Compter des cartes planaires (colorées)

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I. Trees and maps
Plane trees
Plane trees
Plane trees
Plane trees: rooted version

Mark a corner
Plane trees: rooted version

Mark a corner
Plane trees: rooted version

Mark a corner
Plane trees: rooted version
Plane maps
Plane maps

degree 3
Plane maps

degree 3
Plane maps: rooted version

degree 3

5

2

3
Plane maps: rooted version

degree 3

root edge

root vertex

root face (outer degree 6)
Coloured plane maps

Proper colouring:

Non-proper colouring:

Monochromatic edge
Who studies plane maps?
Who studies plane maps?

- combinatorialists (enumeration, graph theory)
- probabilists (large random maps; matrix integrals)
- theoretical physicists (quantum gravity)
- algebraists (factorizations in classical groups)
- computational geometers, graph drawers
Who studies plane maps?

- combinatorialists (enumeration, graph theory)
- probabilists (large random maps; matrix integrals)
- theoretical physicists (quantum gravity) Potts model
- algebraists (factorizations in classical groups)
- computational geometers, graph drawers
II. Recursive descriptions and enumeration
A recursive description of trees

Delete the root edge
A recursive description of trees

Delete the root edge

⇒ An ordered pair of trees
A recursive description of trees

Delete the root edge

\[
\Rightarrow \text{An ordered pair of trees}
\]
A recursive description of trees

Delete the root edge

⇒ An ordered pair of trees

Enumeration: let $a(n)$ be the number of rooted plane trees with $n$ edges. Then $a(0) = 1$ and

$$a(n) = \sum_{i+j=n-1} a(i)a(j)$$
A recursive description of trees

Delete the root edge

⇒ An ordered pair of trees

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\[
a(n) = \sum_{i+j=n-1} a(i)a(j)
\]

Generating function: the associated formal power series

\[
A := \sum_{n \geq 0} a(n) t^n = \sum_{T \text{ tree}} t^{e(T)}
\]
A recursive description of trees

Delete the root edge

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$$A := \sum_{n \geq 0} a(n)t^n = \sum_{T \text{ tree}} t^{e(T)}$$

**Functional equation:**

$$A = 1 + tA^2$$
A recursive description of maps: delete the root edge

Let $b(n)$ be the number of rooted plane maps with $n$ edges. Then $b(0) = 1$ and $b(n; e)$ is the number of maps with $n$ edges and outer degree $e$. The diagram illustrates a recursive description of maps.
A recursive description of maps: delete the root edge

Enumeration:

Let $b(n)$ be the number of rooted plane maps with $n$ edges. Then $b(0) = 1$ and $b(n; e)$ is the number of maps with $n$ edges and outer degree $e$. 

[Diagram of a recursive description of maps with arrows pointing to labeled nodes and edges connecting them.]
A recursive description of maps: delete the root edge

Let $b(n)$ be the number of rooted plane maps with $n$ edges. Then $b(0) = 1$ and $b(n; e)$ is the number of maps with $n$ edges and outer degree $e$. 

\[ 
\begin{align*}
\text{Diagram:} & \\
\text{Recursive step:} & \\
\end{align*} \]
A recursive description of maps: delete the root edge

\[ b(n) = b(n-1; d+1) \]

where \( b(n; e) \) is the number of maps with \( n \) edges and outer degree \( e \).
A recursive description of maps: delete the root edge

Enumeration: let \( b(n) \) be the number of rooted plane maps with \( n \) edges. Then \( b(0) = 1 \) and

\[
b(n) = \sum_{i+j=n-1} b(i)b(j) + \sum_{d \geq 0} b(n-1; d)(d+1)
\]

where \( b(n; e) \) is the number of maps with \( n \) edges and outer degree \( e \).
A recursive description of maps: delete the root edge

Enumeration: let $b(n)$ be the number of rooted plane maps with $n$ edges. Then $b(0) = 1$ and

$$b(n; e) = \sum_{\substack{i+j=n-1 \\ c+d=e-2}} b(i; c)b(j; d) + \sum_{d \geq e-1} b(n-1; d)$$

where $b(n; e)$ is the number of maps with $n$ edges and outer degree $e$. 
**Enumeration:** let $b(n; d)$ be the number of rooted plane maps with $n$ edges and outer degree $d$. Then $b(0; e) = \delta_{e,0}$ and

$$b(n; e) = \sum_{i+j=n-1, c+d=e-2} b(i; c)b(j; d) + \sum_{d\geq e-1} b(n - 1; d)$$
Generating function of plane maps

