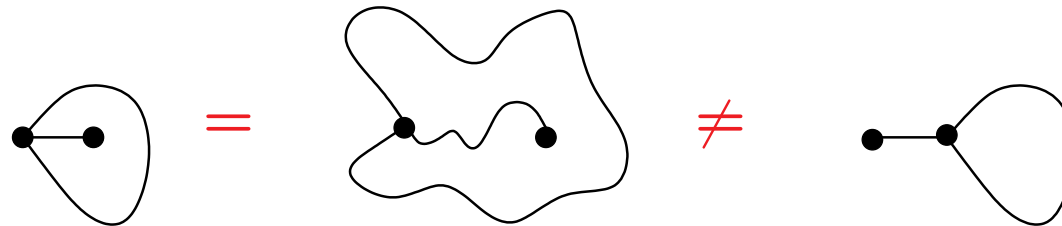
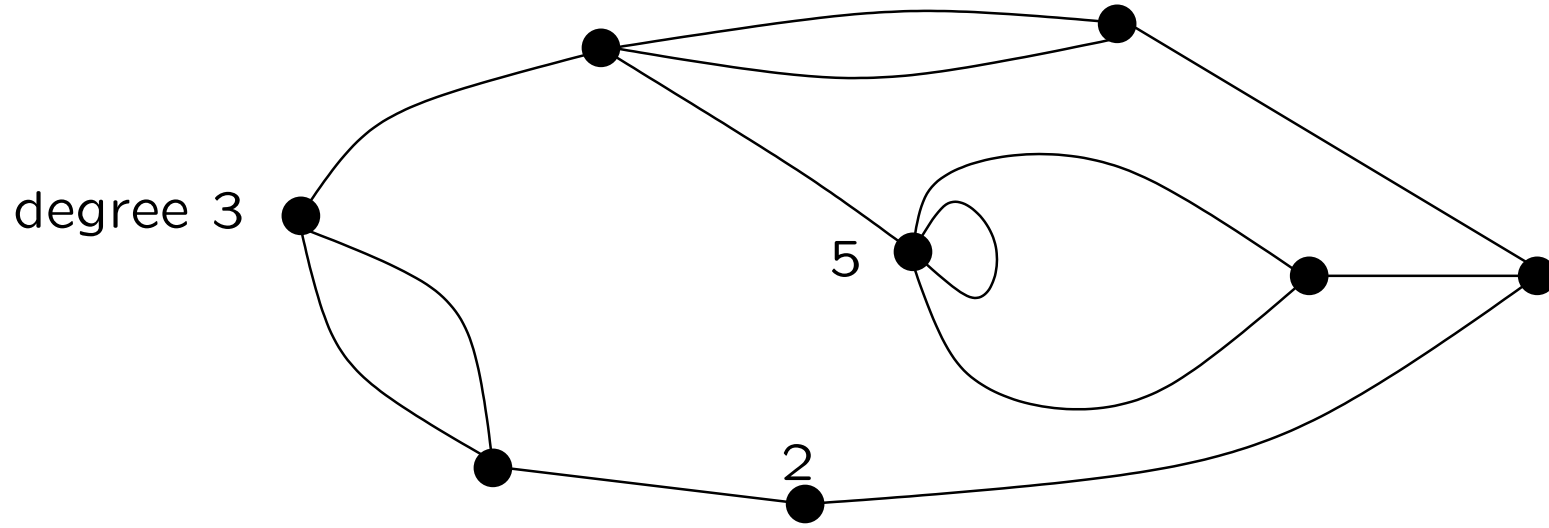


# Counting coloured planar maps

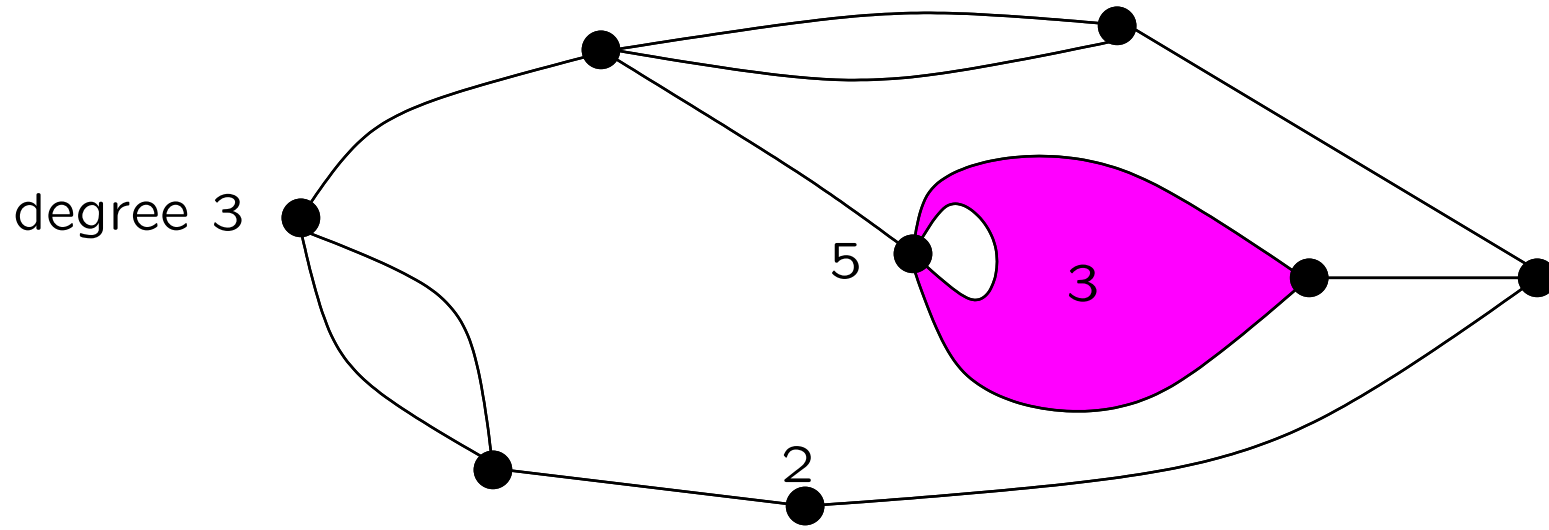
Mireille Bousquet-Mélou, CNRS, Bordeaux, France

