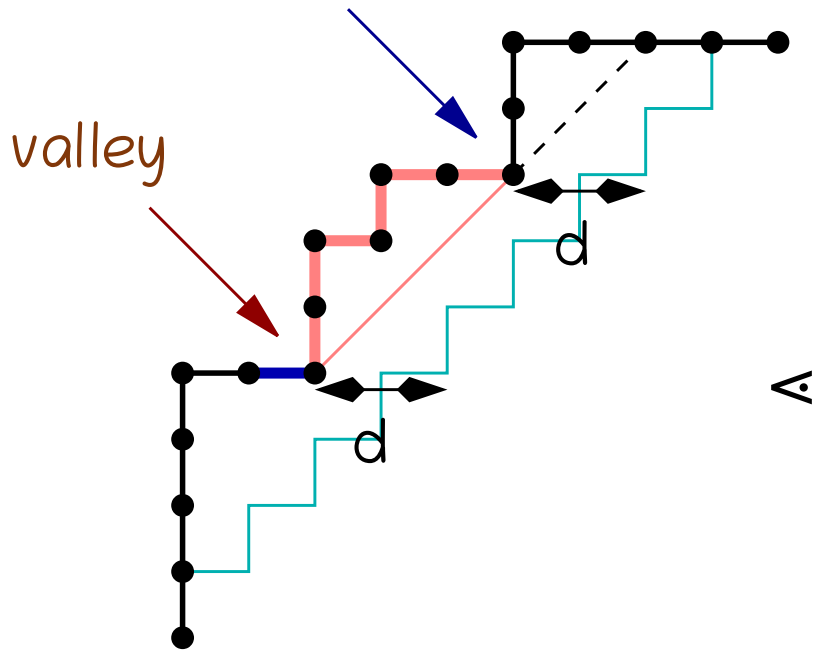
- A poset on Dyck paths with  $n$  up steps
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Swap the East step and the **shortest Dyck path** that follows

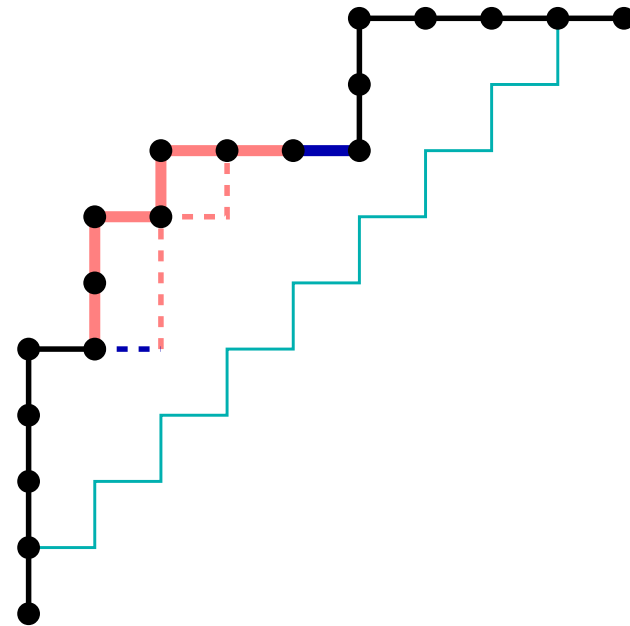


# Extension: the $v$ -Tamari lattice

Next point at horizontal distance  $d=2$  of the forbidden area

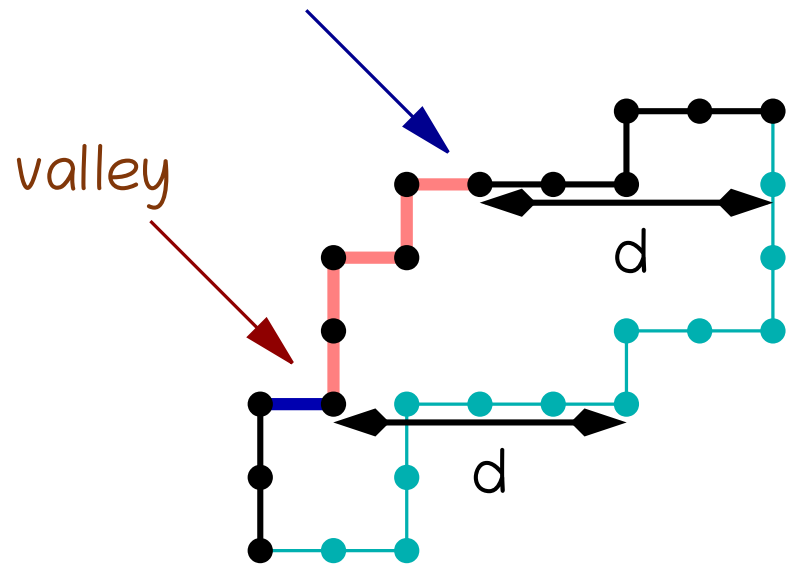


[Préville-Ratelle & Viennot 17]



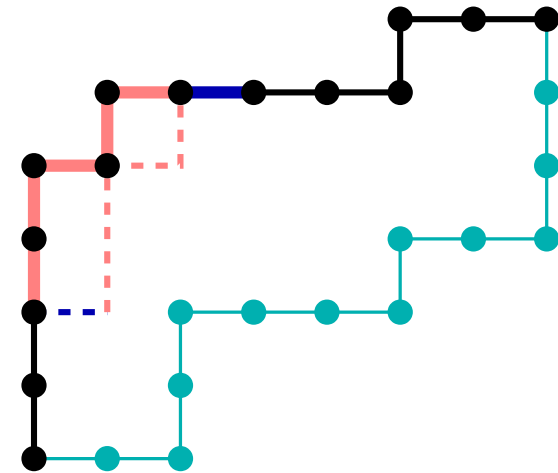
# Extension: the $v$ -Tamari lattice

Next point at horizontal distance  $d=4$  of the forbidden area



[Préville-Ratelle & Viennot 17]