**Enumeration:** let $b(n; d)$ be the number of rooted plane maps with $n$ edges and outer degree $d$. Then $b(0; e) = \delta_{e,0}$ and

$$b(n; e) = \sum_{i+j=n-1 \atop c+d=e-2} b(i; c)b(j; d) + \sum_{d \geq e-1} b(n - 1; d)$$

**Generating function:** the associated bivariate formal power series

$$B(x) := \sum_{n,d \geq 0} b(n; d)t^n x^d = \sum_{M \text{ map}} t^{e(M)} x^{df(M)}$$

Note: $B(1) = \sum_n b(n)t^n$ is the GF we want to compute
Generating function of plane maps

**Enumeration:** let \( b(n; d) \) be the number of rooted plane maps with \( n \) edges and outer degree \( d \). Then \( b(0; e) = \delta_{e,0} \) and

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**Generating function:** the associated bivariate formal power series

\[
B(x) := \sum_{n,d \geq 0} b(n; d)t^nx^d = \sum_{\text{M map}} t^{e(M)} x^{df(M)}
\]

**Note:** \( B(1) = \sum_n b(n)t^n \) is the GF we want to compute

**Functional equation:**

\[
B(x) = 1 + tx^2B(x)^2 + tx \frac{xB(x) - B(1)}{x - 1}
\]

A discrete differential equation

[Tutte 68]
Another description of maps: contract the root edge

Enumeration: let $\tilde{b}(n; e)$ be the number of rooted plane maps with $n$ edges and root degree $e$.

Then $\tilde{b}(0; e) = \delta_{e,0}$ and $\tilde{b}(n; e) = \sum_{i + j = n - 1} c^+ d = e - 2 \tilde{b}(i; c) \tilde{b}(j; d) + \sum_{d \geq e - 1} \tilde{b}(n - 1; d)$

Same recurrence relation

Functional equation:

$$\tilde{B}(y) = 1 + ty^2 \tilde{B}(y) + tyy \tilde{B}(y) - \tilde{B}(1)y - 1$$

where $\tilde{B}(y) := \sum_{n \geq 0} a(n; d) t^n y^d = \sum_{M \text{ map}} t^e(M) y^d(M)$
Another description of maps: contract the root edge

\[ d + 1 \text{ maps} \]

\[ \text{degree of root vertex} = d \]
Another description of maps: contract the root edge

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Another description of maps: contract the root edge

Enumeration: let \( \tilde{b}(n; e) \) be the number of rooted plane maps with \( n \) edges and root degree \( e \). Then \( \tilde{b}(0; e) = \delta_{e,0} \) and

\[
\tilde{b}(n; e) = \sum_{i+j=n-1 \atop c+d=e-2} \tilde{b}(i; c)\tilde{b}(j; d) + \sum_{d \geq e-1} \tilde{b}(n-1; d)
\]

Same recurrence relation
A recursive description of coloured maps: delete/contract the root edge

- Deletion/contraction relation for the chromatic polynomial: the number $\chi_M(q)$ of proper $q$-colourings of the map $M$ satisfies:

$$\chi_M(q) = \chi_{M\setminus e}(q) - \chi_{M\leftrightarrow e}(q)$$
A recursive description of coloured maps: delete/contract the root edge

- Deletion/contraction relation for the chromatic polynomial: the number \( \chi_M(q) \) of proper \( q \)-colourings of the map \( M \) satisfies:

\[
\chi_M(q) = \chi_{M\setminus e}(q) - \chi_{M\leftrightarrow e}(q)
\]

- More generally, if \( P_M(q,\nu) \) counts all \( q \)-colourings of \( M \) with a weight \( \nu \) per monochromatic edge:

\[
P_M(q,\nu) = P_{M\setminus e}(q,\nu) + (\nu - 1) P_{M\leftrightarrow e}(q,\nu)
\]

⇒ Use this to write a recurrence relation (then a functional equation) to count coloured maps, with two additional parameters.