*<http://www.labri.fr/~bousquet>*

# Planar maps

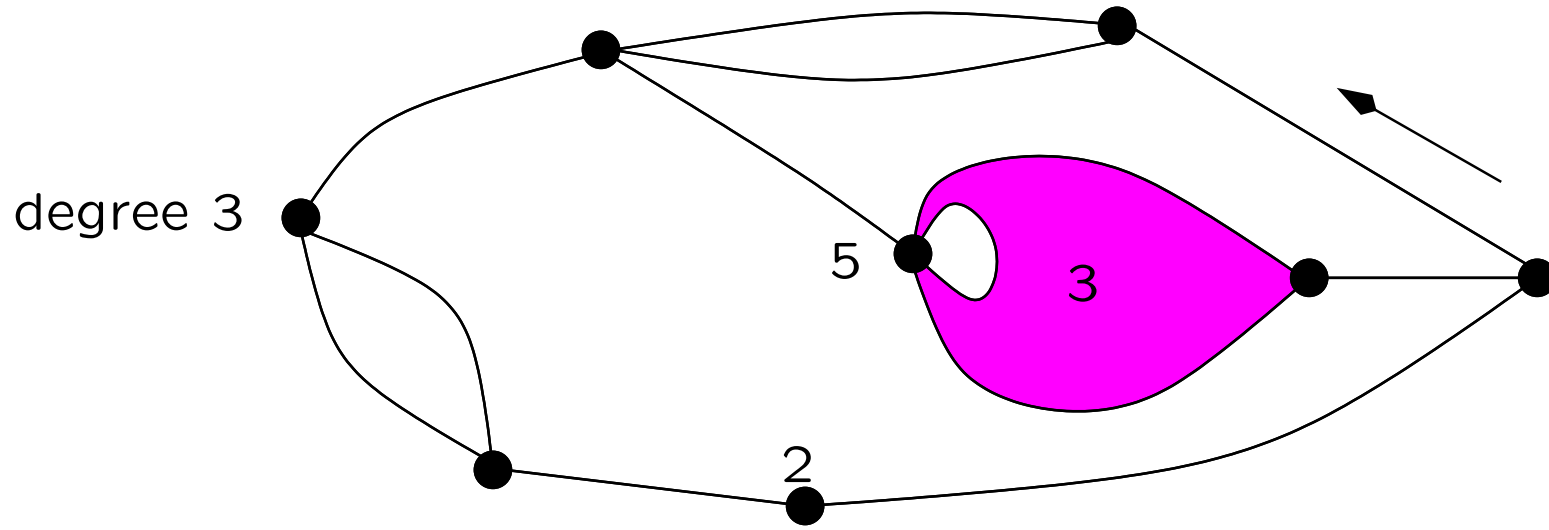


# Planar maps



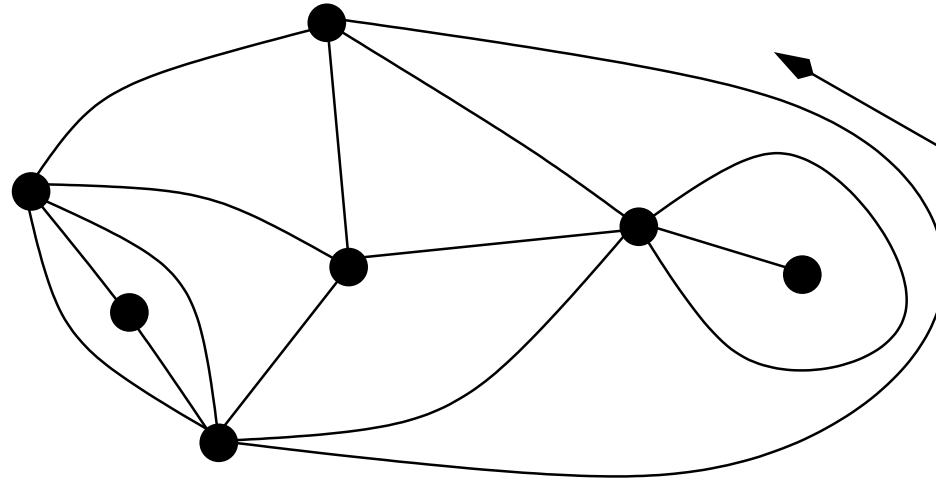
- vertices
- edges
- and faces

# Rooted planar maps



- vertices
- edges
- and faces

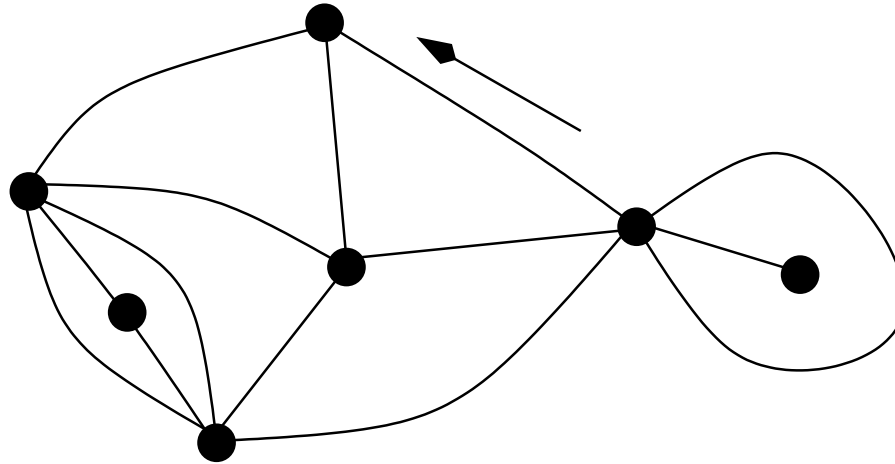
# Triangulations



Every face has degree 3.

Loops and multiple edges are allowed.

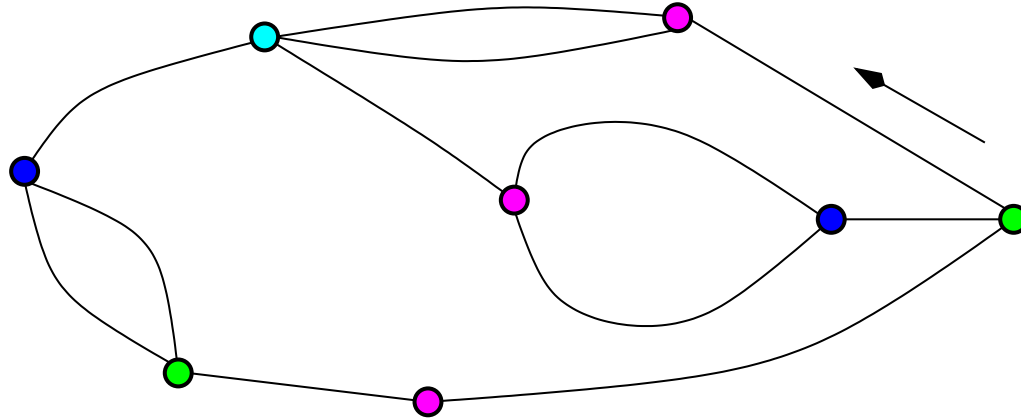
## Quasi-triangulations



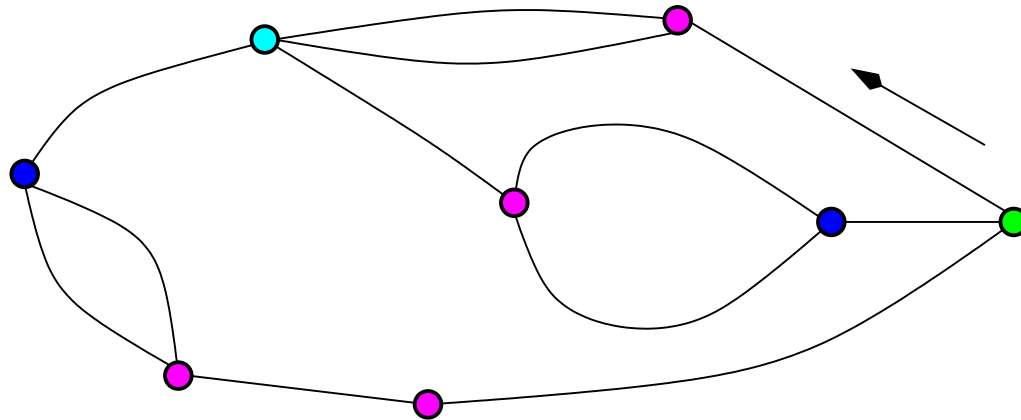
Every **finite** face has degree 3.  
Loops and multiple edges are allowed.