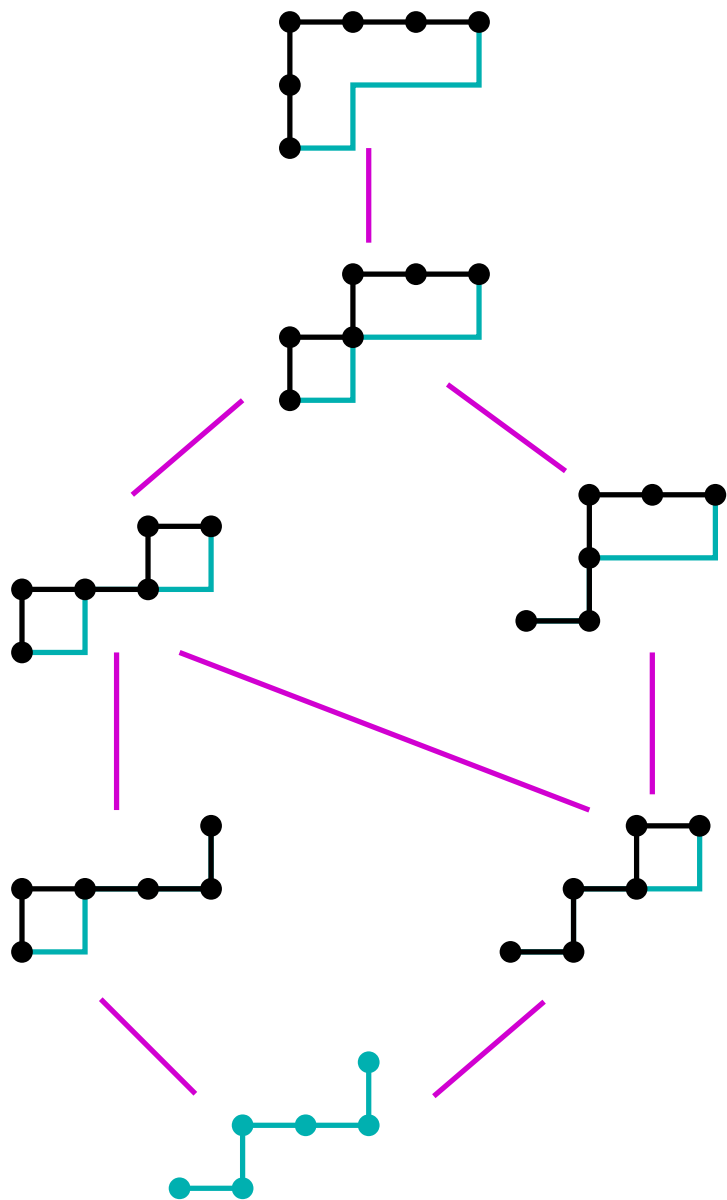
$\Delta$



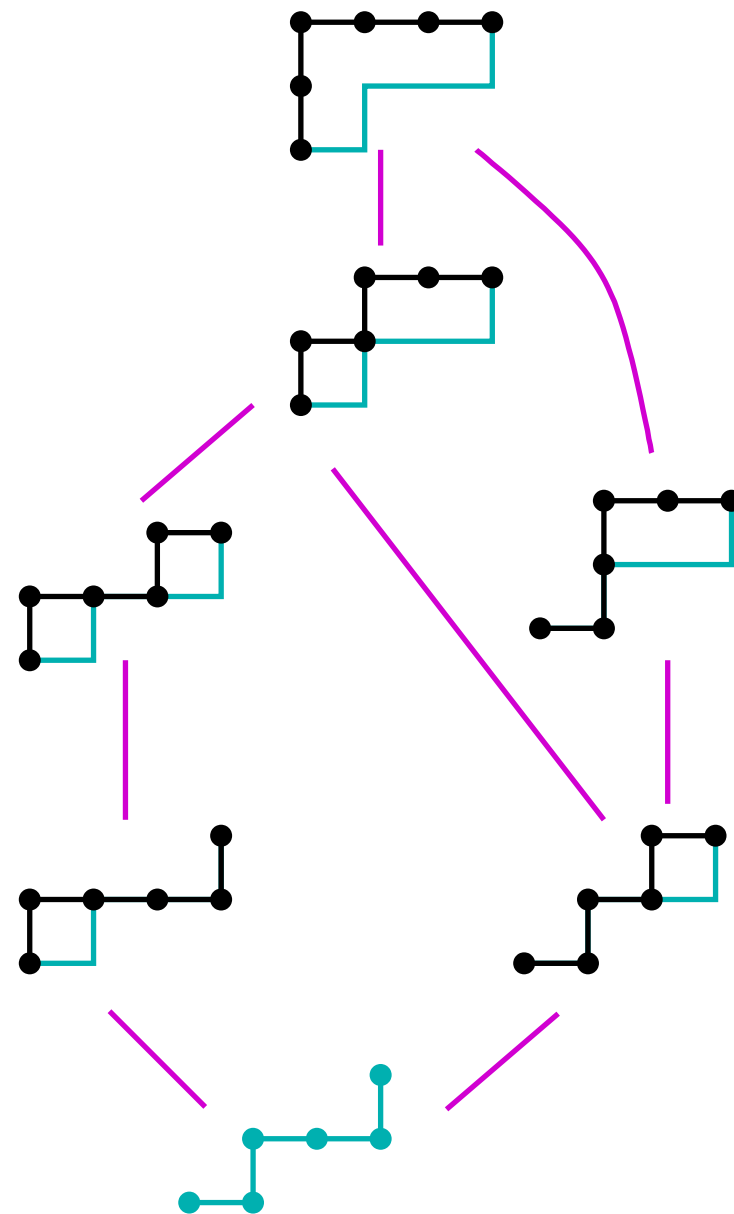
[Ceballos, Padrol, Sarmiento 20]



# The $v$ -Tamari lattice: an example



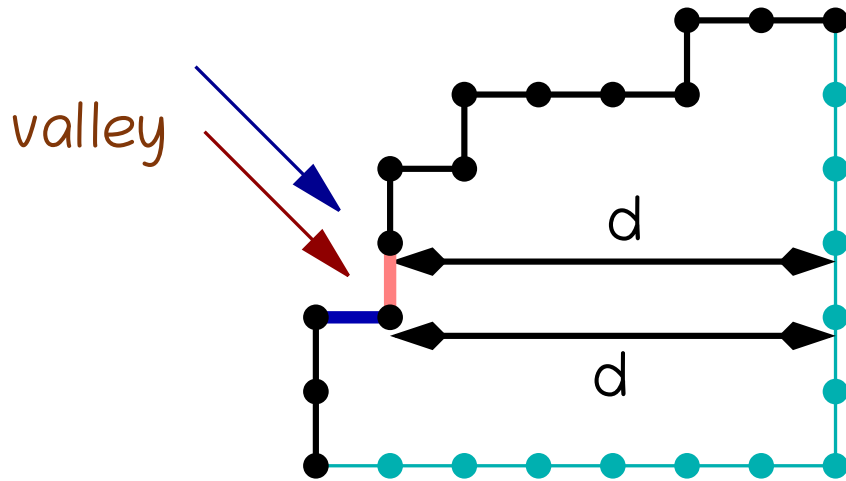
$v$ -Stanley lattice



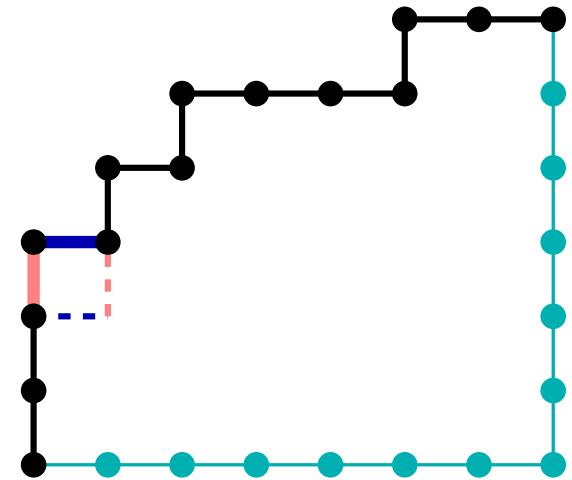
$v$ -Tamari lattice

# A very special case: $v$ -Tamari with $v = E^i N^j$

Next point at horizontal distance  $d=6$  of the forbidden area



$\Delta$



The covering relations are  $EN \rightarrow NE$ :  
one recovers the Stanley lattice  $S_v$

