Counting planar maps, coloured or uncoloured

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A planar map
Planar maps

degree 3

=  ≠
- vertices
- edges
- and faces
Rooted planar maps

- vertices
- edges
- and faces
Every face has degree 3.
Loops and multiple edges are allowed.
Colourings

Proper

Non-proper (general)

Monochromatic edge
Map enumeration

Let $\mathcal{M}$ be a set of planar maps, and let $m(n)$ be the number of maps of $\mathcal{M}$ of size (= edge number) $n$.

- **Objective:** determine the sequence $m(n)$

\[ m(n) = \cdots \quad \text{or} \quad M(t) := \sum_{n \geq 0} m(n) t^n = \sum_{M \in \mathcal{M}} t^{e(M)} = \cdots \]

where $M(t)$ is the generating function of $\mathcal{M}$.

- **Multi-parameter enumeration:**

\[ m(n; k) = \cdots \quad \text{or} \quad M(t; x) := \sum_{n \geq 0} m(n; k) t^n x^k = \cdots \]
Examples

- The generating function of planar maps, counted by edges, is

\[
M(t) = \sum_{M} t^{e(M)} = \sum_{n \geq 0} \frac{2 \cdot 3^n}{n(n + 1)} \binom{2n}{n} t^n = \frac{(1 - 12t)^{3/2} - 1 + 18t}{54t^2}
\]

[Tutte 60’s]
Examples

- The generating function of planar maps, counted by edges, is

$$M(t) = \sum_{M} t^{e(M)} = \sum_{n \geq 0} \frac{2 \cdot 3^n}{n(n+1)} \binom{2n}{n} t^n = \frac{(1 - 12t)^{3/2} - 1 + 18t}{54t^2}$$

[Tutte 60's]

- The generating function of planar triangulations, counted by vertices, is

$$T(t) = \sum_{T} t^{v(T)} = \sum_{n \geq 2} \frac{2 \cdot 4^{n-2} (3n - 6)!!}{n!(n-2)!!} t^n$$

with $k!! = k(k-2)(k-4)\cdots$, and satisfies the cubic equation

$$64T(t)^3 + (1 - 96t)T(t)^2 - t(1 - 30t)T(t) + t^3 (1 - 27t) = 0$$

[Mullin, Nemeth, Schellenberg 70]
Maps equipped with an additional structure

In combinatorics, but mostly in statistical physics

<table>
<thead>
<tr>
<th>How many maps equipped with...</th>
<th>What is the expected partition function of...</th>
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<td>– a spanning tree?</td>
<td>– the Ising model?</td>
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<td>[Mullin 67]</td>
<td>[Boulatov, Kazakov, MBM, Schaeffer,</td>
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<td>– a spanning forest?</td>
<td>– the hard-particle model?</td>
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<td>[Bouttier et al., Sportiello et al.]</td>
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<td>– a self-avoiding walk?</td>
<td>– the Potts model?</td>
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<td>[Duplantier-Kostov 88]</td>
<td>[Eynard-Bonnet 99, Baxter 01,</td>
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<td>MBM-Bernardi 09, Guionnet et al. 10]</td>
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The Potts model on planar maps

- Count all $q$-colourings of some family $\mathcal{M}$ of planar maps, keeping track of the number $m(M)$ of monochromatic edges:

$$M(q, \nu, t) := \sum_{M \text{ } q\text{-coloured}} t^{e(M)} \nu^{m(M)}$$

The Potts generating function of maps.

- In other words,

$$M(q, \nu, t) = \sum_{M} Z_{M}(q, \nu) t^{e(M)}$$

where

$$Z_{M}(q, \nu) = \sum_{c:V(M) \rightarrow \{1,2,\ldots,q\}} \nu^{m(c)}$$

is the Potts partition function of $M$.