# Colourings

Proper



Non-proper (general)



Monochromatic edge

## Combinatorial enumeration

Let  $\mathcal{A}$  be a set of discrete objects equipped with an integer size,  $a \mapsto |a|$ .

Assume that for all  $n$ , the number of objects of  $\mathcal{A}$  of size  $n$  is finite.

Let  $a(n)$  be this number.

- **Objective:** determine the sequence  $a(n)$

$$a(n) = \dots \quad \text{or} \quad A(t) := \sum_{n \geq 0} a(n)t^n = \dots$$

where  $A(t)$  is the **generating function of the objects of  $\mathcal{A}$** .

- **Asymptotics:** the position and nature of the singularities of  $A(t)$  (often) give asymptotic **estimates** of the numbers  $a(n)$ , as  $n \rightarrow \infty$

$$a(n) \sim \dots$$

- **Multi-parameter enumeration:**

$$a(n, k) = \dots \quad \text{or} \quad A(t, u) := \sum_{n \geq 0} a(n, k)t^n u^k = \dots$$



## Examples

- The generating function of planar maps, counted by edges, is

$$M(t) = \sum_M t^{e(M)} = \sum_{n \geq 0} \frac{2 \cdot 3^n}{n(n+1)} \binom{2n}{n} t^n = \frac{(1 - 12t)^{3/2} - 1 + 18t}{54t^2}$$

[Tutte 60's]

- The generating function of planar triangulations, counted by vertices, is

$$T(t) = \sum_T t^{v(T)} = \sum_{n \geq 2} \frac{2 \cdot 4^{n-2} (3n-6)!!}{n!(n-2)!!} t^n$$

with  $k!! = k(k-2)(k-4)\cdots$ , and satisfies the cubic equation

$$64T(t)^3 + (1 - 96t)T(t)^2 - t(1 - 30t)T(t) + t^3(1 - 27t) = 0$$

[Mullin, Nemeth, Schellenberg 70]

## Objectives

**A.** Count **proper  $q$ -colourings** of some family  $\mathcal{M}$  of planar maps:

$$\sum_{M \text{ prop. } q\text{-coloured}} t^{|M|} = \sum_{M \in \mathcal{M}} \chi_M(q) t^{|M|} = \dots$$

## Objectives

**A.** Count **proper  $q$ -colourings** of some family  $\mathcal{M}$  of planar maps:

$$\sum_{M \text{ prop. } q\text{-coloured}} t^{|M|} = \sum_{M \in \mathcal{M}} \chi_M(q) t^{|M|} = \dots$$

**B1.** Count **all  $q$ -colourings** of some family  $\mathcal{M}$  of planar maps, **keeping track of the number  $m(M)$  of monochromatic edges**:

$$\sum_{M \text{ } q\text{-coloured}} t^{|M|} z^{m(M)} = \dots$$

## Objectives

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**B1.** Count **all  $q$ -colourings** of some family  $\mathcal{M}$  of planar maps, **keeping track of the number  $m(M)$  of monochromatic edges**:

$$\sum_{M \text{ } q\text{-coloured}} t^{|M|} z^{m(M)} = \dots$$

**B2.** Equivalently, find

$$\sum_{M \in \mathcal{M}} T_M(x, y) t^{|M|} = \dots$$

where  $T_M(x, y)$  is the **Tutte polynomial of  $M$** .

$$(x - 1)(y - 1)^{v(M)} T_M(x, y) = \sum_{q\text{-colourings of } M} (y - 1)^{m(M)}$$

with  $q = (x - 1)(y - 1)$

## Some motivations

**A.** Count proper  $q$ -colourings of some family  $\mathcal{M}$  of planar maps:

$$\sum_{M \text{ prop. } q\text{-coloured}} t^{|M|} = \sum_{M \in \mathcal{M}} \chi_M(q) t^{|M|} = \dots$$

• How many  $q$ -colourings has a random map of size  $n$ ?

$q = 4$ : quantitative (and averaged) version of the 4-colour theorem

**B1.** Count all  $q$ -colourings of some family  $\mathcal{M}$  of planar maps, keeping track of the number  $m(M)$  of monochromatic edges:

$$\sum_{M \text{ } q\text{-coloured}} t^{|M|} z^{m(M)} = \dots$$

• Solution of the **Potts model** on planar maps ([Eynard-Bonnet 99], [Baxter 01])

• **A rich problem, hard but not completely out of reach**

# I. Problems in uncoloured enumeration

The generating functions counting (uncoloured) families of planar maps are in general algebraic

## Planar maps (tend to) have algebraic generating functions

$M(t_1, \dots, t_k)$  map GF: there exists a polynomial  $Q$  such that

$$Q(t_1, \dots, t_k; M(t_1, \dots, t_k)) = 0.$$

Example:

$$M(t) = \frac{(1 - 12t)^{3/2} - 1 + 18t}{54t^2} \quad \Rightarrow \quad (54t^2 M(t) + 1 - 18t)^2 = (1 - 12t)^3$$

Arquès Bauer Bédard Bender Bernardi Bessis Bodirsky Bousquet-Mélou  
Boulatov Bouttier Brézin Brown Canfield Chauve Cori Di Francesco  
Duplantier Fusy Gao Goupil Goulden Guitter t'Hooft Itzykson Jackson  
Jacquard Kazakov Kostov Krikun Labelle Lehman Leroux Liskovets Liu Machì  
Mehta Mullin Parisi Poulalhon Richmond Robinson  
Schaeffer Schellenberg Strehl **Tutte** Vainshtein Vauquelin Visentin Walsh  
Wanless Wormald Zinn-Justin Zuber Zvonkine...