Remark. The polynomial \( P_M(q,\nu) \) is equivalent to the Tutte polynomial of \( M \).
A functional equation for coloured maps

- Let

\[ C(x, y) = \frac{1}{q} \sum_{M} P_{M}(q, \nu) t^{e(M)} x^{dv(M)} y^{df(M)} \]

\[ = \frac{1}{q} \sum_{M \text{ q–coloured}} t^{e(M)} x^{dv(M)} y^{df(M)} \nu^{m(M)} \]

where \( dv(M) \) (resp. \( df(M) \)) is the degree of the root-vertex (resp. root-face) and \( m(M) \) the number of monochromatic edges.
A functional equation for coloured maps

Let

\[ C(x, y) = \frac{1}{q} \sum_{M} P_M(q, \nu) t^{e(M)} x^{dv(M)} y^{df(M)} \]

where \( dv(M) \) (resp. \( df(M) \)) is the degree of the root-vertex (resp. root-face) and \( m(M) \) the number of monochromatic edges.

**Proposition [Tutte 68]**

\[
C(x, y) = 1 + xyt ((\nu - 1)(y - 1) + qy) C(x, y) C(1, y) \\
+ xyt(x\nu - 1) C(x, y) C(x, 1) \\
+ xyt(\nu - 1) \frac{x C(x, y) - C(1, y)}{x - 1} + xyt \frac{y C(x, y) - C(x, 1)}{y - 1}
\]

A discrete partial differential equation (in two variables)

This equation has been sleeping for 40 years
Our bestiary

- Plane trees
  \[ A = 1 + tA^2 \]

- Plane maps
  \[ B(x) = 1 + tx^2 B(x)^2 + tx \frac{x B(x) - B(1)}{x - 1} \]

- Coloured plane maps
  \[ C(x, y) = 1 + xyt ((\nu - 1)(y - 1) + qy) C(x, y) C(1, y) + xyt(x\nu - 1) C(x, y) C(x, 1) + xyt(\nu - 1) \frac{x C(x, y) - C(1, y)}{x - 1} + xyt y C(x, y) - C(x, 1) \]
Recursive constructions are robust

- Plane triangulations (every finite face has degree 3):
  \[ B(x) = tx(1 + xB(x))^2 + t \frac{B(x) - B(0)}{x} \]

- Properly \(q\)-coloured triangulations [Tutte 73]:
  \[ T(x, y) = x(q - 1) + xyz T(x, y) T(1, y) + xz \frac{T(x, y) - T(x, 0)}{y} - x^2yz \frac{T(x, y) - T(1, y)}{x - 1} \]
II. Guessing

Recurrences produce coefficients
Given the first terms of a series \((a(0), a(1), a(2), \ldots)\), Gfun can guess:

- linear rec. relations with polynomial coefficients for \(a(n)\)

\[
p_\ell(n)a(n + \ell) + \cdots + p_0(n)a(n) = 0
\]

- equivalently, linear diff. equations with polynomial coefficients for the GF \(A(t)\)

\[
P_d(t)A^{(d)}(t) + \cdots + P_0(t)A(t) = 0
\]

- also polynomial equations with polynomial coefficients for \(A(t)\)

\[
P_d(t)A(t)^d + \cdots + P_0(t) = 0
\]

Gfun: [Salvy & Zimmermann 94]
Guessing with Maple and Gfun

Given the first terms of a series \((a(0), a(1), a(2), \ldots)\), Gfun can guess:

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Gfun: [Salvy & Zimmermann 94]
Guessing the number/generating function of plane maps

```maple
> b:=proc(n,e) option remember:
if n=0 and e=0 then 1 elif n=0 or e=0 then 0
else add(add(b(i,c)*b(n-1-i,e-2-c),i=0..n-1)
,c=0..e-2) + add(b(n-1,d),d=e-1..2*n):
fi: end:

> liste:=[seq(b(n,1),n=1..10)];
    liste := [1, 2, 9, 54, 378, 2916, 24057, 208494, 1876446, 17399772]

> with(gfun):
> listtorec(liste,bb(n));

[\{(-12 \, n - 6) \, bb(n) + (n + 3) \, bb(n + 1), \, bb(0) = 1\}]

> listtodiffeq(liste,B(t));

\[
\begin{aligned}
\{6 B(t)+(30 \, t-3) \left(\frac{d}{dt} B(t)\right)+(12 \, t^2-t) \left(\frac{d^2}{dt^2} B(t)\right)\},
B(0) = 1,
\end{aligned}
\]

> listtoalgeq(liste,B(t));

\[
[16 \, t - 1 + (-18 \, t + 1) \, B(t) + 27 \, t^2 \, B(t)^2, \text{ogf}]
\]
```
Is it true that