Example: When $M$ has one edge and two vertices, $Z_{M}(q, \nu) = q\nu + q(q - 1)$

\[ i \quad \nu \quad i \quad \quad i \quad j \neq i \quad \text{proper} \]
The Potts model on planar maps

- Count all $q$-colourings of some family $\mathcal{M}$ of planar maps, keeping track of the number $m(M)$ of monochromatic edges:

$$M(q, \nu, t) := \sum_{M \ q-\text{coloured}} t^{e(M)} \nu^{m(M)}$$

The Potts generating function of maps.

- In particular,

$$M(q, 0, t) := \sum_{M \ q-\text{prop. coloured}} t^{e(M)} = \sum_{M} \chi_{M}(q)t^{e(M)}$$

counts properly coloured maps.
The Potts model on planar maps

• Count all $q$-colourings of some family $\mathcal{M}$ of planar maps, keeping track of the number $m(M)$ of monochromatic edges:

$$M(q, \nu, t) := \sum_{M \text{ $q$-coloured}} t^{e(M)} \nu^{m(M)}$$

The Potts generating function of maps.

• Equivalently, find

$$\sum_{M \in \mathcal{M}} T_M(x, y) t^{e(M)} = \ldots$$

where $T_M(x, y)$ is the Tutte polynomial of $M$. Connection:

$$(x - 1)(y - 1)^{\nu(M)} T_M(x, y) = \sum_{q\text{-colourings of } M} \nu^{m(M)}$$

with $q = (x - 1)(y - 1)$ and $\nu = y - 1$. 
Three approaches for counting maps

- [Tutte 60's]: recursive description of maps
  ⇒ functional equations for the GF

- [Schaeffer 97]: bijections with trees

- [Brézin-Itzykson-Parisi-Zuber 78]: matrix integrals

\[
\lim_{N \to \infty} \frac{1}{N^2} \log \int dHe \ N \text{tr} \left( -\frac{H^2}{2} + t \frac{H^3}{3} \right).
\]

\(H\): hermitian \(N \times N\) matrix
I.1. Uncoloured enumeration: the recursive approach
Planar maps (tend to) have algebraic generating functions

\[ M(t_1, \ldots, t_k) \text{ map GF: there exists a polynomial } Q \text{ such that} \]

\[ Q(t_1, \ldots, t_k; M(t_1, \ldots, t_k)) = 0. \]

Example:

\[ M(t) = \frac{(1 - 12t)^{3/2} - 1 + 18t}{54t^2} \Rightarrow (54t^2M(t) + 1 - 18t)^2 = (1 - 12t)^3 \]
Recursive description of planar maps: deleting the root-edge

Let

\[ F(t; x) \equiv F(x) = \sum_{M} t^{e(M)} x^{df(M)} = \sum_{d \geq 0} F_d(t) x^d \]

where \( e(M) \) is the number of edges and \( df(M) \) the degree of the outer face.
Recursive description of planar maps: deleting the root-edge

Let

\[ F(t; x) \equiv F(x) = \sum_{M} t^{e(M)} x^{d_f(M)} = \sum_{d \geq 0} F_d(t) x^d \]

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where \( e(M) \) is the number of edges and \( df(M) \) the degree of the outer face.

\[ F(x) = 1 + tx^2 F(x)^2 + t \sum_{d \geq 0} F_d(t) (x^{d+1} + x^d + \cdots + x) \]

\[ = 1 + tx^2 F(x)^2 + \frac{tx \frac{xF(x) - F(1)}{x - 1}}{x} \]

[Tutte 68] A quadratic equation with one catalytic variable, \( x \)
Recursive description of planar maps: contracting the root-edge

Let

\[ F(t; y) \equiv F(y) = \sum_M t^{e(M)} y^{dv(M)} = \sum_{d \geq 0} F_d(t) y^d \]

where \( e(M) \) is the number of edges and \( dv(M) \) the degree of the root vertex.

\[ F(y) = 1 + ty^2F(y)^2 + \]

\[ = 1 + ty^2F(y)^2 + \]

\[ t \sum_{d \geq 0} F_d(t) \left( y^{d+1} + y^d + \cdots + y \right) \]

\[ = 1 + ty^2F(y)^2 + \]

\[ ty \frac{yF(y) - F(1)}{y - 1} \]