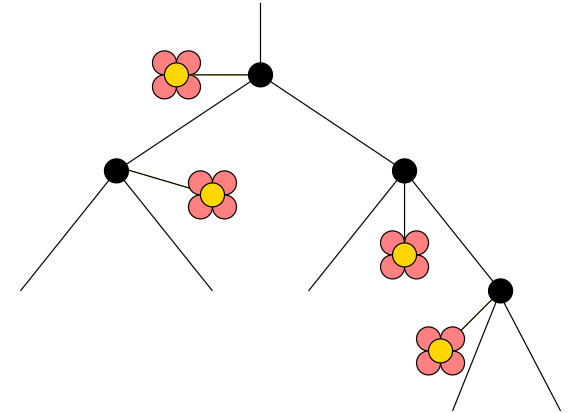
## Three approaches for counting maps

- [Tutte 60's]: recursive description of maps

⇒ functional equations for the GF

⇒ algebraic solution

- [Schaeffer 97]: bijection with trees



- [Brézin-Itzykson-Parisi-Zuber 78]: matrix integrals

$$\lim_{N \rightarrow \infty} \frac{1}{N^2} \log \int dH e^{N \text{tr} \left( -\frac{H^2}{2} + t \frac{H^3}{3} \right)}.$$

$H$ : hermitian  $N \times N$  matrix

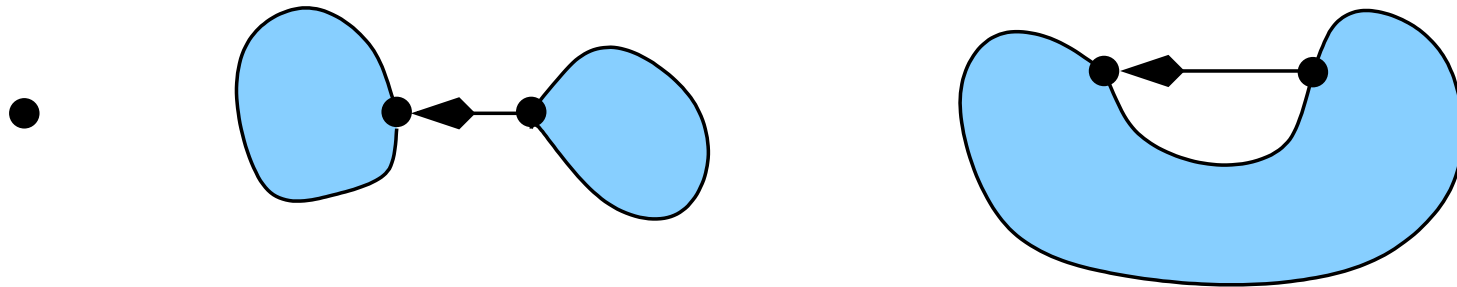


## Recursive description of planar maps: deleting the root-edge

Let

$$F(t, u) \equiv F(u) = \sum_M t^{e(M)} u^{d(M)} = \sum_{d \geq 0} F_d(t) u^d$$

where  $e(M)$  is the number of edges and  $d(M)$  the degree of the outer face.

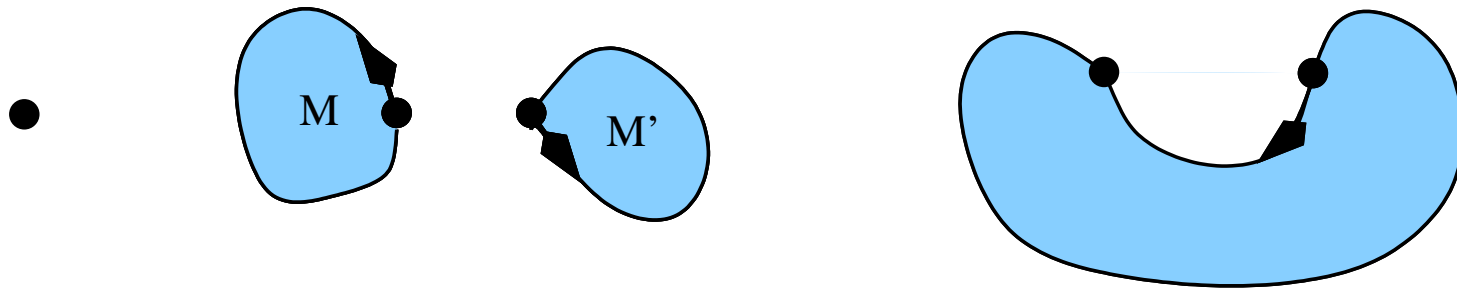


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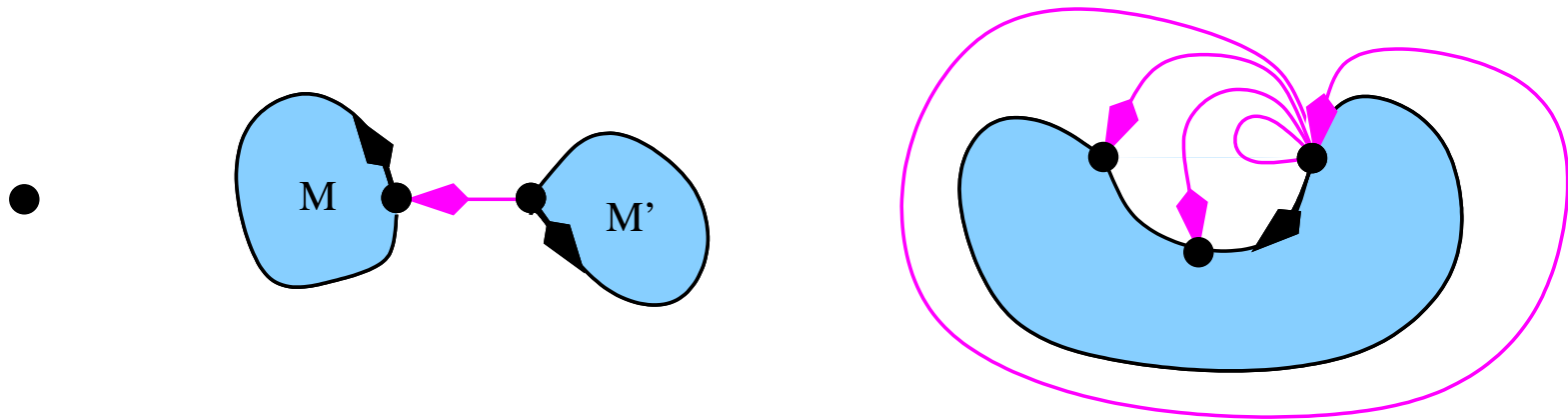


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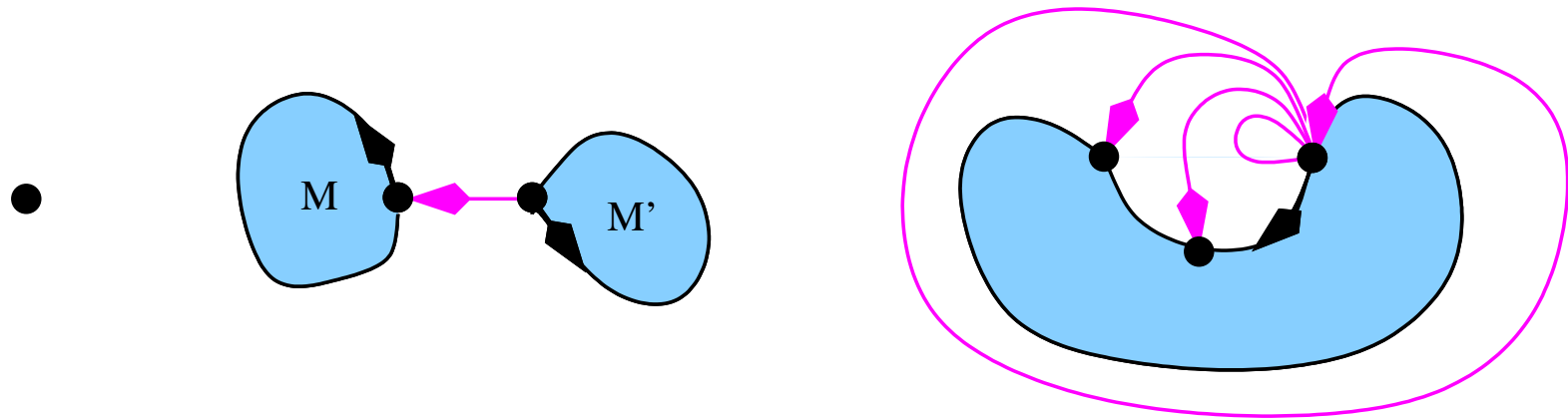