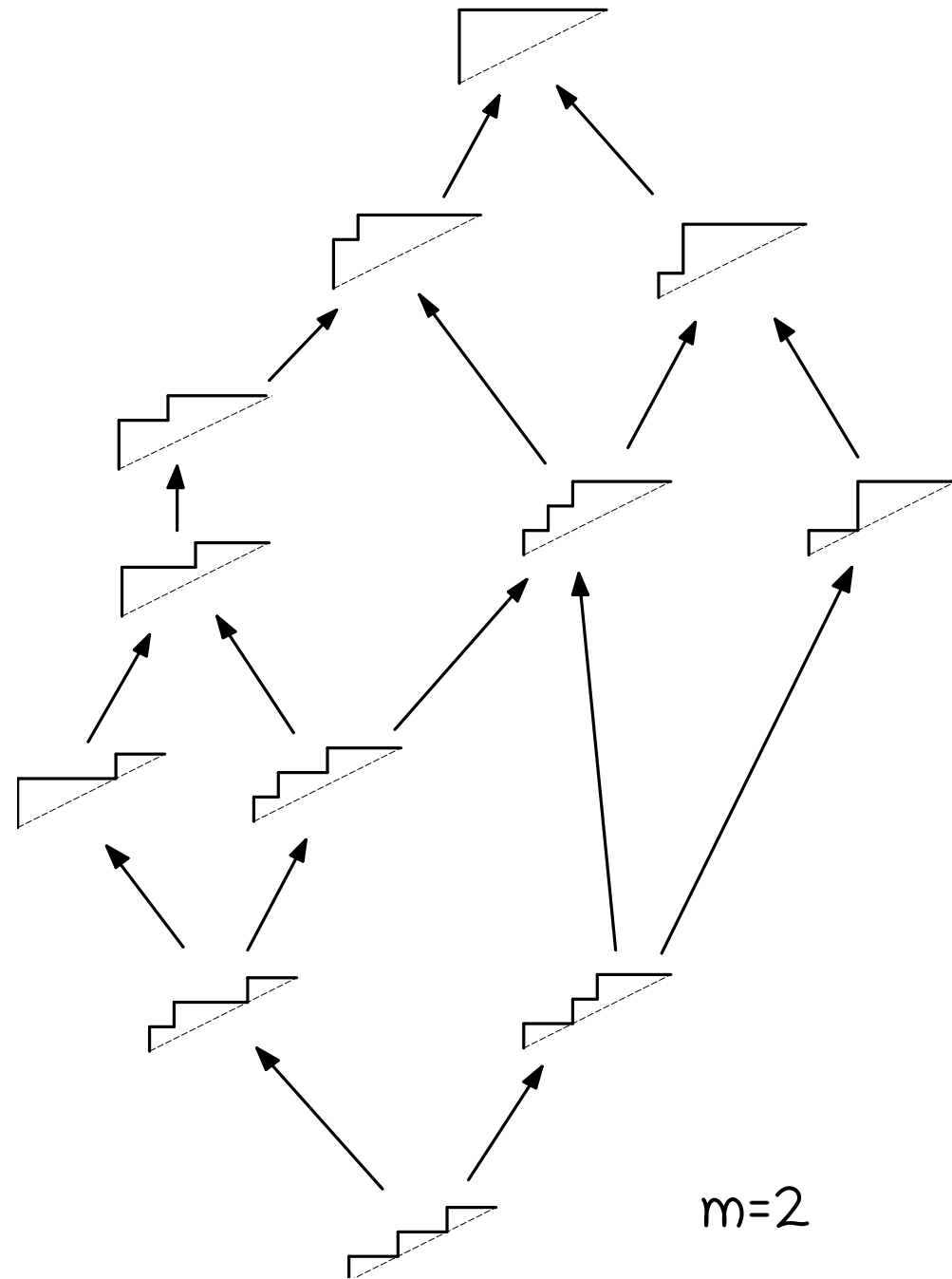
# An interesting case: $v$ -Tamari, with $v = (NE^m)^n$

Conjectured relation with  
coinvariant polynomials in 3 sets  
of variables

[Bergeron & Préville-Ratelle 12]

[Haiman 94]

Also called  $m$ -Tamari lattice



# Intervals in the $m$ -Tamari lattices

[Chapoton 06 ( $m=1$ ) -- MBM, Fusy & Préville-Ratelle 11]

For  $v=(NE^m)^n$ , the number of intervals in the  $v$ -Tamari lattice is:

$$t_n^{(m)} = \text{Int}(T_v) = \frac{m+1}{n(mn+1)} \binom{(m+1)^2n+m}{n-1}$$

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$$t_n^{(1)} = \frac{2}{n(n+1)} \binom{4n+1}{n-1}$$

$$t_n^{(2)} = \frac{3}{n(2n+1)} \binom{9n+2}{n-1}$$

$$t_n^{(3)} = \frac{4}{n(3n+1)} \binom{16n+3}{n-1}$$

# In the planar maps literature...

$$\frac{2}{(n+1)(2n+1)} \binom{3n}{n}$$

non-separable

$$\frac{2^n}{(n+1)(2n+1)} \binom{3n}{n}$$

cubic

$$\frac{2 \cdot 3^n}{(n+1)(n+2)} \binom{2n}{n}$$

all maps

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$$\frac{2}{(n+1)(3n+2)} \binom{4n+1}{n} = \frac{2}{n(n+1)} \binom{4n+1}{n-1} = t_n^{(1)}$$

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# Intervals in the m-Tamari lattices

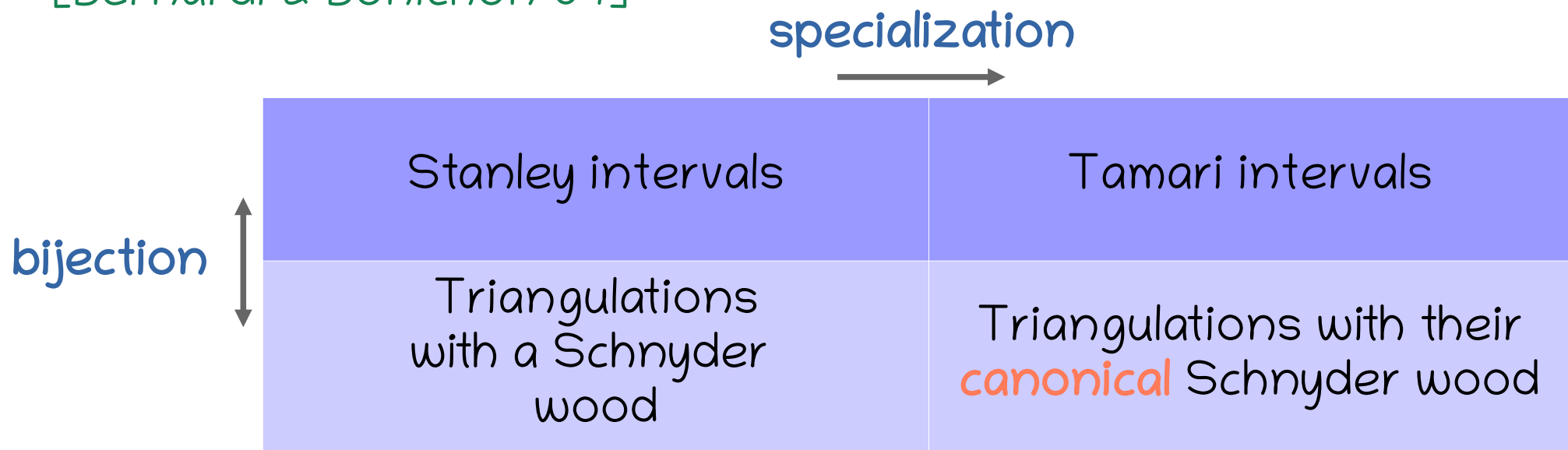
[Chapoton 06 (m=1) -- MBM, Fusy & Préville-Ratelle 11]

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**Bijection proof (m=1)**

[Bernardi & Bonichon 07]





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**Bijjective proof ( $m=1$ )**

[Bernardi & Bonichon 07]

+ another bijection [Fang 18]

For  $m>1$ , which type of maps do these numbers count?