\[ B(x) = 1 + tx^2 B(x)^2 + tx \frac{x B(x) - B(1)}{x - 1} \]

implies

\[ 16t - 1 + (1 - 18t)B(1) + 27t^2 B(1)^2 = 0 \]

that is,

\[ B(1) = \frac{(1 - 12t)^{3/2} - 1 + 18t}{54t^2} = \sum_{n \geq 0} 2 \cdot 3^n \frac{(2n)!}{n!(n + 2)!} t^n \]
Oh?

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• Cf. for plane trees:

\[ A = 1 + tA^2 \quad \text{and} \quad A = \frac{1 - (1 - 4t)^{1/2}}{2t} = \sum_{n \geq 0} \frac{(2n)!}{n!(n + 1)!} t^n \]
Guessing with Maple and Gfun

Given the first terms of a series \((a(0), a(1), a(2), \ldots)\), Gfun can guess:

- linear rec. relations with polynomial coefficients for \(a(n)\)
  \[
p_\ell(n)a(n + \ell) + \cdots + p_0(n)a(n) = 0
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  \[
P_d(t)A^{(d)}(t) + \cdots + P_0(t)A(t) = 0
  \]
- also polynomial equations with polynomial coefficients for \(A(t)\)
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- also polynomial equations with polynomial coefficients for \(A(t)\)
  \[
P_d(t)A(t)^d + \cdots + P_0(t) = 0
  \]

More generally

- linear relations with polynomial coefficients between given series \(A_0(t), \ldots, A_d(t)\)
  \[
P_d(t)A_d(t) + \cdots + P_0(t)A_0(t) = 0
  \]

(Important) example: non-linear differential equations
Guessing the number of properly \( q \)-coloured triangulations

- Triangulations with \( n \) vertices and outer-degree 2:

\[
(c(n))_{n \geq 2} = q - 1, (q - 1)(q - 2), (4q - 9)(q - 1)(q - 2), 3(q - 1)(q - 2)(8q^2 - 37q + 43), (176q^3 - 1245q^2 + 2951q - 2344)(q - 1)(q - 2) \ldots
\]

\[
T(x, y) = x(q - 1) + xyz T(x, y) T(1, y) + xz \frac{T(x, y) - T(x, 0)}{y} - x^2 yz \frac{T(x, y) - T(1, y)}{x - 1}
\]

\[
\text{Differential equation:} \\
2(1 - q) t + (t + 10C - 6tC') C'' + (4 - q)(20C - 18tC' + 9t^2 C'') = 0
\]

\[
\text{Recurrence relation:} \\
(n + 1)(n + 2)c(n + 2) = (q - 4)(3n - 1)(3n - 2)c(n + 1) + 2n \sum_{i=1}^{n} i(i + 1)(3n - 3i + 1)c(i + 1)c(n + 2 - i)
\]

with \( c(2) = q - 1 \).

[Tutte 84] — A combinatorial mystery
Guessing the number of properly $q$-coloured triangulations

- Triangulations with $n$ vertices and outer-degree 2:

  $$(c(n))_{n \geq 2} = q - 1, (q - 1)(q - 2), (4q - 9)(q - 1)(q - 2),$$

  $$3(q - 1)(q - 2)(8q^2 - 37q + 43), (176q^3 - 1245q^2 + 2951q - 2344)(q - 1)(q - 2) \ldots$$

- Differential equation:

  $$2(1 - q)t + (t + 10C - 6tC')C'' + (4 - q)(20C - 18tC' + 9t^2C'') = 0$$

- Recurrence relation:

  $$(n + 1)(n + 2)c(n + 2) = (q - 4)(3n - 1)(3n - 2)c(n + 1)$$

  $$+ 2 \sum_{i=1}^{n} i(i + 1)(3n - 3i + 1)c(i + 1)c(n + 2 - i),$$

  with $c(2) = q - 1$.