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where  $e(M)$  is the number of edges and  $d(M)$  the degree of the outer face.



$$\begin{aligned}
 F(u) &= 1 + tu^2 F(u)^2 + t \sum_{d \geq 0} F_d(t) (u^{d+1} + u^d + \dots + u) \\
 &= 1 + tu^2 F(u)^2 + tu \frac{uF(u) - F(1)}{u - 1}
 \end{aligned}$$

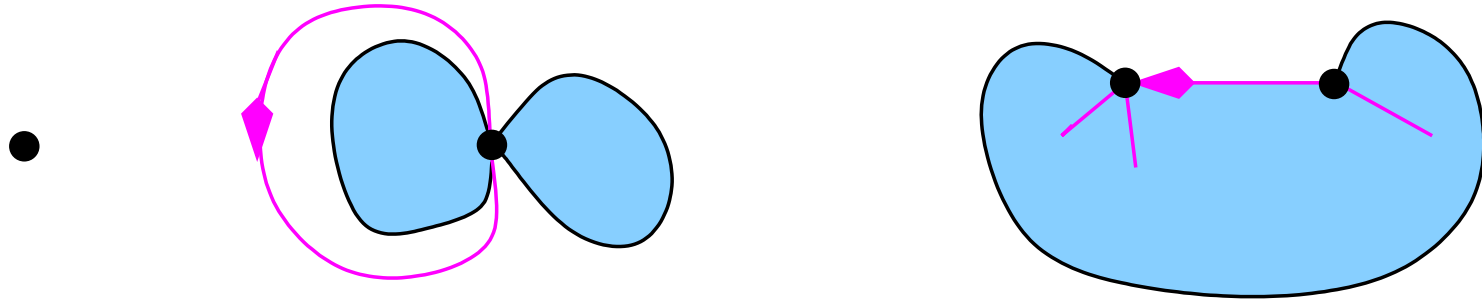
[Tutte 68]

## Recursive description of planar maps: contracting the root-edge

Let

$$F(t, v) \equiv F(v) = \sum_M t^{e(M)} v^{\delta(M)} = \sum_{d \geq 0} F_d(t) v^d$$

where  $e(M)$  is the number of edges and  $\delta(M)$  the degree of the root vertex.



$$\begin{aligned}
 F(v) &= 1 + tv^2 F(v)^2 + t \sum_{d \geq 0} F_d(t) (v^{d+1} + v^d + \dots + v) \\
 &= 1 + tv^2 F(v)^2 + tv \frac{vF(v) - F(1)}{v - 1}
 \end{aligned}$$

The same equation... (duality)

## Not so bad!

$$F(t; u) \equiv F(u) = 1 + tu^2 F(u)^2 + tu \frac{uF(u) - F(1)}{u - 1}$$

- Gives the number of  $n$ -edge maps in polynomial time
- The method is **extremely** robust (maps with prescribed degrees, triangulations, bipartite maps...)
- Question: Asymptotics of the number of  $n$ -edge maps from the equation?

## But not so good either...

$$F(t; u) \equiv F(u) = 1 + tu^2 F(u)^2 + tu \frac{uF(u) - F(1)}{u - 1} \quad (1)$$

should be compared to

$$F(t; 1) \equiv F(1) = \frac{(1 - 12t)^{3/2} - 1 + 18t}{54t^2} = \sum_{n \geq 0} \frac{2 \cdot 3^n}{n(n+1)} \binom{2n}{n} t^n$$

In particular,  $F(t; u)$  and  $F(t; 1)$  are algebraic, and  $F(t; 1)$  has a nice expansion. How does it follow from (1)?

- Guess and check [Tutte 60's]
- Brown's quadratic method [Brown 65]
- A generalization to polynomial equations with one catalytic variable [mbm-Jehanne 06]

## Polynomial equations with one catalytic variable

- **General framework:** A polynomial equation:

$$P(F(u), F_1, \dots, F_k, t, u) = 0 \quad (1)$$

where  $F(u) \equiv F(t, u)$  is a series in  $t$  with **polynomial** coefficients in  $u$ , and  $F_i \equiv F_i(t)$  does not depend on  $u$ .

- **Example**

$$F(u) = 1 + tuF(u)^3 + tu(2F(u) + F(1)) \frac{F(u) - F(1)}{u - 1} + tu \frac{F(u) - F(1) - (u - 1)F'(1)}{(u - 1)^2}$$

- **Results**

1. The solution of every **proper** equation of this type is **algebraic**.
2. A practical (but heavy) strategy allows to solve specific examples (that is, to derive from (1) an algebraic equation for  $F(u)$ , or  $F_1, \dots, F_k$ ).



## Example

$$F(t; u) \equiv F(u) = 1 + tu^2 F(u)^2 + tu \frac{uF(u) - F(1)}{u - 1}$$

↓

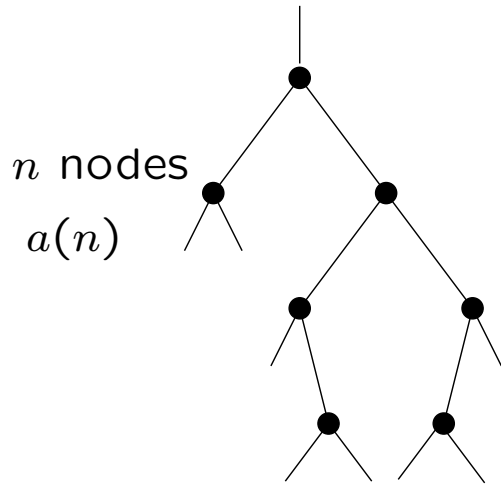
$$F(t; 1) = \frac{(1 - 12t)^{3/2} - 1 + 18t}{54t^2} = \sum_{n \geq 0} \frac{2 \cdot 3^n}{n(n+1)} \binom{2n}{n} t^n$$

Polynomial equations with one catalytic variable

[mbm-Jehanne 06]

## Interlude: A proof from The Book?

Algebraic series usually arise from tree enumeration:

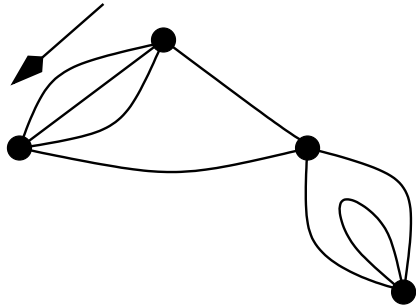


$$A(t) = 1 + tA(t)^2$$

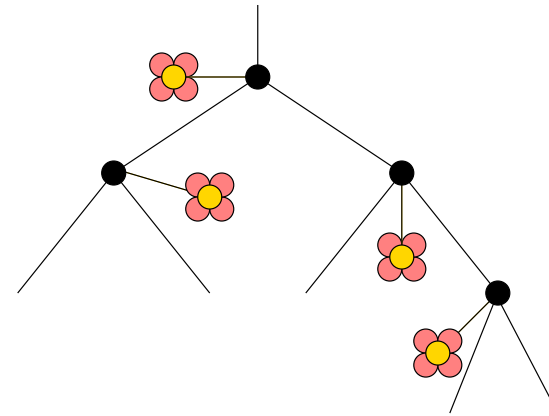
Are there correspondences between maps and trees?