[Fusy & Humbert 19]

# Intervals in the $v$ -Tamari lattices

Summing over some  $v$

[Fang & Préville-Ratelle 17, Fang 17(a)]

$$\sum_{i+j=n} \sum_{v \rightsquigarrow (i,j)} \text{Int}(T_v) = \dots$$

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**synchronized intervals** in the ordinary

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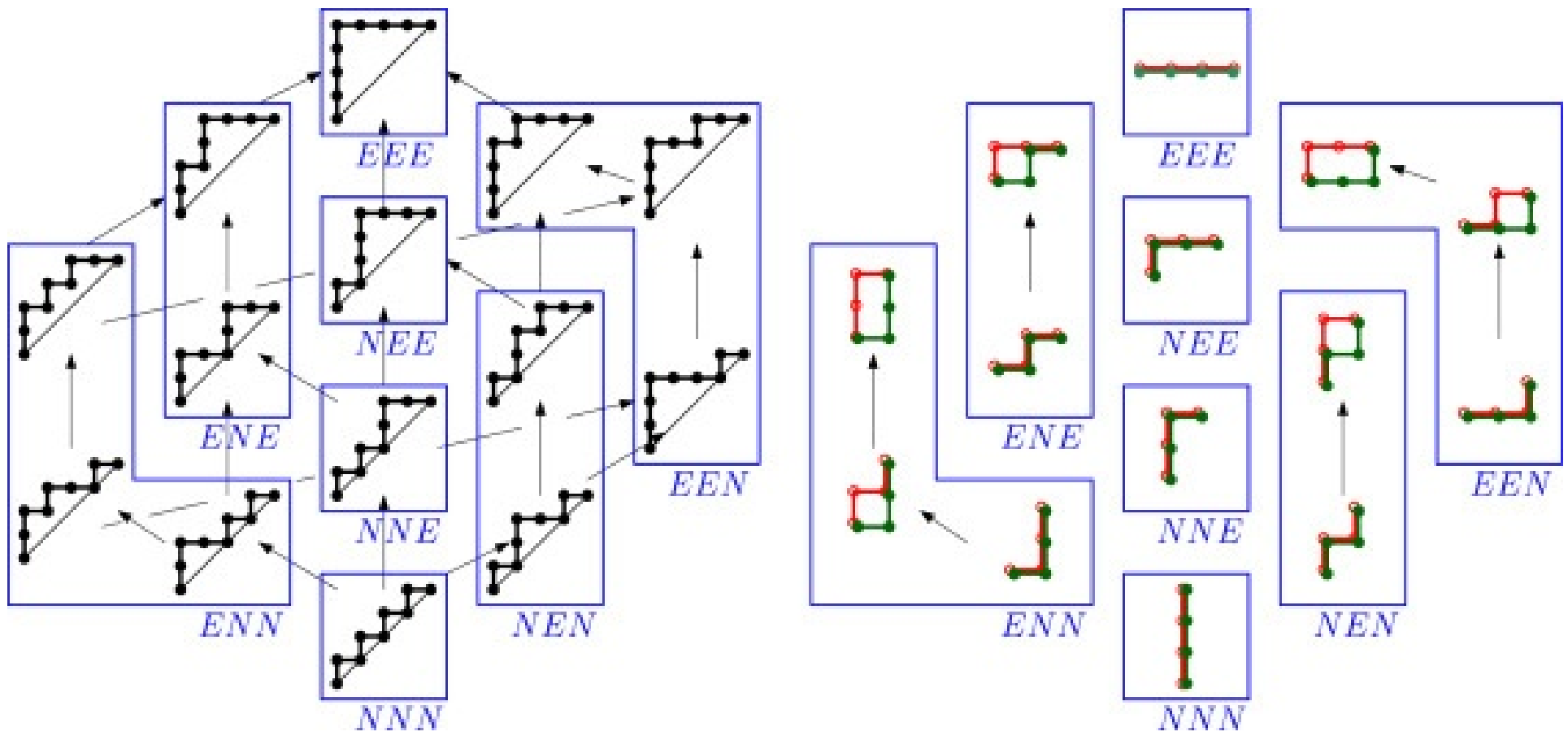
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More precisely,

$$\sum_{v \rightsquigarrow (i,j)} \text{Int}(T_v) = \frac{1}{(i+1)(j+1)} \binom{2i+j+1}{j} \binom{2j+i+1}{i}$$

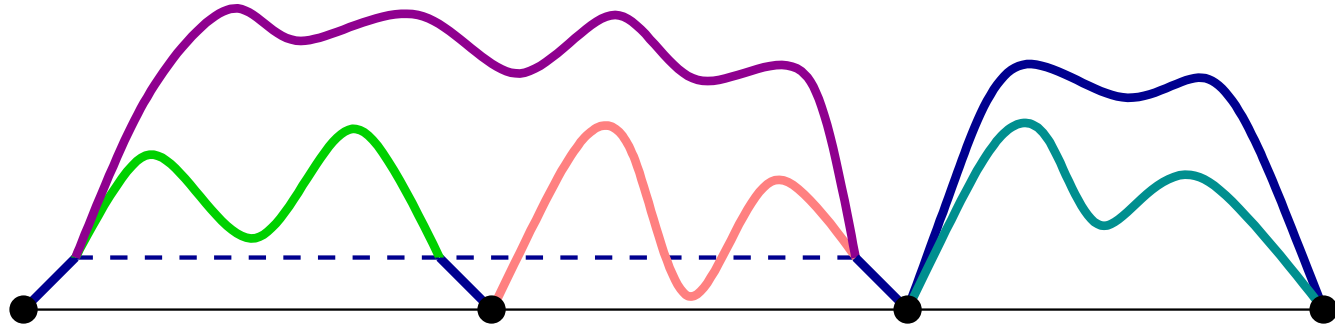
Bijection with **non-separable planar maps** (no cut-point) with  $(i+2)$  vertices and  $(j+2)$  faces.