[Tutte 84] — A combinatorial mystery
III. Solving discrete differential equations?

\[ B(x) = 1 + tx^2 B(x)^2 + tx \frac{xB(x) - B(1)}{x - 1} \]
Our bestiary

- **Plane trees**: no derivative (polynomial equation)
  \[
  A = 1 + tA^2
  \]

- **Plane maps**: discrete ordinary diff. eq.
  \[
  B(x) = 1 + tx^2 B(x)^2 + tx \frac{xB(x) - B(1)}{x - 1}
  \]

- **Coloured plane maps**: discrete partial diff. eq.
  \[
  C(x, y) = 1 + xyt ((\nu - 1)(y - 1) + qy) C(x, y)C(1, y) \\
  +xyt(x\nu - 1) C(x, y)C(x, 1) \\
  +xyt(\nu - 1) \frac{xC(x, y) - C(1, y)}{x - 1} + xyt \frac{yC(x, y) - C(x, 1)}{y - 1}
  \]
A hierarchy of formal power series

- Rational series: \( R(t) = \frac{P(t)}{Q(t)} \)
- Algebraic series: \( P(t, A(t)) = 0 \)
- Differentially finite series (D-finite): \( \sum_{i=0}^\infty P_i(t) A(i)(t) = 0 \)
- D-algebraic series: \( P(t, A(t), A'(t), \ldots, A^{(d)}(t)) = 0 \)
A hierarchy of formal power series

- Rational series
  \[ A(t) = \frac{P(t)}{Q(t)} \]

- Algebraic series
  \[ P(t, A(t)) = 0 \]

- Differentially finite series (D-finite)
  \[ \sum_{i=0}^{d} P_i(t) A^{(i)}(t) = 0 \]

- D-algebraic series
  \[ P(t, A(t), A'(t), \ldots, A^{(d)}(t)) = 0 \]
Our bestiary

- **Plane trees**: no derivative (polynomial equation)
  \[ A = 1 + xA^2 \]

- **Plane maps**: discrete ordinary diff. eq.
  \[ B(x) = 1 + tx^2 B(x)^2 + tx \frac{xB(x) - B(1)}{x - 1} \]

- **Coloured plane maps**: discrete partial diff. eq.
  \[
  C(x, y) = 1 + xyt \left((\nu - 1)(y - 1) + qy\right) C(x, y)C(1, y) \\
  + xyt(x\nu - 1)C(x, y)C(x, 1) \\
  + xyt(\nu - 1)\frac{xC(x, y) - C(1, y)}{x - 1} + xyt\frac{yC(x, y) - C(x, 1)}{y - 1}
  \]
Theorem [mbm-Jehanne 2006] (after [Knuth 72] and [Brown 65])

Every series $A(t; x) ≡ A(x)$ solution of a (proper) discrete ordinary differential equation of any degree and order is algebraic:

$$P(t, x, A(t; x)) = 0$$

for some polynomial $P$. 
Discrete ordinary differential equations are simple

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- Plane maps

  $$B(x) = 1 + tx^2 B(x)^2 + tx \frac{xB(x) - B(1)}{x - 1}$$
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- **Plane maps**

\[
B(x) = 1 + tx^2B(x)^2 + tx \frac{xB(x) - B(1)}{x - 1}
\]

\[
\Rightarrow 27t^2B(1)^2 + (1 - 18t)B(1) + 16t - 1 = 0
\]
Discrete **ordinary** differential equations are simple

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\[
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\]

for some polynomial \( P \).

\- Plane 3-constellations (degree 3, order 2)

\[
A(x) = 1 + txA(x)^3 \\
+ tx \frac{(A(x) - A(1))(2A(x) + A(1))}{x - 1} \\
+ \frac{tx (A(x) - A(1) - (x - 1)A'(1))}{(x - 1)^2}.
\]

\[\Rightarrow A(1) = 1 - 47t + 3t^2 + 3t(22 - 9t)A(1) + 9t(9t - 2)A(1)^2 - 81t^2A(1)^3\]
Discrete **partial** differential equations are not that easy!