The same equation... (duality)
Polynomial equations with one catalytic variable

Examples:

$$F(x) = 1 + tx^2 F(x)^2 + tx \frac{x F(x) - F(1)}{x - 1}$$

$$F(x) = 1 + tx F(x)^3 + tx (2F(x) + F(1)) \frac{F(x) - F(1)}{x - 1}$$

$$+ tx \frac{F(x) - F(1) - (x - 1) F'(1)}{(x - 1)^2}$$

Solution:

- Guess and check [Tutte 60’s]
- Brown’s quadratic method [Brown 65]
- A generalization to all polynomial equations with one catalytic variable [mbm-Jehanne 06]
Polynomial equations with one catalytic variable

- **General framework:** A polynomial equation:

\[
P(F(x), F_1, \ldots, F_m, t, x) = 0
\]  

where \( F(x) \equiv F(t; x) \) is a series in \( t \) with **polynomial** coefficients in \( x \), and \( F_i \equiv F_i(t) \) does not depend on \( x \).

- **Results**

1. The solution of every **proper** equation of this type is **algebraic:** There exists \( Q \) such that \( Q(t, x, F(t; x)) = 0 \).

2. A practical (but heavy) strategy allows to solve specific examples (that is, to derive from (1) an algebraic equation for \( F(x) \), or \( F_1, \ldots, F_m \)).

[MBM-Jehanne 06]
Example

\[ F(t; x) \equiv F(x) = 1 + tx^2 F(x)^2 + tx \frac{xF(x) - F(1)}{x - 1} \]

\[ \Downarrow \]

\[ F(t; 1) = \frac{(1 - 12t)^{3/2} - 1 + 18t}{54t^2} \]

with two lines of Maple.

Equivalently,

\[ F(t; 1) = A(t) - tA(t)^3 \quad \text{where} \quad A(t) = 1 + 3tA(t)^2 \]

⇒ Many map families have an algebraic generating function
I.2. Uncoloured enumeration: bijections

The algebraicity of planar maps generating functions can be explained via bijections with plane trees
Algebraic series count plane trees

Algebraic series usually arise from tree enumeration:

\[ n \text{ nodes} \]
\[ a(n) \]

\[
A(t) = 1 + tA(t)^2
\]

Are there correspondences between maps and trees?
Four-valent maps: a proof from The Book?

- **Bijection** between 4-valent planar maps \((n\) vertices) and **balanced** blossoming trees \((n\) nodes) [Schaeffer 97]

\[
\begin{align*}
\text{A planar map (GF } M & \text{)} \\
\text{A blossoming tree (GF } A = 1 + 3tA^2) & \text{)
\end{align*}
\]

- **An unbalanced blossoming tree** is, in essence, a 3-tuple of blossoming subtrees [Bouttier et al. 02]:

\[
\Rightarrow M = A - tA^3.
\]
Blossoming trees and 4-valent maps
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Bijections between maps and trees...

- Improve our understanding of the nature of planar maps

- Lead to new results, e.g. on the geometry of random planar maps. For instance, the average diameter of a random map of size $n$ scales like $n^{1/4}$ [Chassaing-Schaeffer 02]

- These bijections are the starting point of many recent results in probability theory on the asymptotic properties of large random maps [Le Gall, Miermont, Marckert, Paulin...]

II.1. Coloured enumeration: the recursive approach

(joint work with Olivier Bernardi)

\[ M(q, \nu, t) := \sum_{M \text{ q–coloured}} t^{e(M)} \nu^{m(M)} \]

- Other approaches: [Eynard-Bonnet 99], [Guionnet et al. 10]
Forget algebraicity!

**Theorem [Tutte 73]:** For planar triangulations,

\[ \sum_{T} \chi'_T(1) t^\nu(T) = \sum_{n} (-1)^n b(n) t^{n+2} \]

where

\[ b(n) = \frac{2 (3n)!}{n!(n+1)!(n+2)!} \sim 27^n n^{-4}, \]

and this asymptotic behaviour prevents the series \( B(t) := \sum b_n t^n \) from being algebraic.