## Interlude: A proof from The Book?

- Bijection between 4-valent planar maps ( $n$  vertices) and **balanced** blossoming trees ( $n$  nodes) [Schaeffer 97]



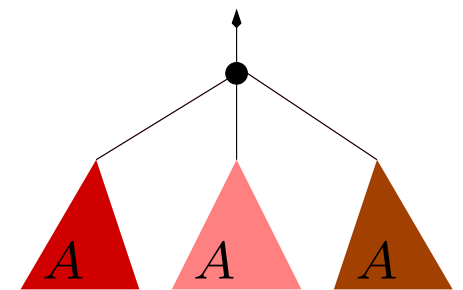
A planar map (GF  $M$ )



A blossoming tree (GF  $A = 1 + 3tA^2$ )

- An **unbalanced** blossoming tree is, in essence, a 3-tuple of blossoming subtrees [Bouttier et al. 02]:

$$\Rightarrow M = A - tA^3.$$

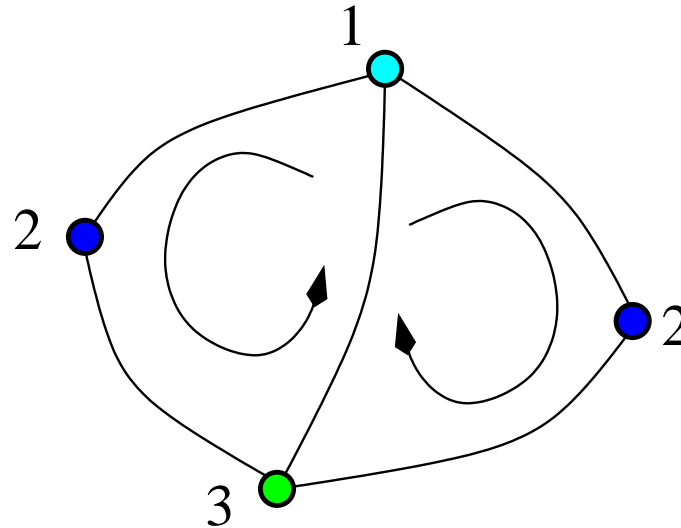


## II. "Easy" problems in coloured enumeration

(sketched)

Some coloured problems belong, in a more or less hidden way, to the world of un-coloured questions, and in particular, give rise to **algebraic** generating functions

## 3-Colourings of triangulations



A triangulation has a proper 3-colouring iff it is face-bicolourable (i.e. **Eulerian**).  
In this case it admits exactly six 3-colourings.

Counting proper 3-colourings of triangulations



Counting Eulerian triangulations

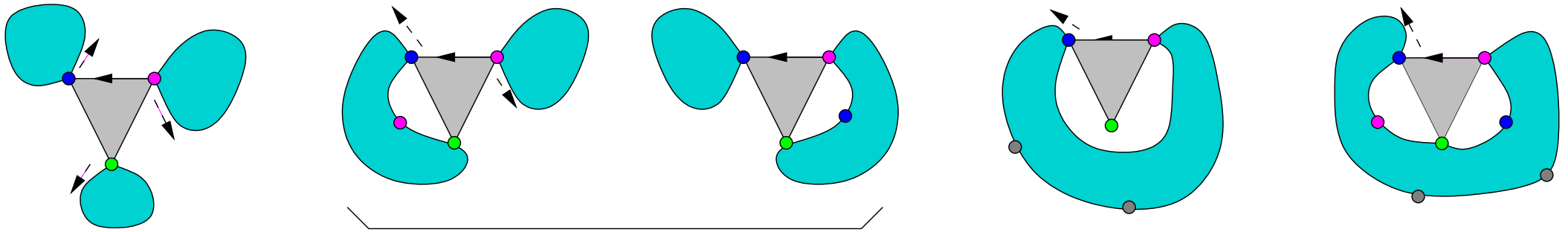
[Di Francesco-Eynard-Guitter 98], [Eynard-Kristjansen 98]

## Face-bicolored (quasi)-triangulations

Let

$$F(t, u) \equiv F(u) = \sum_M t^{|M|} u^{d(M)} = \sum_{d \geq 0} F_d(t) u^d$$

where  $|M|$  is the number of black faces and  $d(M)$  the degree of the outer face.



$$F(u) = 1 + tuF(u)^3 + 2tF(u)(F(u) - 1) + t(F(u) - 1) + t \frac{F(u) - 1 - uF_1}{u}$$

⇒ Polynomial equation with one catalytic variable

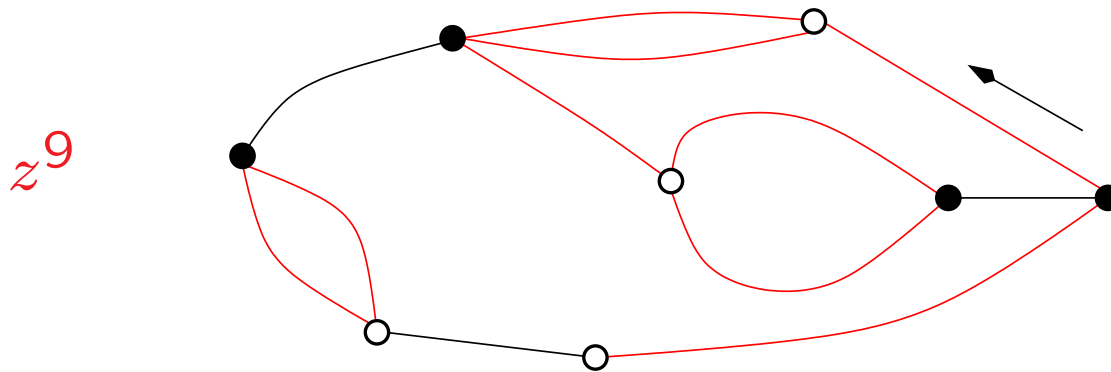
## More generally...

**Theorem:** Given two finite sets  $D_{\circ} \subset \mathbb{N}$  and  $D_{\bullet} \subset \mathbb{N}$ , the generating function of face-bicolored planar maps such that all white (resp. black) faces have their degree in  $D_{\circ}$  (resp.  $D_{\bullet}$ ) is algebraic.