Even in the linear case

- **Algebraic**
  \[ A(x, y) = 1 + txyA + t\Delta_x A + t\Delta_y A \]

- **D-finite but not algebraic**
  \[ A(x, y) = 1 + t(x + y)A + t\Delta_x A + t\Delta_y A \]

- **Not D-finite**
  \[ A(x, y) = 1 + tx(1 + y)A + t\Delta_x A + t\Delta_y A \]

\[ \Delta_x A = \frac{A(x) - A(0)}{x} \]
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  \[
  A(x, y) = 1 + txyA + t\Delta_x A + t\Delta_y A
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  A(x, y) = 1 + t(x + y)A + t\Delta_x A + t\Delta_y A
  \]

- **Not D-finite**
  \[
  A(x, y) = 1 + tx(1 + y)A + t\Delta_x A + t\Delta_y A
  \]

Now a complete classification: [mbm-Mishna 10], [Bostan-Kauers 10], [Kurkova-Raschel 12], [Mishna-Rechnitzer 07], [Melczer-Mishna 13], [Bostan-Raschel-Salvy 14]
What about coloured maps?

A non-linear discrete differential equation in two variables:

\[
C(x, y) = 1 + xyt ((\nu - 1)(y - 1) + qy) C(x, y) C(1, y) \\
+ xyt(x\nu - 1) C(x, y) C(x, 1) \\
+ xyt(\nu - 1) \frac{x C(x, y) - C(1, y)}{x - 1} + xyt \frac{y C(x, y) - C(x, 1)}{y - 1}.
\]

... sleeping since 1968 [Tutte]
For the GF $T(x, y)$ of properly $q$-coloured triangulations:

$$T(x, y) = x(q - 1) + xyz T(x, y) T(1, y)$$

$$+ xz \frac{T(x, y) - T(x, 0)}{y} - x^2 yz \frac{T(x, y) - T(1, y)}{x - 1}$$
In the footsteps of W. Tutte

- For the GF $T(x, y)$ of properly $q$-coloured triangulations:

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[Tutte 73] Chromatic sums for rooted planar triangulations: the cases $\lambda = 1$ and $\lambda = 2$

[Tutte 73] Chromatic sums for rooted planar triangulations, II: the case $\lambda = \tau + 1$

[Tutte 73] Chromatic sums for rooted planar triangulations, III: the case $\lambda = 3$

[Tutte 73] Chromatic sums for rooted planar triangulations, IV: the case $\lambda = \infty$

[Tutte 74] Chromatic sums for rooted planar triangulations, V: special equations

[Tutte 78] On a pair of functional equations of combinatorial interest

[Tutte 82] Chromatic solutions

[Tutte 82] Chromatic solutions II

[Tutte 84] Map-colourings and differential equations

[Tutte 95]: Chromatic sums revisited
In the footsteps of W. Tutte

• For the GF $T(x, y)$ of properly $q$-coloured triangulations:

$$T(x, y) = x(q - 1) + xyz T(x, y) T(1, y)$$

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**Theorem [Tutte]**

• For $q = 2 + 2 \cos \frac{2\pi}{m}$, $q \neq 0, 4$, the series $T(1, y)$ satisfies a discrete differential equation in one variable $y$. 
In the footsteps of W. Tutte

- For the GF $T(x, y)$ of properly $q$-coloured triangulations:

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T(x, y) = x(q - 1) + xyz T(x, y) T(1, y) \\
+ xz \frac{T(x, y) - T(x, 0)}{y} - x^2 yz \frac{T(x, y) - T(1, y)}{x - 1}
\]

**Theorem [Tutte]**

- For $q = 2 + 2 \cos \frac{2\pi}{m}$, $q \neq 0, 4$, the series $T(1, y)$ satisfies a discrete differential equation in one variable $y$.
- When $q$ is generic, the generating function of properly $q$-coloured planar triangulations is differentially algebraic in $t$:

\[
2(1 - q)t + (t + 10H - 6tH')H'' + (4 - q)(20H - 18tH' + 9t^2 H'') = 0
\]

with $H(t) = t^2 T(1, 0)$ and $t = \sqrt{z}$. 

Our results

Let $C(x, y)$ be the generating function of $q$-coloured planar maps:

$$C(x, y) = \frac{1}{q} \sum_{M \text{ q–coloured}} t^{e(M)} x^{dv(M)} y^{df(M)} \nu^m(M),$$

where $dv(M)$ (resp. $df(M)$) is the degree of the root-vertex (resp. root-face) and $m(M)$ the number of monochromatic edges.