+ Another bijective proof [Fusy & Humbert 19]

# **III. Recursive proofs and discrete differential equations**

# Recursive description of Tamari intervals

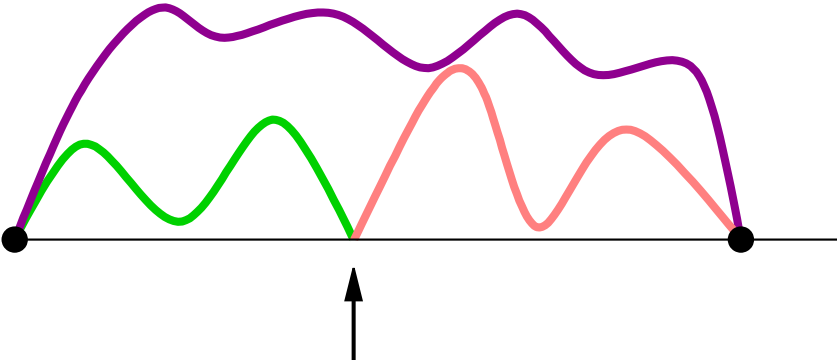
Consider the original Tamari lattice (i.e.,  $v=(NE)^n$ )



Tamari interval

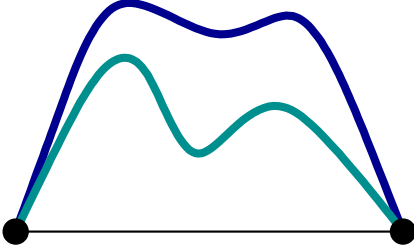
bijection

=



interval with a marked contact in the bottom path

\*



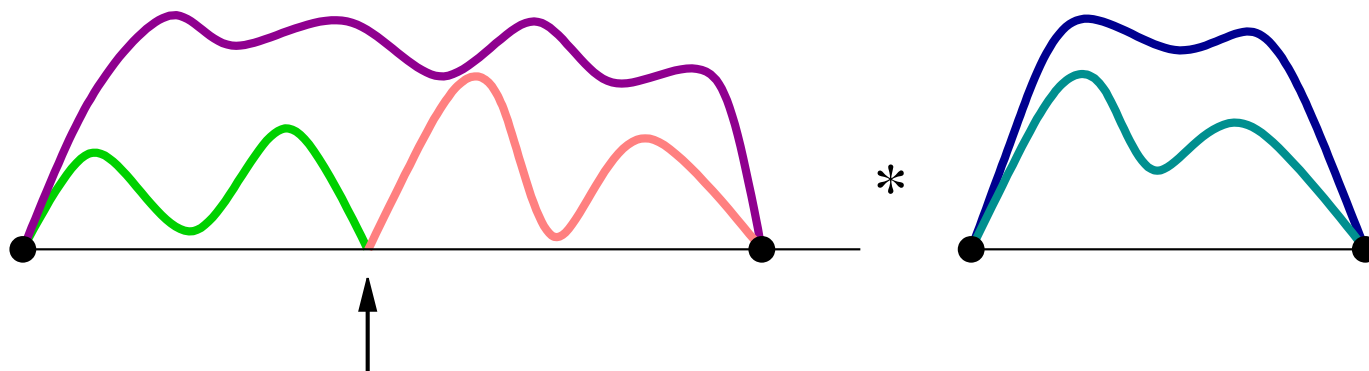
arbitrary Tamari interval



# A discrete differential equation

Consider the **original Tamari lattice** (i.e.,  $v=(NE)^n$ )

bijection



Then  $F(x) \equiv F(x; t) = \sum_{n \geq 0} t^n \sum_{[u,v] \subset T_n} x^{\text{contacts}(u)}$  satisfies:

$$F(x) = x + tx \frac{F(x) - F(1)}{x - 1} F(x)$$

$$= x + tx \Delta F(x) F(x),$$

with  $\Delta F(x) := \frac{F(x) - F(1)}{x - 1}$ .

Discrete derivative

# A discrete differential equation (m-Tamari)

Consider the **m-Tamari lattice** (i.e.,  $v=(NE^m)^n$ )

⇒ Similar decomposition of intervals

The series  $F(x) \equiv F(x; t) = \sum_{n \geq 0} t^n \sum_{[u, v] \subset T_n} x^{\text{contacts}(u)}$  satisfies

$$F(x) = x + tx (F(x)\Delta)^{(m)} F(x),$$

with  $\Delta F(x) := \frac{F(x) - F(1)}{x - 1}$ . [mbm, Fusy, Préville-Ratelle 11]

The operator  $F(x)\Delta$  is applied  $m$  times.

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The operator  $F(x)\Delta$  is applied m times.

**Example: m=2**

$$F(x) = x + \frac{xt}{x-1} F(x) \left( F(x) \frac{F(x) - F(1)}{x-1} - F(1)F'(1) \right).$$

# Discrete differential equations

**Thm.** The solution of any discrete differential equation is **algebraic**  
[mbm & Jehanne 06]

This holds for

$$F(x) = x + tx (F(x)\Delta)^{(m)} F(x),$$

But we had to **guess & check** the value of  $F(x)$ .

[mbm, Fusy, Prévaille-Ratelle 11]

In particular,

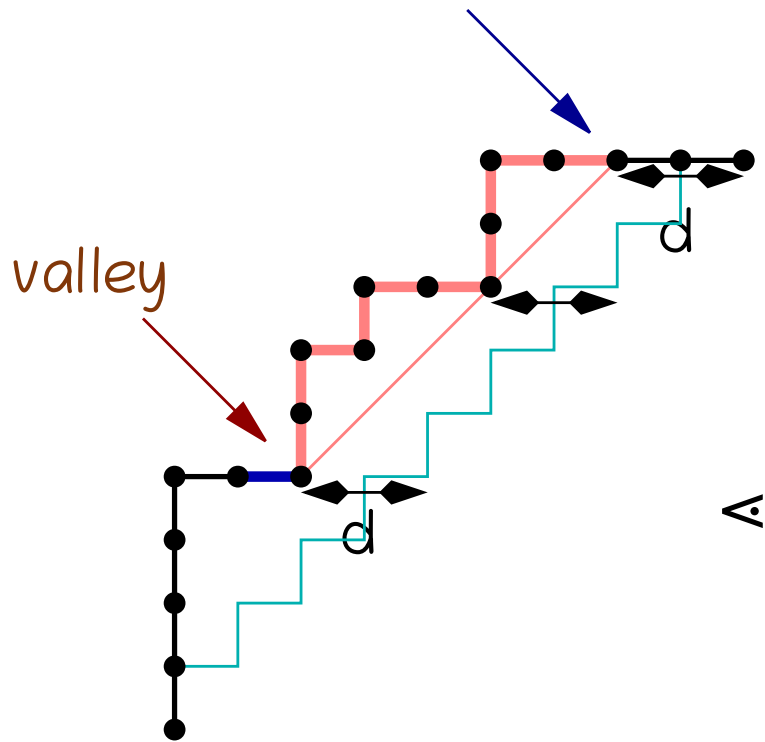
$$F(1) = \sum_{n \geq 0} t^n \frac{m+1}{n(mn+1)} \binom{(m+1)^2 n + m}{n-1}$$

