Walks in the quadrant: Tutte’s invariant method

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with

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Counting quadrant walks

Let $S$ be a finite subset of $\mathbb{Z}^2$ (set of steps) and $p_0 \in \mathbb{N}^2$ (starting point).

Example. $S = \{10, \bar{1}0, 1\bar{1}, \bar{1}1\}$, $p_0 = (0, 0)$
Counting quadrant walks

Let $S$ be a finite subset of $\mathbb{Z}^2$ (set of steps) and $p_0 \in \mathbb{N}^2$ (starting point).

- What is the number $q(n)$ of $n$-step walks starting at $p_0$ and contained in $\mathbb{N}^2$?
- For $(i, j) \in \mathbb{N}^2$, what is the number $q(i, j; n)$ of such walks that end at $(i, j)$?

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The associated generating function:

$$Q(x, y; t) = \sum_{n \geq 0} \sum_{i, j \geq 0} q(i, j; n)x^i y^j t^n.$$

What is the nature of this series?
A hierarchy of formal power series

- Rational series
  \[ A(t) = \frac{P(t)}{Q(t)} \]

- Algebraic series
  \[ P(t, A(t)) = 0 \]

- Differentially finite series (D-finite)
  \[ \sum_{i=0}^{d} P_i(t) A^{(i)}(t) = 0 \]

- D-algebraic series
  \[ P(t, A(t), A'(t), \ldots, A^{(d)}(t)) = 0 \]

Multi-variate series: one DE per variable
The case of half-plane walks

Example: steps $S = \{01, \bar{1}0, 1\bar{1}\}$, upper half plane

- Generating function:

$$H(y; t) \equiv H(y) = \sum_{n \geq 0} \sum_{j \geq 0} h(j; n)y^j t^n$$

$h(j; n)$: number of $n$-step walks in the half plane ending at ordinate $j$
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- Step-by-step construction:

$$H(y) = 1 + t(y + 1 + \bar{y})H(y) - t\bar{y}H(0)$$

with $\bar{y} = 1/y$. 
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$$H(y) = 1 + t(y + 1 + \bar{y})H(y) - t\bar{y}H(0)$$

or

$$(1 - t(y + 1 + \bar{y}))H(y) = 1 - t\bar{y}H(0),$$
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- The polynomial \( 1 - t(y + \bar{x} + x\bar{y}) \) is the kernel of this equation

- The equation is linear, with one catalytic variable \( y \) (tautological at \( y = 0 \)) [Zeilberger 00]
The case of half-plane walks: the kernel method

- The equation (with $\bar{y} = 1/y$):

$$\left(1 - t(y + 1 + \bar{y})\right)y H(y) = y - tH(0),$$

The right-hand side also vanishes:

$$H(0) = H(0; t) = Y_0 t.$$
The case of half-plane walks: the kernel method

- The equation (with $\bar{y} = 1/y$):
  \[(1 - t(y + 1 + \bar{y}))y \, H(y) = y - tH(0),\]
- **Cancel the kernel** by an appropriate choice of $y$:
  
  \[Y_0 = \frac{1 - t - \sqrt{(1 - t)^2 - 4t^2}}{2t} = t + t^2 + 2t^3 + O(t^4)\]

  The right-hand side also vanishes:
  \[H(0) = H(0; t) = \frac{Y_0}{t}.\]
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  The right-hand side also vanishes:
  \[ H(0) = H(0; t) = \frac{Y_0}{t}. \]

**Theorem**

The generating function of walks in the upper half-plane is an algebraic series (for any step set and any starting point).

[Gessel 80, Duchon 00, mbm-Petkovšek 00, Banderier & Flajolet 02...]
Polynomial equations with one catalytic variable

**Theorem [mbm-Jehanne 06]**

Let $P(t, y, S(y; t), A_1(t), \ldots, A_k(t))$ be a proper polynomial equation in one catalytic variable $y$ (it defines uniquely $S(y; t), A_1(t), \ldots, A_k(t)$ as formal power series in $t$). Then each of these series is algebraic.