However, it satisfies a **linear differential equation**.
Catalytic variables

The Potts generating function of planar maps, being transcendental, cannot be described with one catalytic variable
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HOWEVER

it can be described with two catalytic variables
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it can be described with two catalytic variables

WHY IS THAT SO?

• The recursive description of the Potts partition function

  \[ Z_G(q, \nu) = Z_{G\setminus e}(q, \nu) + (\nu - 1) Z_{G/e}(q, \nu) \]

calls for a recursive description of maps by contraction and deletion of edges.

• This is possible if one keeps track of the degree of the outer face, and the degree of the root-vertex.
Equations with two catalytic variables

• Let

\[ M(x, y) \equiv M(q, \nu, t; x, y) = \frac{1}{q} \sum_{M} Z_{M}(q, \nu) t^{e(M)} x^{d_{v}(M)} y^{d_{f}(M)}, \]

where \( d_{v}(M) \) (resp. \( d_{f}(M) \)) is the degree of the root-vertex (resp. root-face).

• The Potts generating function of planar maps satisfies:

\[
M(x, y) = 1 + xyt \left((\nu - 1)(y - 1) + qy\right) M(x, y) M(1, y) \\
+ xyt (\nu - 1) M(x, y) M(x, 1) \\
+ xyt (x \nu - 1) \frac{x M(x, y) - M(1, y)}{x - 1} + xyt \frac{y M(x, y) - M(x, 1)}{y - 1}.
\]

[Tutte 68]

This equation has been sleeping for 40 years
In the footsteps of W. Tutte

- For the GF $T(q, t; x, y) \equiv T(x, y)$ of properly $q$-coloured triangulations:

$$T(x, y) = xy^2 q(q-1) + \frac{xt}{yq} T(1, y) T(x, y) + xt \frac{T(x, y) - y^2 T_2(x)}{y} - x^2 yt \frac{T(x, y) - T(1, y)}{x-1}$$

where $T_2(x)$ is the coefficient of $y^2$ in $T(x, y)$.

[Tutte 73] Chromatic sums for rooted planar triangulations: the cases $\lambda = 1$ and $\lambda = 2$
[Tutte 73] Chromatic sums for rooted planar triangulations, II: the case $\lambda = \tau + 1$
[Tutte 73] Chromatic sums for rooted planar triangulations, III: the case $\lambda = 3$
[Tutte 73] Chromatic sums for rooted planar triangulations, IV: the case $\lambda = \infty$
[Tutte 74] Chromatic sums for rooted planar triangulations, V: special equations
[Tutte 78] On a pair of functional equations of combinatorial interest
[Tutte 82] Chromatic solutions
[Tutte 82] Chromatic solutions II
[Tutte 84] Map-colourings and differential equations

[Tutte 95]: Chromatic sums revisited
In the footsteps of W. Tutte

• For the GF $T(q, t; x, y) \equiv T(x, y)$ of properly $q$-coloured triangulations:

$$T(x, y) = xy^2q(q-1) + \frac{xt}{yq} T(1, y)T(x, y) + xt\frac{T(x, y) - y^2T_2(x)}{y} - x^2yt\frac{T(x, y) - T(1, y)}{x-1}$$

where $T_2(x)$ is the coefficient of $y^2$ in $T(x, y)$.

**Theorem** [Tutte]

• For $q = 2 + 2 \cos \frac{2\pi}{m}$, $q \neq 4$, the series $T(1, y) \equiv T(t; 1, y)$ satisfies a polynomial equation with one catalytic variable $y$. 
In the footsteps of W. Tutte

• For the GF $T(q, t; x, y) \equiv T(x, y)$ of properly $q$-coloured triangulations:

$$T(x, y) = xy^2q(q-1) + \frac{x}{yq}T(1, y)T(x, y) + xT(x, y) - y^2T_2(x) - x^2yt\frac{T(x, y) - T(1, y)}{x - 1}$$

where $T_2(x)$ is the coefficient of $y^2$ in $T(x, y)$.