## IV. Greedy Tamari posets

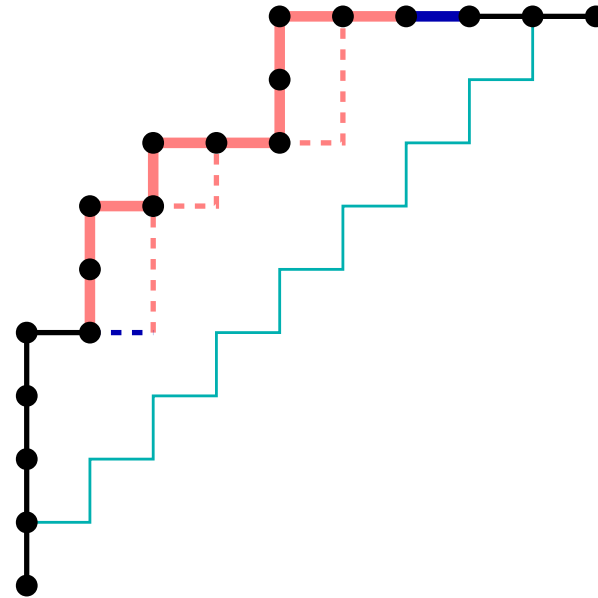
$\text{Greedy}(v) \subset \text{Tamari}(v) \subset \text{Stanley}(v)$

# Be greedy when swapping!

First point at horizontal distance  $d=2$  of the forbidden area followed by E (or final)

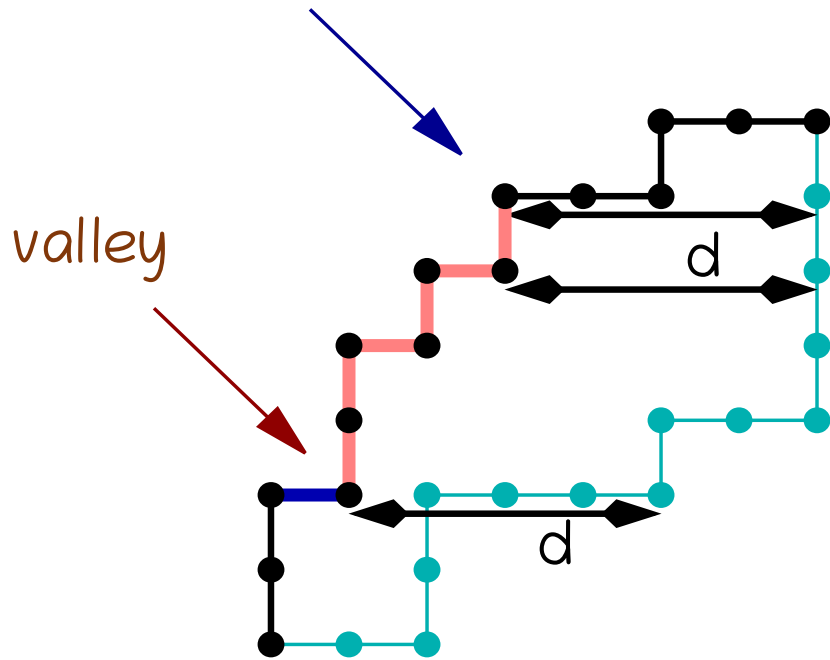


[Dermenjjan 23]



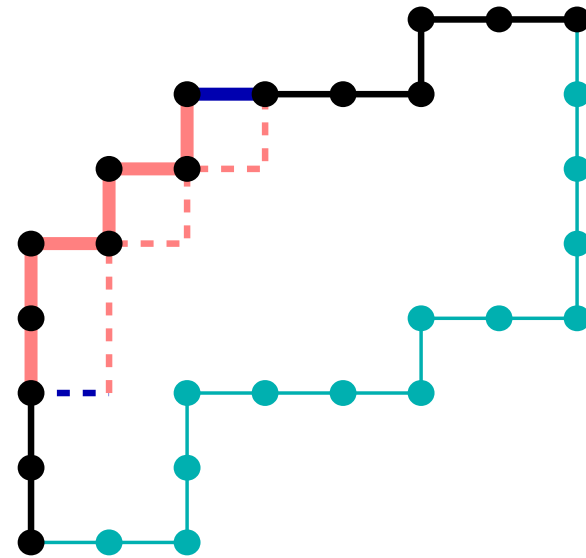
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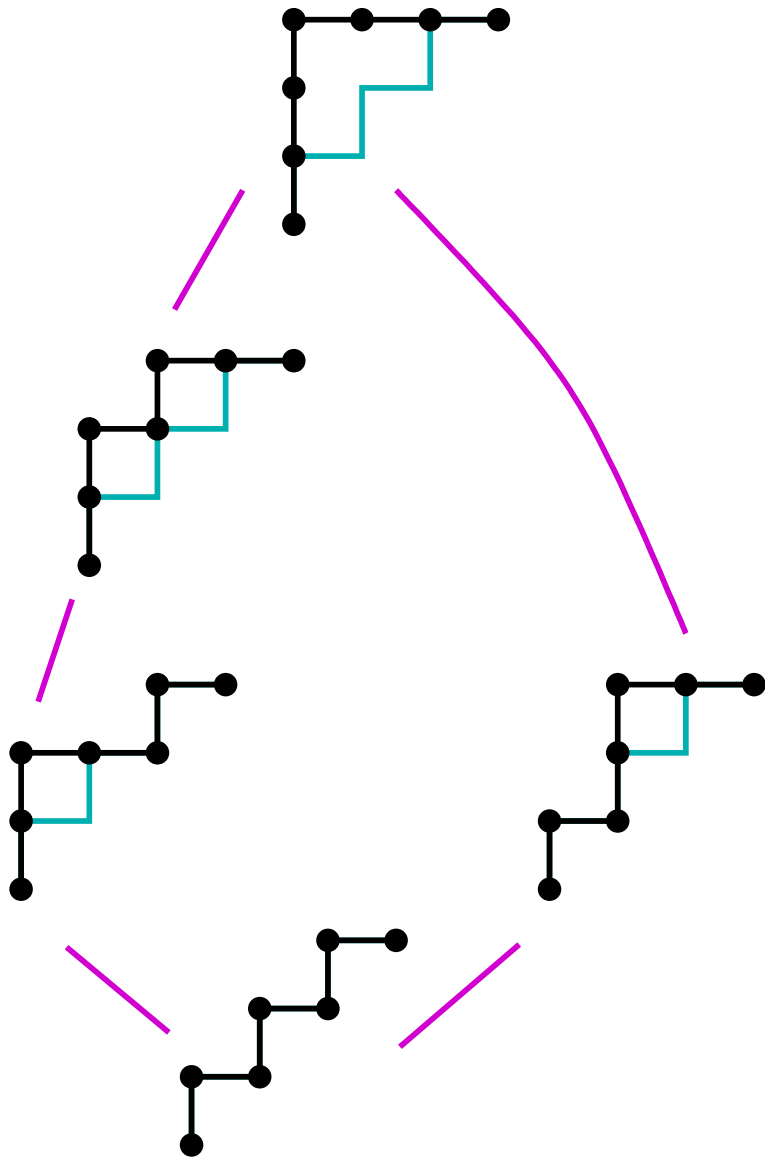


[Dermenjian 23]