[mbm-Schaeffer 02], [Bouttier et al. 05], [mbm-Jehanne 06]

**Corollary:** The Ising model on planar maps

The generating function of 2-coloured planar maps of bounded vertex-degree, counted by their size **and by the number of monochromatic edges**, is algebraic.



### III. The real thing: coloured enumeration

The enumeration of coloured planar maps is inherently harder (and richer) than the enumeration of uncoloured maps

(joint work with Olivier Bernardi)



## Forget algebraicity!

**Theorem [Tutte 73]:** For planar triangulations,

$$\sum_T \chi'_T(1) t^{v(T)} = \sum_n (-1)^n b(n) t^{n+2}$$

where

$$b(n) = \frac{2(3n)!}{n!(n+1)!(n+2)!} \sim \frac{27^n}{n^4}$$

This series is **not algebraic**.

However, it satisfies a linear differential equation.

## Catalytic variables

The  $q$ -chromatic generating function, being transcendental, cannot be described with **one** catalytic variable

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The  $q$ -chromatic generating function, being transcendental, cannot be described with **one** catalytic variable

HOWEVER

it can be described with **two** catalytic variables

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The  $q$ -chromatic generating function, being transcendental, cannot be described with **one** catalytic variable

HOWEVER

it can be described with **two** catalytic variables

WHY IS THAT SO?

- The recursive description of the chromatic (or Tutte) polynomial,

$$\chi_M(q) = \chi_{M \setminus e}(q) - \chi_{M/e}(q)$$

calls for a recursive description of maps by contraction **and** deletion of edges.

- This is possible if one keeps track of the degree of the outer face, **and** the degree of the root-vertex.

## Equations with two catalytic variables

- $q$ -Coloured (quasi-)triangulations:

$$T(t, q; u, v) \equiv T(u, v) = \sum_M \chi_M(q) t^{f(M)} u^{d(M)-2} v^{\delta(M)}$$

$$T(u, v) = q(q-1)vt + \frac{uv}{q}T(u, v)T(u, 1) + vt \frac{T(u, v) - T(0, v)}{u} - uv^2t \frac{T(u, v) - T(u, 1)}{v-1}$$

[Tutte 73]

- Planar maps weighted by their Tutte polynomial ( $\simeq$   $q$ -colourings counted by monochromatic edges)

$$M(t, x, y; u, v) \equiv M(u, v) = \sum_M T_M(x, y) t^{e(M)} u^{d(M)} v^{\delta(M)}$$

$$M(u, v) = 1 + uvt(ux - 1)M(u, v)M(u, 1) + uvt(vy - 1)M(u, v)M(1, v) \\ + uvt \frac{uM(u, v) - M(1, v)}{u-1} + uvt \frac{vM(u, v) - M(u, 1)}{v-1}$$

[Tutte 68]

## Not so bad!

$$T(u, v) = q(q-1)vt + \frac{uv}{q}T(u, v)T(u, 1) + vt \frac{T(u, v) - T(0, v)}{u} - uv^2t \frac{T(u, v) - T(u, 1)}{v-1}$$

- Gives the number of  $q$ -coloured triangulations of size  $n$  in polynomial time
- The method is **robust** [Liu 84], [Bernardi 06]
- Question: Asymptotics of the number of  $q$ -coloured triangulations of size  $n$  from the equation?

But not so good either...

$$T(u, v) = q(q-1)vt + \frac{uv}{q}T(u, v)T(u, 1) + vt \frac{T(u, v) - T(0, v)}{x} - uv^2t \frac{T(u, v) - T(u, 1)}{y-1}$$

A **polynomial** (quadratic) equation with **two** catalytic variables

- The series  $T(t; 0, 1)$ , which counts  $q$ -coloured triangulations by their size, is not algebraic. But does satisfy a linear differential equation? a non-linear one?

(yes)

## What tools do we have?

- A systematic method for solving **polynomial** equations with **one** catalytic variable [mbm-Jehanne 06]

$$F(t; u) \equiv F(u) = 1 + tu^2 F(u)^2 + tu \frac{uF(u) - F(1)}{u - 1}$$

- An [almost] systematic method for solving **linear** equations with **two** catalytic variables

[mbm-Petkovšek 03], [mbm 05], [Mishna 07 ], [Rechnitzer et al. 08]...

$$T(u, v) = v + \frac{vt(T(u, v) - T(0, v))}{u} - \frac{tuv^2(T(u, v) - T(u, 1))}{v - 1}$$

$$T(u, v) = q(q-1)vt + \frac{uv}{q}T(u, v)T(u, 1) + vt \frac{T(u, v) - T(0, v)}{x} - uv^2t \frac{T(u, v) - T(u, 1)}{y - 1}$$

???



# 1. From polynomial to linear equations: differentiation w.r.t. $q$

- The tricky non-linear terms:  $T(t, q; u, v)T(t, q; u, 1)$

- Differentiate w.r.t.  $q$ :

$$T'(t, q; u, v)T(t, q; u, 1) + T(t, q; u, v)T'(t, q; u, 1)$$

- Specialize at some trivial value of  $q$ , for instance  $q = 1$ , using:

$$T(t, 1; u, v) = 0 \text{ or } 1$$

## Example: the derivative of the chromatic polynomial, for (quasi)-triangulations

Let

$$T'(t; u, v) = \sum_T \chi'_T(\mathbf{1}) t^{v(T)} u^{d(M)-2} v^{\delta(M)}$$

Then

$$T'(u, v) = v + \frac{vt (T'(u, v) - T'(\mathbf{0}, v))}{u} - \frac{tuv^2 (T'(u, v) - T'(u, \mathbf{1}))}{v - 1}$$

A **linear** equation with **two** catalytic variables

$\Rightarrow$  [almost] systematic machinery

## Example: bipolar orientations of triangulations

**Theorem** [Tutte 73]: For planar triangulations,

$$\sum_T \chi'_T(1) t^{v(T)} = \sum_n (-1)^n b(n) t^{n+2}$$

where

$$b(n) = \frac{2(3n)!}{n!(n+1)!(n+2)!} \sim \frac{27^n}{n^4}$$

In particular, this series is not algebraic. However, it satisfies a linear differential equation

### Remarks

- More systematic proof [mbm]
- $b(n)$  counts **bipolar orientations** of triangulations [Zaslavsky-Greene 83]
- Bijective proof via Young tableaux [Bonichon-mbm-Fusy 08]

## Bipolar orientations of general planar maps

Theorem [Baxter 01]:

$$M'(t; 1, 1) = \sum_M (-1)^{v(M)} \chi'_M(1) t^{e(M)} = \sum_{n \geq 0} b(n) t^{n+1}$$

with

$$b(n) = \frac{2}{n(n+1)^2} \sum_{m=0}^{n-1} \binom{n+1}{m} \binom{n+1}{m+1} \binom{n+1}{m+2} \sim \kappa \frac{8^n}{n^4}$$

Again, this series is not algebraic. However, it satisfies a linear differential equation.