**Theorem** [Tutte]

• For $q = 2 + 2 \cos \frac{2\pi}{m}$, $q \neq 4$, the series $T(1, y) \equiv T(t; 1, y)$ satisfies a polynomial equation with one catalytic variable $y$.

• When $q$ is generic, the generating function of properly $q$-coloured planar triangulations is differentially algebraic:

$$2q^2(1-q)t + (qt + 10H - 6tH')H'' + q(4-q)(20H - 18tH' + 9t^2H''') = 0$$

with $H(t) = t^2T_2(q, \sqrt{t}; 1)/q$. 
Adapt this to other equations!

[Tutte 73] Chromatic sums for rooted planar triangulations: the cases $\lambda = 1$ and $\lambda = 2$
[Tutte 73] Chromatic sums for rooted planar triangulations, II: the case $\lambda = \tau + 1$
[Tutte 73] Chromatic sums for rooted planar triangulations, III: the case $\lambda = 3$
[Tutte 73] Chromatic sums for rooted planar triangulations, IV: the case $\lambda = \infty$
[Tutte 74] Chromatic sums for rooted planar triangulations, V: special equations
[Tutte 78] On a pair of functional equations of combinatorial interest
[Tutte 82] Chromatic solutions
[Tutte 82] Chromatic solutions II
[Tutte 84] Map-colourings and differential equations

[Tutte 95]: Chromatic sums revisited
Our results

- Let $M(q, \nu, t; x, y)$ be the Potts generating function of planar maps:

  $$M(x, y) \equiv M(q, \nu, t; x, y) = \frac{1}{q} \sum_{M} Z_M(q, \nu) t^{e(M)} x^{d_{v}(M)} y^{d_{f}(M)},$$

where $d_{v}(M)$ (resp. $d_{f}(M)$) is the degree of the root-vertex (resp. root-face).

Theorem

- For $q = 2 + 2 \cos \frac{j \pi}{m}$, $q \neq 0, 4$, the series $M(q, \nu, t; 1, y) \equiv M(1, y)$ satisfies a polynomial equation with one catalytic variable $y$, and the complete Potts generating function $M(q, \nu, t; x, y)$ is algebraic.

- When $q$ is generic, $M(q, \nu, t; 1, 1)$ is differentially algebraic:

  (an explicit system of differential equations)

Example: The Ising model on planar maps ($q = 2$)

Let $A$ be the series in $t$, with polynomial coefficients in $\nu$, defined by

$$ A = t \frac{(1 + 3 \nu A - 3 \nu A^2 - \nu^2 A^3)^2}{1 - 2 A + 2 \nu^2 A^3 - \nu^2 A^4}. $$

Then the Ising generating function of planar maps is

$$ M(2, \nu, t; 1, 1) = \frac{1 + 3 \nu A - 3 \nu A^2 - \nu^2 A^3}{(1 - 2 A + 2 \nu^2 A^3 - \nu^2 A^4)^2} \, P(\nu, A) $$

where

$$ P(\nu, A) = \nu^3 A^6 + 2 \nu^2 (1 - \nu) A^5 + \nu (1 - 6 \nu) A^4 
- \nu (1 - 5 \nu) A^3 + (1 + 2 \nu) A^2 - (3 + \nu) A + 1. $$

~~ Asymptotics: Phase transition at $\nu_c = \frac{3 + \sqrt{5}}{2}$, critical exponents...
Example: properly 3-coloured planar maps \((q = 3, \nu = 0)\)

Let \(A\) be the quartic series in \(t\) defined by

\[
A = t \frac{(1 + 2A)^3}{(1 - 2A^3)}.
\]

Then the generating function of properly 3-coloured planar maps is

\[
M(3, 0, t; 1, 1) = \frac{(1 + 2A)(1 - 2A^2 - 4A^3 - 4A^4)}{(1 - 2A^3)^2}
\]

\sim Asymptotics: A random loopless planar map with \(n\) edges has approximately \((1.42...)^n\) proper 3-colourings.
Our results: when $q$ is generic

• Let $M(q, \nu, t; x, y)$ be the Potts generating function of planar maps:

$$M(x, y) \equiv M(q, \nu, t; x, y) = \frac{1}{q} \sum_M Z_M(q, \nu) t^e(M) x^{dv(M)} y^{df(M)},$$

where $dv(M)$ (resp. $df(M)$) is the degree of the root-vertex (resp. root-face).