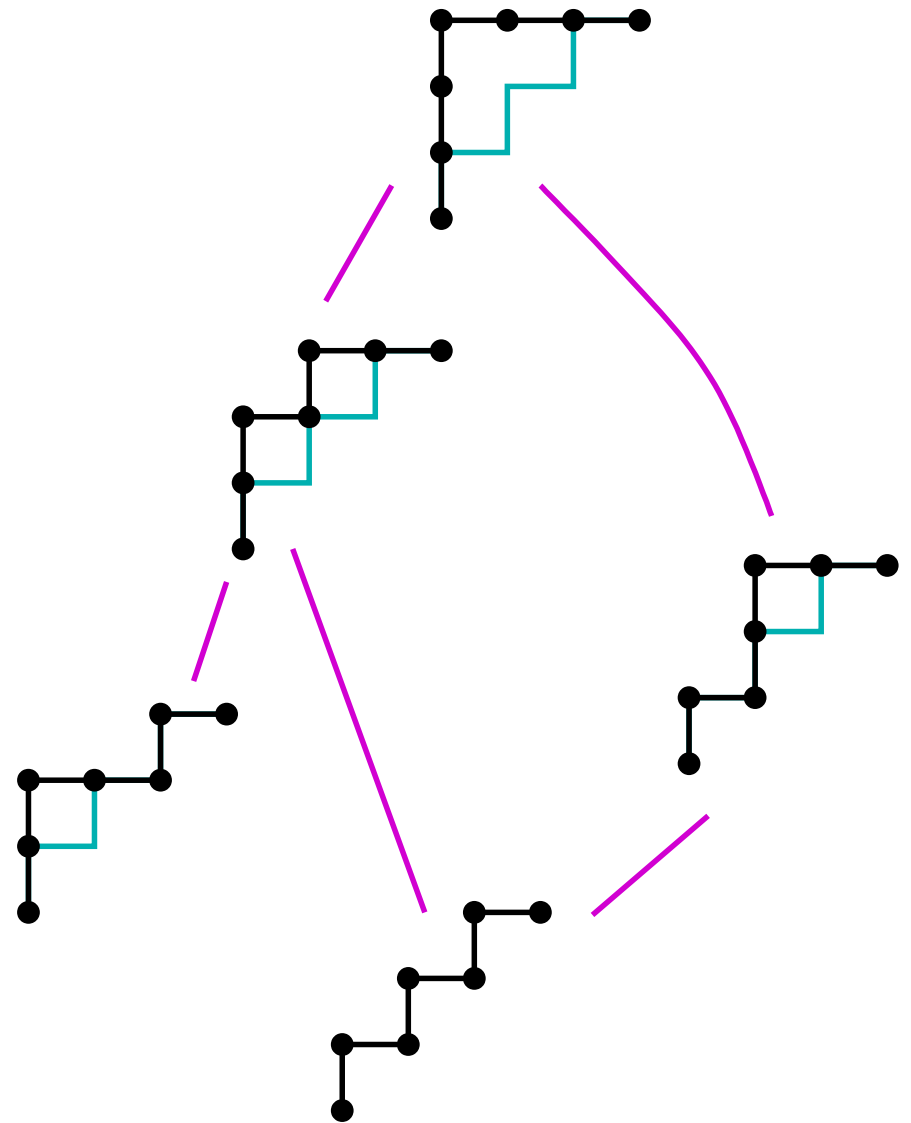
Λ



# Tamari vs. greedy Tamari: Dyck paths



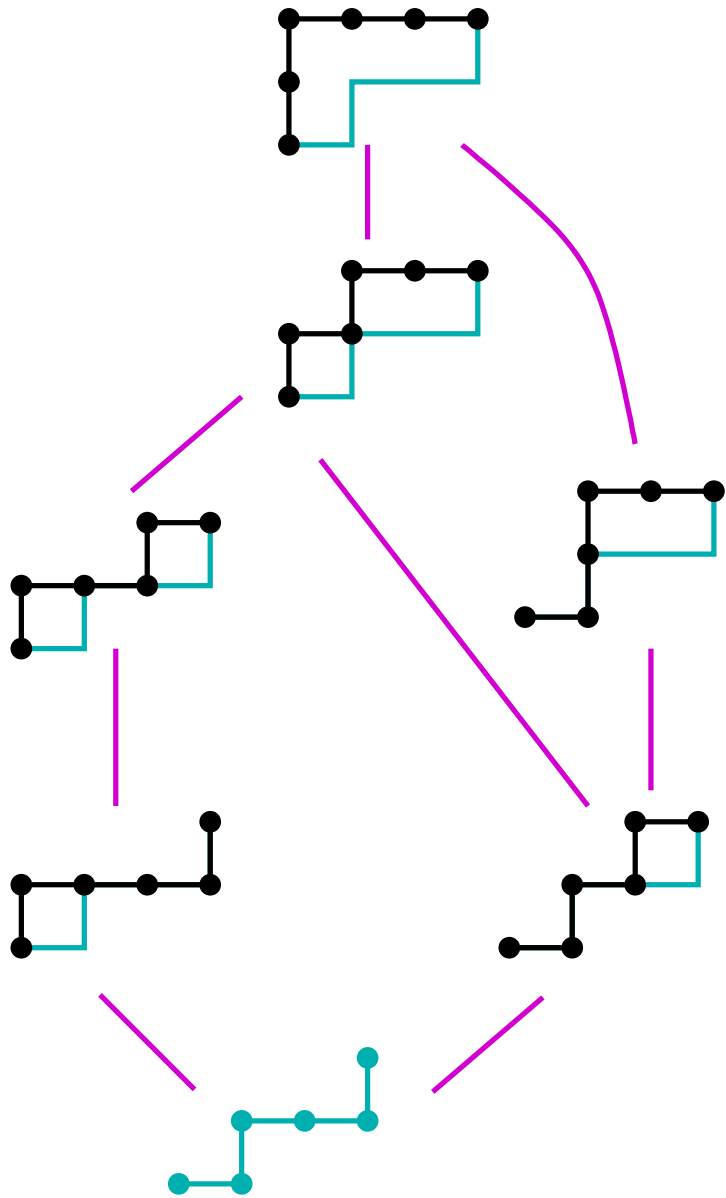
Tamari  $T_3$



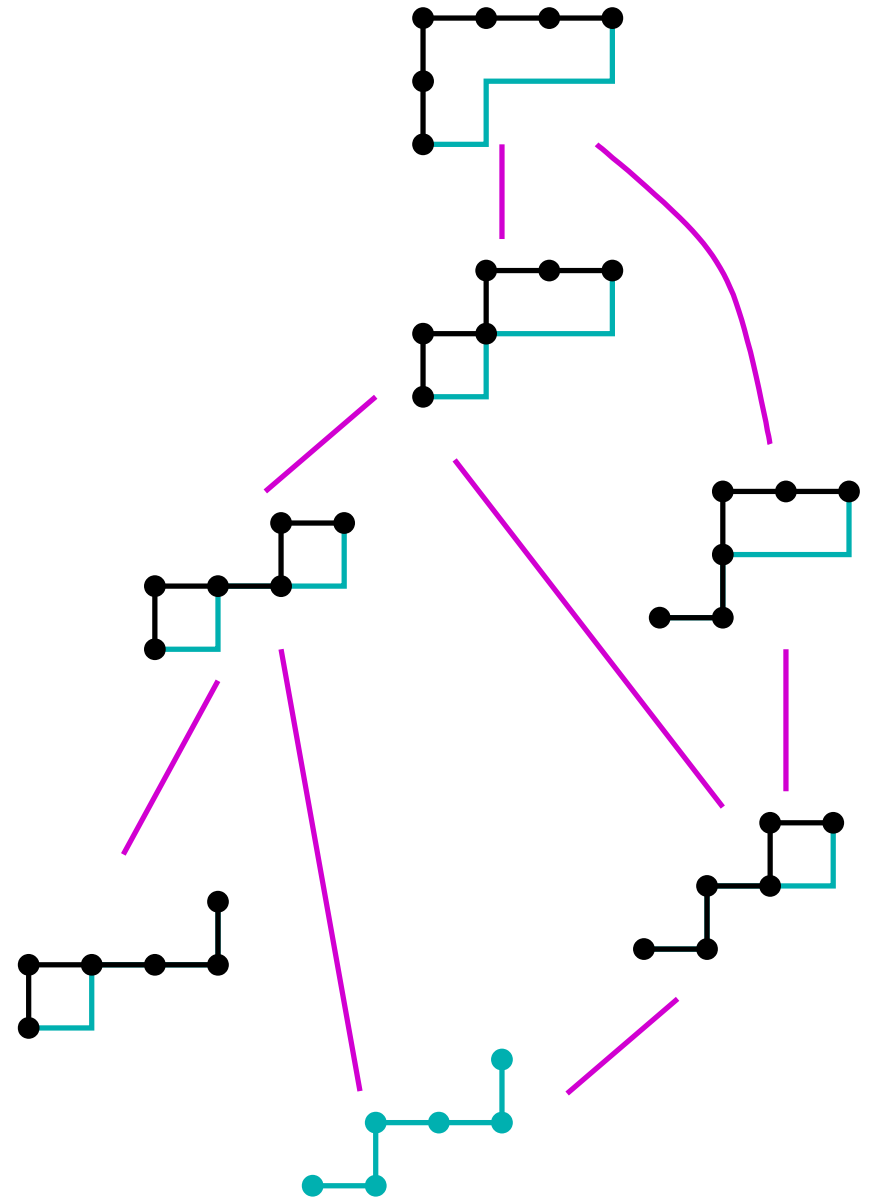
Greedy Tamari  $G_3$



# Tamari vs. greedy Tamari



Tamari  $T_v$



Greedy Tamari  $G_v$

# The number of **greedy** m-Tamari intervals

[MBM & Chapoton 23(a)]

For  $v=(NE^m)^n$ , the number of intervals in the greedy Tamari order is:

$$g_n^{(m)} = \frac{(m+2)(m+1)^{n-1}}{(mn+1)(mn+2)} \binom{(m+1)n}{n}.$$

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**Proof:** a recursive construction of intervals, involving a new “catalytic” parameter: the **final descent** of the upper path.

**Discrete differential equation:**

$$x^2 G(x) = t (x + x^2 G(x) \Delta)^{(m+2)} (1).$$

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**Bijjective proof?** We know what these numbers count!

# Planar constellations

[MBM & Schaeffer 00]

The number of planar  $(m+1)$ -constellations with  $n$  black faces is:

$$g_n^{(m)} = \frac{(m+2)(m+1)^{n-1}}{(mn+1)(mn+2)} \binom{(m+1)n}{n}.$$

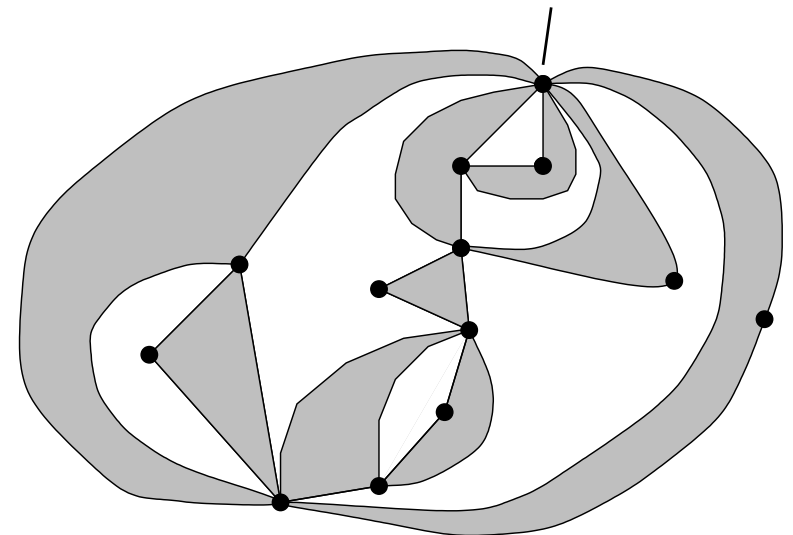
**Constellation:** a rooted planar map with bicolored faces

- black faces of degree  $(m+1)$
- white faces of degree  $k(m+1)$ , for some  $k > 0$ .

[Lando & Zvonkine 04]

Bijection with  
greedy intervals?

$m+1=3$



## **V. Final remarks**

# Refined enumeration of intervals

**Conjecture:** a bijection between

- greedy  $m$ -Tamari intervals  $[v,w]$  where  $w$  has  $n_i$  ascents of size  $i$  for each  $i$ ,
- $(m+1)$ -constellations with  $n_i$  white faces of degree  $i(m+1)$ .