The proof is constructive.

**Example 1:** for $S(y; t) = H(y; t)$, and $A_1 = S(0; t) = H(0; t)$,

$$(1 - t(y + 1 + \tilde{y}))yS(y; t) = y - tA_1(t).$$
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The proof is constructive.

**Example 2:** for \( S(y; t) = Q(0, y; t) \) and \( A_1(t) = Q(0, 0; t) \),

\[
\frac{t}{y^2} - \frac{1}{y} - ty = t \left( tyS(y; t) + \frac{1}{y} \right)^2 - \left( tyS(y; t) + \frac{1}{y} \right) - 2t^2 A_1(t).
\]
Polynomial equations with one catalytic variable

Theorem [mbm-Jehanne 06]

Let $P(t, y, S(y; t), A_1(t), \ldots, A_k(t))$ be a proper polynomial equation in one catalytic variable $y$ (it defines uniquely $S(y; t), A_1(t), \ldots, A_k(t)$ as formal power series in $t$). Then each of these series is algebraic.

The proof is constructive.

$\Rightarrow$ Algebraicity follows from (a special case of) an Artin approximation theorem with “nested” conditions [Popescu 86, Swan 98]
Example: $S = \{01, \bar{1}0, 1\bar{1}\}$

$$Q(x, y; t) = 1 + t(y + \bar{x} + x\bar{y})Q(x, y) - t\bar{x}Q(0, y) - tx\bar{y}Q(x, 0)$$

with $\bar{x} = 1/x$ and $\bar{y} = 1/y$.

\[Q(x, y; t) \equiv Q(x, y) = \sum_{n \geq 0} \sum_{i, j \geq 0} q(i, j; n)x^i y^j t^n\]
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Example: $S = \{01, \bar{1}0, 1\bar{1}\}$

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or

$$(1 - t(y + \bar{x} + x\bar{y}))xyQ(x, y) = xy - tyQ(0, y) - tx^2Q(x, 0)$$
Back to quadrant walks: a functional equation

Example: \( S = \{01, 0\bar{1}, 1\bar{1}\} \)

\[
Q(x, y; t) = 1 + t(y + \bar{x} + x\bar{y})Q(x, y) - t\bar{x}Q(0, y) - tx\bar{y}Q(x, 0)
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or

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(1 - t(y + \bar{x} + x\bar{y}))Q(x, y) = 1 - t\bar{x}Q(0, y) - tx\bar{y}Q(x, 0),
\]

or

\[
(1 - t(y + \bar{x} + x\bar{y}))xyQ(x, y) = xy - tyQ(0, y) - tx^2Q(x, 0)
\]

- The polynomial \( 1 - t(y + \bar{x} + x\bar{y}) \) is the kernel of this equation
- The equation is linear, with two catalytic variables \( x \) and \( y \) (tautological at \( x = 0 \) or \( y = 0 \))
Equations with **two** catalytic variables are harder...

- **Algebraic** [Kreweras 65, Gessel 86]
  \[(1 - t(\bar{x} + \bar{y} + xy))xyQ(x, y) = xy - tyQ(0, y) - txQ(x, 0)\]

- **D-finite, but transcendental** [Gessel 90]
  \[(1 - t(y + \bar{x} + x\bar{y}))xyQ(x, y) = xy - tyQ(0, y) - tx^2Q(x, 0)\]

- **Not D-finite, but D-algebraic** [Bernardi, mbm & Raschel 17]
  \[(1 - t(x + \bar{x} + y + x\bar{y}))xyQ(x, y) = xy - tyQ(0, y) - tx^2Q(x, 0)\]