Theorem [Bernardi-mbm]

For $q = 2 + 2 \cos \frac{j \pi}{m}$, $q \neq 0, 4$, the series $C(1, y) \equiv C(y)$ satisfies a discrete ordinary differential equation in variable $y$.

[Bernardi-mbm 14? 15?] Counting coloured planar maps: differential equations
Our results

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**Theorem** [Bernardi-mbm]

- For $q = 2 + 2 \cos \frac{j \pi}{m}$, $q \neq 0, 4$, the series $C(1, y) \equiv C(y)$ satisfies a discrete ordinary differential equation in variable $y$, and the generating function $C(x, y)$ is algebraic.

[Bernardi-mbm 14? 15?] Counting coloured planar maps: differential equations
Our results

- Let $C(x, y)$ be the generating function of $q$-coloured planar maps:

$$C(x, y) = \frac{1}{q} \sum_{M \text{ q–coloured}} t^{e(M)} x^{d^v(M)} y^{d_f(M)} \nu^m(M),$$

where $d^v(M)$ (resp. $d_f(M)$) is the degree of the root-vertex (resp. root-face) and $m(M)$ the number of monochromatic edges.

**Theorem** [Bernardi-mbm]

- For $q = 2 + 2 \cos \frac{j \pi}{m}$, $q \neq 0, 4$, the series $C(1, y) \equiv C(y)$ satisfies a discrete ordinary differential equation in variable $y$, and the generating function $C(x, y)$ is algebraic.
- When $q$ is generic, $C(1, 1)$ is differentially algebraic in $t$: (an explicit system of differential equations)

[Bernardi-mbm 14? 15?] Counting coloured planar maps: differential equations
Let $A$ be the series in $t$, with polynomial coefficients in $\nu$, defined by

$$A = t \frac{(1 + 3\nu A - 3\nu A^2 - \nu^2 A^3)^2}{1 - 2A + 2\nu^2 A^3 - \nu^2 A^4}.$$ 

Then the generating function of bicoloured planar maps is

$$C(2, \nu, t; 1, 1) = \frac{1 + 3\nu A - 3\nu A^2 - \nu^2 A^3}{(1 - 2A + 2\nu^2 A^3 - \nu^2 A^4)^2} P(\nu, A)$$

where

$$P(\nu, A) = \nu^3 A^6 + 2\nu^2 (1 - \nu) A^5 + \nu(1 - 6\nu) A^4 - \nu(1 - 5\nu) A^3 + (1 + 2\nu) A^2 - (3 + \nu) A + 1.$$ 

$\rightsquigarrow$ Asymptotics: Phase transition at $\nu_c = \frac{3 + \sqrt{5}}{2}$, critical exponents...
Fix $n$ (large) and choose a bi-coloured planar map $M$ with $n$ edges at random, with probability proportional to

$$\nu^m(M).$$

When $\nu = 0$, the colouring is proper.
When $\nu$ is small, we favour maps with few monochromatic edges.
When $\nu$ is large, we favour maps with many monochromatic edges.
$q = 3, \nu = 0$: Properly 3-coloured planar maps

Let $A$ be the quartic series in $t$ defined by

$$A = t \frac{(1 + 2A)^3}{1 - 2A^3}.$$

Then the generating function of properly 3-coloured planar maps is

$$C(3, 0, t; 1, 1) = \frac{(1 + 2A)(1 - 2A^2 - 4A^3 - 4A^4)}{(1 - 2A^3)^2}.$$
Simple formulas cry for bijective proofs

Plane trees

\[ a(n) = \frac{1}{n+1} \binom{2n}{n} = \frac{(2n)!}{n!(n+1)!} \]

Plane maps

\[ b(n) = \frac{2 \cdot 3^n}{(n+2)(n+1)} \binom{2n}{n} = 2 \cdot 3^n \frac{(2n)!}{n!(n+2)!} \]
A tree with $n$ edges is a Dyck path of length $2n$
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add a tail...

$$a(n) =$$
A tree with $n$ edges is a Dyck path of length $2n$

add a tail...
mark a step ($(2n + 1)$ choices)...