The size of the first ascent matches the degree of the root face.

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Equation for  $m$ -constellations [Fang 16]:

$$C(x) = 1 + xt (C(x) + \Omega)^{(m)} (1),$$

where

$$\Omega(x^k) = \sum_{i=1}^k p_i x^{k-i},$$

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cf. Enumeration of  $m$ -Tamari intervals  $[v,w]$  by ascent lengths of  $w$  [mbm, Chapuy & Préville-Ratelle 13]

# q-Analogues

Count intervals  $[w_1, w_2]$  and record the **length of the longest chain** from  $w_1$  to  $w_2$ .

→ Stanley lattices  $S_v$  : some known results (e.g.,  $v = E^i N^j$ )

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$$x^2 G(x) = t (x + x^2 G(x)\Delta_q)^{(m+2)} (1).$$

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# More questions

→ Summing over  $v$  the number of intervals in the greedy posets  $G_v$ :

$$\sum_{v \rightsquigarrow (i,j)} \text{Int}(G_v) = ???$$

Algebraicity? D-finiteness? Asymptotics?

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→ **Asymptotic properties** of random intervals? [Chapuy 23]

→ Intervals in **Cambrian lattices**? [Reading 06, Stump et al. 15]



Thanks for your  
attention

# Des tamaris gloutons ?



Des tamaris



Tamarinier, feuilles et fruits

# Des tamaris gloutons ?



Des tamaris



Tamarinier, feuilles et fruits



Un tamarin empereur

# Des tamaris gloutons ?



Des tamaris



Tamarinier, feuilles et fruits



Un tamarin empereur



Un glouton

# Des tamaris gloutons ?



Des tamaris



Dov Tamari (1911-2006)



Un tamarin empereur



inier, feuilles et fruits



Un glouton