- **Not D-algebraic (in \(y\))** [Dreyfus, Hardouin & Singer 17]
  \[(1 - t(x\bar{y} + \bar{x} + \bar{y} + y))xyQ(x, y) = xy - tyQ(0, y) - tx(1 + x)Q(x, 0)\]
Properly coloured triangulations ($q$ colours):

$$T(x, y; t) \equiv T(x, y) = x(q - 1) + xyt T(x, y) T(1, y) + xt \frac{T(x, y) - T(x, 0)}{y} - x^2yt \frac{T(x, y) - T(1, y)}{x - 1}.$$
An old equation [Tutte 73]

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Isn’t this reminiscent of quadrant equations?

$$Q(x, y; t) \equiv Q(x, y) = 1 + txy Q(x, y)$$

$$+ t \frac{Q(x, y) - Q(0, y)}{x} + t \frac{Q(x, y) - Q(x, 0)}{y}$$
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Theorem [Tutte 73-84]

- For \( q = 4 \cos^2 \frac{\pi}{m} \), \( q \neq 0, 4 \), the series \( T(1, y) \) satisfies an equation with one catalytic variable \( y \).
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- For $q = 4 \cos^2 \frac{\pi}{m}$, $q \neq 0, 4$, the series $T(1, y)$ satisfies an equation with one catalytic variable $y$. This implies that it is algebraic [mbm-Jehanne 06].
- For any $q$, the generating function of properly $q$-coloured planar triangulations is differentially algebraic:

$$2(1 - q)w + (w + 10H - 6wH')H'' + (4 - q)(20H - 18wH' + 9w^2 H'') = 0$$

with $H(w) = w T(1, 0; \sqrt{w})$. 
In this talk

I. Adapt Tutte’s method to quadrant walks: new and uniform proofs of algebraicity.

II. Extension to an analytic context: some walks with an infinite group (hence not D-finite) are still D-algebraic.
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quadrant models: 79

\[ |G| < \infty: 23 \quad \text{D-finite} \quad \text{OS} = 0: 4 \quad \text{algebraic} \]

\[ |G| = \infty: 56 \quad \text{Not D-finite} \quad \text{OS} \neq 0: 19 \quad \text{transcendental} \]
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| \begin{align*}
\text{dec. 4} & \quad \text{no dec. 19} \\
\text{algebraic} & \quad \text{transcendental}
\end{align*} |

| Not D-finite |
| \begin{align*}
\text{dec. 9} & \quad \text{no dec. 47} \\
\text{D-algebraic} & \quad ???
\end{align*} |
I. New proofs for algebraic models
Kreweras’ model

- The equation (with $\bar{x} = 1/x$ and $\bar{y} = 1/y$):

\[
(1 - t(\bar{x} + \bar{y} + xy))xyQ(x, y) = xy - txQ(x, 0) - tyQ(0, y)
= xy - R(x) - S(y)
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  \]

- If we take $x = t + ut^2$, both roots of the kernel
  \[
  Y_{0,1} = \frac{x - t \pm \sqrt{(x - t)^2 - 4t^2x^3}}{2tx^2}
  \]
  are (Laurent) series in $t$ with rational coefficients in $u$, and can be legally substituted for $y$ in $Q(x, y)$. 

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  are (Laurent) series in $t$ with rational coefficients in $u$, and can be legally substituted for $y$ in $Q(x, y)$. This gives four equations:

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- Four equations relating $x$, $R(x)$, $Y_0$, $Y_1$, $S(Y_0)$, $S(Y_1)$ and $t$:

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- Eliminate $x$ from the kernel equations:

$$\frac{t}{Y_0^2} - \frac{1}{Y_0} - tY_0 = \frac{t}{Y_1^2} - \frac{1}{Y_1} - tY_1$$
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- Eliminate $R(x)$ from the RHS equations, and then $x$:

  $$S(Y_0) + \frac{1}{Y_0} = S(Y_1) + \frac{1}{Y_1}$$
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- Four equations relating $x$, $R(x)$, $Y_0$, $Y_1$, $S(Y_0)$, $S(Y_1)$ and $t$:

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- Eliminate $R(x)$ from the RHS equations, and then $x$:

$$S(Y_0) + \frac{1}{Y_0} = S(Y_1) + \frac{1}{Y_1}$$

The two following functions are invariants, in the sense of Tutte:

$$I(y) = \frac{t}{y^2} - \frac{1}{y} - ty \quad \text{and} \quad J(y) = S(y) + \frac{1}{y}.$$
The invariant lemma

We have

$$I(Y_0) = I(Y_1) \quad \text{and} \quad J(Y_0) = J(Y_1)$$

with

$$I(y) = \frac{t}{y^2} - \frac{1}{y} - ty \quad \text{and} \quad J(y) = tyQ(0, y) + \frac{1}{y}.$$
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There are few invariants: \( I(y) \) must be a polynomial in \( J(y) \) whose coefficients are series in \( t \).
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There are few invariants: \( I(y) \) must be a polynomial in \( J(y) \) whose coefficients are series in \( t \).

\[ I(y) = \frac{t}{y^2} - \frac{1}{y} - ty = t \left( tyQ(0, y) + \frac{1}{y} \right)^2 - \left( tyQ(0, y) + \frac{1}{y} \right) + c \]

Expanding at \( y = 0 \) gives the value of \( c \).
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\[ I(y) = \frac{t}{y^2} - \frac{1}{y} - ty = t \left( tyQ(0, y) + \frac{1}{y} \right)^2 - \left( tyQ(0, y) + \frac{1}{y} \right) - 2t^2 Q(0, 0). \]

Expanding at \( y = 0 \) gives the value of \( c \).
The invariant lemma

We have

\[ I(Y_0) = I(Y_1) \quad \text{and} \quad J(Y_0) = J(Y_1) \]

with

\[ I(y) = \frac{ty}{y^2} - \frac{1}{y} - ty \quad \text{and} \quad J(y) = tyQ(0, y) + \frac{1}{y}. \]

The invariant lemma

There are few invariants: \( I(y) \) must be a polynomial in \( J(y) \) whose coefficients are series in \( t \).

\[ I(y) = \frac{t}{y^2} - \frac{1}{y} - ty = t \left( tyQ(0, y) + \frac{1}{y} \right)^2 - \left( tyQ(0, y) + \frac{1}{y} \right) - 2t^2 Q(0, 0). \]

Expanding at \( y = 0 \) gives the value of \( c \).

Polynomial equation with one catalytic variable \( \Rightarrow Q(0, y; t) \) is algebraic
Other set steps: are there invariants?

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High time for the complete picture!
The series $Q(x, y; t)$ is D-finite iff the group $G$ is finite. It is algebraic iff, in addition, the orbit sum is zero.

[D-finite] [mbm-Mishna 10], [Bostan-Kauers 10]
[non-singular non-D-finite] [Kurkova-Raschel 12]
singular non-D-finite [Mishna-Rechnitzer 07], [Melczer-Mishna 13]
The series $Q(x, y; t)$ is D-finite iff the group $G$ is finite. It is algebraic iff, in addition, the orbit sum is zero.
Classification of quadrant walks with small steps

quadrant models: 79

∃ K-invariant  no K-invariant

D-finite  Not D-finite

∃ RHS-invariant  no RHS-invariant

algebraic  transcendental

Theorem

The series $Q(x, y; t)$ is D-finite iff the group $G$ is finite. It is algebraic iff, in addition, the orbit sum is zero.