### Remarks

- Systematic proof [mbm 03]
- Bijections with Baxter permutations and lattice paths configurations [Felsner-Fusy-Noy-Orden 08], [Fusy-Poulalhon-Schaeffer 08] [Bonichon-mbm-Fusy 08] — See next talk!

## 2. From two to one catalytic variable: Tutte's invariants

For  $q$ -coloured (quasi)-triangulations:

$$T(u, v) = q(q-1)vt + \frac{uv}{q}T(u, v)T(u, \mathbf{1}) + vt \frac{T(u, v) - T(\mathbf{0}, v)}{x} - uv^2t \frac{T(u, v) - T(u, \mathbf{1})}{y-1}$$

One is interested in  $T(\mathbf{0}, \mathbf{1})$

**Theorem:** For  $q = 2 + 2 \cos \frac{2\pi}{m}$ ,  $q \neq 4$ , the series  $T(0, v) \equiv T(t; 0, v)$  satisfies a polynomial equation with one catalytic variable  $v$

The proof is long!

[Tutte 73]: Chromatic sums for rooted planar triangulations: the cases  $\lambda = 1$  and  $\lambda = 2$

[Tutte 73]: Chromatic sums for rooted planar triangulations, II : the case  $\lambda = \tau + 1$

[Tutte 73]: Chromatic sums for rooted planar triangulations, III : the case  $\lambda = 3$

[Tutte 73]: Chromatic sums for rooted planar triangulations, IV : the case  $\lambda = \infty$

[Tutte 74]: Chromatic sums for rooted planar triangulations, V : special equations

[Tutte 78]: On a pair of functional equations of combinatorial interest

[Tutte 82]: Chromatic solutions

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+ [mbm-Jehanne 06] on such equations

↪ For these values of  $q$ , the chromatic generating function of triangulations is algebraic.

**Examples:**  $q = 0, 1, 2, 3, 2 + \sqrt{2}, 2 + \sqrt{3} \dots$

[Bernardi-mbm]

## What about general planar maps?

For  $q$ -coloured planar maps:

$$M(u, v) = 1 + uvt(u(q-1) + 1)M(u, v)M(u, \mathbf{1}) + \frac{uvt(uM(u, v) - M(\mathbf{1}, v))}{u-1} \\ - uvtM(u, v)M(\mathbf{1}, v) - \frac{uvt(vM(u, v) - M(u, \mathbf{1}))}{v-1}$$

One is interested in  $M(\mathbf{1}, \mathbf{1})$

**Theorem:** For  $q = 2 + 2 \cos \frac{p\pi}{m}$ ,  $q \neq 4$ , the series  $M(\mathbf{1}, v) \equiv M(t; \mathbf{1}, v)$  satisfies a polynomial equation with one catalytic variable  $v$

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↪ For these values of  $q$ , the chromatic generating function of planar maps is algebraic.

**Examples:**  $q = 0, 1, 2, 3, 2 + \sqrt{2}, 2 + \sqrt{3} \dots$

[Bernardi-mbm]

## Example: 3-coloured planar maps are “algebraic”

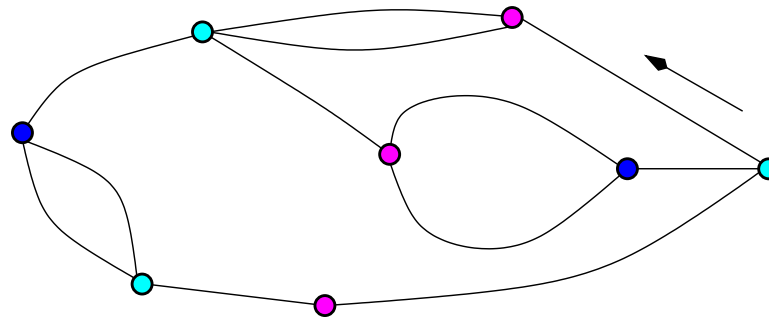
**Theorem:** the generating function of 3-coloured planar maps is

$$M(t; 1, 1) = \frac{(1 + 2A)(1 - 2A^2 - 4A^3 - 4A^4)}{(1 - 2A^3)^2}$$

where  $A \equiv A(t)$  is the quartic series in  $t$  defined by

$$A = t \frac{(1 + 2A)^3}{(1 - 2A^3)}$$

↪ **Asymptotics:** A random loopless planar map with  $n$  edges has approximately  $(1.42\dots)^n$  3-colourings  
[Bernardi-mbm]





### 3. Going further: a differential equation for coloured triangulations

**Theorem [Tutte 84]:** The generating function of  $q$ -coloured planar triangulations is **differentially algebraic**:

$$2q^2(1 - q)t + (qt + 10H - 6tH')H'' + q(4 - q)(20H - 18tH' + 9t^2H'') = 0$$

with

$$H(t) = t^2T(\sqrt{t}; 0, 1)/q.$$

**Idea:** from the polynomial equation with one catalytic variable satisfied by  $T(t; 0, v)$ , work out a differential equation instead of an algebraic one.

[Tutte 73]: Chromatic sums for rooted planar triangulations: the cases  $\lambda = 1$  and  $\lambda = 2$

[Tutte 73]: Chromatic sums for rooted planar triangulations, II : the case  $\lambda = \tau + 1$

[Tutte 73]: Chromatic sums for rooted planar triangulations, III : the case  $\lambda = 3$

[Tutte 73]: Chromatic sums for rooted planar triangulations, IV : the case  $\lambda = \infty$

[Tutte 74]: Chromatic sums for rooted planar triangulations, V : special equations

[Tutte 78]: On a pair of functional equations of combinatorial interest

[Tutte 82]: Chromatic solutions

[Tutte 82]: **Chromatic solutions II**

[Tutte 84]: **Map-colourings and differential equations**

## What about general planar maps?

**Theorem:** The generating function of  $q$ -coloured planar maps is **differentially algebraic**.

↪ A **system** of 6 differential equations...

For  $q = 4$ , a single differential equation, of order 3, and degree 11...

**Idea:** from the polynomial equation with one catalytic variable satisfied by  $M(t; 1, v)$ , work out a differential equation instead of an algebraic one.

[Bernardi-mbm]

## More questions

### A. Equations with two catalytic variables

- A simpler differential equation for  $q$ -coloured planar maps?
- Extend the “chromatic” results to the Tutte polynomial
- A better understanding of [Eynard-Bonnet 99]...

### B. More combinatorics

- Understand the algebraicity of 3-coloured planar maps:

$$M(t; 1, 1) = \frac{(1 + 2A)(1 - 2A^2 - 4A^3 - 4A^4)}{(1 - 2A^3)^2} \quad \text{with} \quad A = t \frac{(1 + 2A)^3}{(1 - 2A^3)}$$

- Understand the combinatorics of Tutte’s differential equation for  $q$ -coloured triangulations:

$$2q^2(1 - q)t + (qt + 10H - 6tH')H'' + q(4 - q)(20H - 18tH' + 9t^2H'') = 0$$

### C. Asymptotic analysis

- Asymptotic number of  $q$ -coloured triangulations?

(done for  $q \in (28/11, 4) \cup [5, \infty)$  [Odlyzko-Richmond 83])