**Theorem**

• For $q = 2 + 2 \cos \frac{j\pi}{m}$, $q \neq 4$, the series $M(q, \nu, t; 1, y) \equiv M(1, y)$ satisfies a polynomial equation with one catalytic variable $y$, and the complete Potts generating function $M(q, \nu, t; x, y)$ is algebraic.

• When $q$ is generic, $M(q, \nu, t; 1, 1)$ is differentially algebraic:

  (an explicit system of differential equations)
An explicit system of differential equations

Let \( D(t,v) = q\nu + (\nu - 1)^2 - q(\nu + 1)v + (q + t(\nu - 1)(q - 4)(q + \nu - 1))v^2. \)

- There exists a unique 8-tuple \((P_1(t), \ldots, P_4(t), Q_1(t), Q_2(t), R_1(t), R_2(t))\) of series in \( t \) with polynomial coefficients in \( q \) and \( \nu \) such that

\[
\frac{1}{v^2 R} \frac{\partial}{\partial v} \left( \frac{v^4 R^2}{PD^2} \right) = \frac{1}{Q} \frac{\partial}{\partial t} \left( \frac{Q^2}{PD^2} \right),
\]

where

\[
P(t,v) = P_4(t)v^4 + P_3(t)v^3 + P_2(t)v^2 + P_1(t)v + 1,
\]
\[
Q(t,v) = Q_2(t)v^2 + Q_1(t)v + 1,
\]
\[
R(t,v) = R_2(t)v^2 + R_1(t)v + q + \nu - 3,
\]

with the initial conditions (at \( t = 0 \)):

\[
P(0,v) = (1 - v)^2 \quad \text{and} \quad Q(0,v) = 1 - v.
\]
An explicit system of differential equations (cont’d)

- The Potts generating function of planar maps, $M(1, 1) \equiv M(q, \nu, t; 1, 1)$, satisfies

$$12 t^2 \left(q\nu + (\nu - 1)^2\right) M(q, \nu, t; 1, 1) =$$

$$8 t (q + \nu - 3) Q_1(t) - Q_1(t)^2 + P_2(t) - 2 Q_2(t) - 4 t (2 - 3 \nu - q) - 12 t^2 (q + \nu - 3)^2.$$  

Questions
1. Use the structure of

$$\frac{1}{v^2 R \partial v} \left( \frac{v^4 R^2}{PD^2} \right) = \frac{1}{Q \partial t} \left( \frac{Q^2}{PD^2} \right),$$

to obtain a single differential equation (or an expression?) for $M(q, \nu, t; 1, 1)$.

2. Relate this to elliptic functions, and to the papers of [Bonnet & Eynard 99], and [Guionnet, Jones, Shlyakhtenko & Zinn-Justin 10]
An analogous system for triangulations

Let \( D(t, v) = qv^2 + (\nu - 1)(4(\nu - 1) + q)v + (q\nu(\nu - 1)(q - 4)t + (\nu - 1)^2)v^2. \)

- There exists a unique 7-tuple \((P_1(t), \ldots, P_3(t), Q_1(t), Q_2(t), R_0(t), R_1(t))\) of series in \( t \) with polynomial coefficients in \( q \) and \( \nu \) such that

\[
\frac{1}{v^2R} \frac{\partial}{\partial v} \left( \frac{v^5R^2}{PD^2} \right) = \frac{1}{Q} \frac{\partial}{\partial t} \left( \frac{Q^2}{PD^2} \right),
\]

where

\[
\begin{align*}
P(t, v) & = P_3(t)v^3 + P_2(t)v^2 + P_1(t)v + 1, \\
Q(t, v) & = Q_2(t)v^2 + Q_1(t)v + 2\nu, \\
R(t, v) & = R_1(t)v + R_0(t),
\end{align*}
\]

with the initial conditions (at \( t = 0 \)):

\[
P(0, v) = 1 + \frac{v}{4} \quad \text{and} \quad Q(0, v) = 2\nu + v.
\]