$(2n + 1)a(n) =$
A tree with \( n \) edges is a Dyck path of length \( 2n \)

add a tail...
mark a step \((2n + 1)\) choices)... and read the path cyclically starting from that step

\[
(2n + 1)a(n) = \binom{2n + 1}{n}
\]
A tree with $n$ edges is a Dyck path of length $2n$

\[ (2n + 1)a(n) = \binom{2n + 1}{n} \quad \Rightarrow \quad a(n) = \frac{(2n)!}{n!(n + 1)!} \]
A plane map with $n$ edges is a 4-valent map with $n$ vertices.
A plane map with \( n \) edges is a 4-valent map with \( n \) vertices

\[ b(n) \]

[Schaeffer 97]
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$$b(n)$$

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A plane map with $n$ edges is a 4-valent map with $n$ vertices or a balanced blossoming binary tree with $n$ vertices

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A plane map with $n$ edges is a 4-valent map with $n$ vertices or a **balanced blossoming binary tree** with $n$ vertices.

$$b(n) = \frac{1}{(n+1)} \binom{2n}{n}$$

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A plane map with $n$ edges is a 4-valent map with $n$ vertices or a balanced blossoming binary tree with $n$ vertices.

$\frac{3^n}{(n+1)} \binom{2n}{n}$

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\[
b(n) = \frac{3^n}{(n + 1)} \binom{2n}{n} \times \frac{2}{n + 2}
\]

[Schaeffer 97]
The strength of bijections

- Improve our understanding of the nature of planar maps
  - nice numbers
  - algebraic generating functions

Compact encoding

Random generation

Give a handle on other parameters: the average diameter of a random map of size $n$ scales like $n^{1/4}$ [Chassaing-Schaeffer 02]

The starting point of many recent results in probability theory on the asymptotic properties of large random maps [Le Gall, Miermont, Marckert, Paulin...].
The strength of bijections

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  - nice numbers
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Plane maps ($n$ edges) $\iff$ balanced blossoming trees ($n$ nodes)

- Blossoming trees:
  \[ B_l = 1 + 3tB_l^2 \]

- An unbalanced blossoming tree is a 3-tuple of blossoming subtrees [Bouttier et al. 02]:
  \[ B = B_l - tB_l^3. \]
The strength of bijections

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  - recurrence relations

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More bijections

- Many, many families of uncoloured maps are now well-understood [Arquès, Bernardi, mbm, Bouttier, Chapuy, Cori, Di Francesco, Fusy, Guitter, Poulalhon, Schaeffer, Vauquelin]

- What about coloured maps? (D-algebraic)
More bijections

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- What about coloured maps? (D-algebraic)

The Ising model: 2-coloured planar maps (algebraic)
[mbm-Schaeffer 02], [Bouttier et al. 04]
More bijections

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- What about coloured maps? (D-algebraic)

Maps equipped with a spanning tree ($q = 0, \nu = 1$; D-finite) [Mullin 67], [Bernardi 07]
More bijections

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- What about coloured maps? (D-algebraic)

Maps equipped with a spanning forest
\((q = 0; \text{D-algebraic})\)
[Bouttier et al. 07], [mbm-Courtiel 13a]
More bijections

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- What about coloured maps? (D-algebraic)

Maps equipped with a bipolar orientation
$$((-1)^{v(M)}\chi'_M(1); \text{D-finite})$$
[Felsner-Fusy-Noy-Orden 08],
[Fusy-Poulalhon-Schaeffer 08],
[Bonichon-mbm-Fusy 08]
More bijections?

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• What about coloured maps? (D-algebraic)

Properly 3-coloured planar maps
(algebraic; OPEN)

\[
C(3, 0, t; 1, 1) = \frac{(1 + 2A)(1 - 2A^2 - 4A^3 - 4A^4)}{(1 - 2A^3)^2}
\]

with

\[
A = t \frac{(1 + 2A)^3}{1 - 2A^3}.
\]
More bijections?

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- What about coloured maps? (D-algebraic)

Properly $q$-coloured triangulations
(D-algebraic; OPEN)

\[(n + 1)(n + 2)c(n + 2) = (q - 4)(3n - 1)(3n - 2)c(n + 1) + 2 \sum_{i=1}^{n} i(i + 1)(3n - 3i + 1)c(i + 1)c(n + 2 - i)\]
Pour résumer...

Objets aléatoires
Transitions de phase

Combinatoire énumérative

Calcul formel
Divination

Séries formelles

\[ B(x) = 1 + tx^2 B(x)^2 + tx \frac{xB(x) - B(1)}{x-1} \]
Joyeux Anniversaire