- Expression of the Potts GF of triangulations in terms of the \( P_i \) and \( Q_i \)
... and for properly \( q \)-coloured triangulations \((\nu = 0)\)

Let \( D(\nu) = \nu + 4 - q \).

- There exists a unique 4-tuple \((P_1, P_2, P_3, Q_1)\) of zeries in \( t \) with polynomial coefficients in \( q \) such that

\[
-\frac{4t}{\nu} \frac{\partial}{\partial \nu} \left( \frac{v^3}{P} \right) = \frac{1}{Q} \frac{\partial}{\partial t} \left( \frac{Q^2}{PD} \right).
\]

where

\[
P(t, v) = P_3(t)v^3 + P_2(t) + P_1(t)v + 1,
Q(t, v) = Q_1(t)v + 1,
\]

with the initial conditions (at \( t = 0 \)):

\[
P(0, v) = 1 + v/4 \quad \text{and} \quad Q(0, v) = 1.
\]

- From the system, one can derive Tutte’s differential equation,

\[
2q^2(1 - q)t + (qt + 10H - 6tH')H'' + q(4 - q)(20H - 18tH' + 9t^2H''') = 0
\]

with \( H(t) = t^2T_2(q, \sqrt{t}; 1)/q \).
II.2. Coloured enumeration: bijections?

Some bijections exist in special cases... but most remain to be found.
Some existing bijections

• Maps equipped with a spanning tree \((T_M(1, 1))\)  
  [Mullin 67], [Bernardi 07]

• Maps equipped with a bipolar orientation \((-1)^{v(M)}\chi_M'(1)\)  
  [Felsner-Fusy-Noy-Orden 08],  
  [Fusy-Poulalhon-Schaeffer 08],  
  [Bonichon-mbm-Fusy 08]

• The Ising model on planar maps (case \(q = 2\))  
  [MBM-Schaeffer 02], [Bouttier et al.]
Bijective counting of maps equipped with a spanning tree

$n$ edges, $k + 1$ vertices ($\Rightarrow k$ edges in the tree)
Bijective counting of maps equipped with a spanning tree

$n$ edges, $k + 1$ vertices ($\Rightarrow k$ edges in the tree)
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$n$ edges, $k + 1$ vertices ($\Rightarrow k$ edges in the tree)

A shuffle of two plane trees

$\begin{align*}
\binom{2n}{2k} C_k C_{n-k}
\end{align*}$

with $C_k = \binom{2k}{k} / (k + 1)$ counts rooted trees with $k$ edges.
Many questions are left...

A. More combinatorics

- Understand algebraic series, e.g., for 3-coloured planar maps:

\[
M(3, 0, t; 1, 1) = \frac{(1 + 2A)(1 - 2A^2 - 4A^3 - 4A^4)}{(1 - 2A^3)^2} \quad \text{with} \quad A = t \frac{(1 + 2A)^3}{(1 - 2A^3)}
\]

- Understand differential equations, e.g., for properly \( q \)-coloured triangulations:

\[
2q^2(1 - q)t + (qt + 10H - 6tH')H'' + q(4 - q)(20H - 18tH' + 9t^2H'') = 0
\]

B. Equations with two catalytic variables

- Elimination in the systems of differential equations
- Connections with elliptic functions
- Connections with [Eynard-Bonnet 99] and [Guionnet et al. 10]

C. Asymptotics

- Asymptotic number of properly \( q \)-coloured maps?
  (done for triangulations \( q \in (28/11, 4) \cup [5, \infty) \) [Odlyzko-Richmond 83])
- More generally, phase transitions and critical exponents of the Potts model