[mbm-Mishna 10], [Bostan-Kauers 10]  D-finite
[Kurkova-Raschel 12]  non-singular non-D-finite
[Mishna-Rechnitzer 07], [Melczer-Mishna 13]  singular non-D-finite
Algebraic models: a uniform approach

All models with a finite group and a zero orbit sum have two invariants $\Rightarrow$ uniform solution via the solution of an equation with one catalytic variable.
All models with a finite group and a zero orbit sum have two invariants \( \Rightarrow \) uniform solution via the solution of an equation with one catalytic variable

This applies as well to weighted algebraic models [Kauers, Yatchak 14(a)]:
II. The non-D-finite case

quadrant models: 79

∃ K-invariant  no K-invariant

D-finite

∃ RHS-invariant  no RHS-invariant

algebraic  transcendental

Not D-finite
II. The non-D-finite case

quadrant models: 79

- ∃ K-invariant
  - D-finite
    - ∃ RHS-invariant
      - algebraic
    - no RHS-invariant
  - no RHS-invariant
    - transcendental

- no K-invariant
  - Not D-finite
    - ∃ RHS-invariant
    - no RHS-invariant
A weaker (and analytic) notion of invariants

- Still require that $I(Y_0) = I(Y_1)$, where $Y_0, Y_1$ are the roots of the kernel

$$xyK(x, y) = xy - xyt \sum_{(i,j) \in S} x^i y^i = a(x)y^2 + b(x)y + c(x)$$

... but only for some complex values of $x$ (and $t$).
- Meromorphicity condition in a domain
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Fayolle, Iasnogorodski, Malyshev [1999]
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Def. A function $I(y; t)$ is a weak invariant if, for $x \in [x_1, x_2]$, $I(Y_0) = I(Y_1)$. 

\[ b(x)^2 - 4a(x)c(x) \]
A weaker (and analytic) notion of invariants

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Def. A function $I(y; t)$ is a weak invariant if, for $x \in [x_1, x_2]$, $I(Y_0) = I(Y_1)$. Moreover, $I(y)$ must be meromorphic inside $\mathcal{L} = Y([x_1, x_2])$, with finite limits on $\mathcal{L}$. 
Can we find weak invariants?

**Theorem** [Fayolle et al. 99, Rasche 12]

For each non-singular model, there exists an (explicit) weak invariant of the form

\[ I(y; t) = \wp(R(y; t), \omega_1(t), \omega_3(t)) \]

where

- \( \wp \) is Weierstrass elliptic function
- its periods \( \omega_1 \) and \( \omega_3 \) are elliptic integrals
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\[
\begin{align*}
\omega_1 &= i \int_{x_1}^{x_2} \frac{dx}{\sqrt{-\delta(x)}}, \\
\omega_2 &= \int_{x_2}^{x_3} \frac{dx}{\sqrt{\delta(x)}}, \\
\omega_3 &= \int_{x_1}^{x(y_1)} \frac{dx}{\sqrt{\delta(x)}}.
\end{align*}
\]

\[
\mathcal{R}(y; t) = \int_{f(y_2)}^{f(y)} \frac{dz}{\sqrt{4z^3 - g_2z - g_3}}
\]

\( g_2, g_3 \) polynomials in \( t \), \( f(y) \) rational in \( y \) and algebraic in \( t \).
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**Proposition [Bernardi-mbm-Raschel 17]**

\( I(y; t) \) is D-algebraic in \( y \) and \( t \).
The invariant lemma

For appropriate values of $x$, we have:

$$I(Y_0) = I(Y_1) \quad \text{and} \quad J(Y_0) = J(Y_1)$$

where $I(y)$ is the weak invariant and $J(y) = t(1 + y)Q(0, y) + 1/y$. 

The invariant lemma [Litvinchuk 00]

There are few invariants: $J(y)$ must be a rational function in $I(y)$. The value of this rational function is found by looking at the poles and zeroes of $J(y)$. 

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\[ J(y) = t(1 + y)Q(0, y) + \frac{1}{y} = \frac{I'(0)}{I(y) - I(0)} - \frac{I'(0)}{I(-1) - I(0)} - 1 \]
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Explicit expression of $Q(0, y; t)$ in terms of the (explicit) weak invariant $I(y)$
The invariant lemma

For appropriate values of $x$, we have:

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There are few invariants: $J(y)$ must be a rational function in $I(y)$. The value of this rational function is found by looking at the poles and zeroes of $J(y)$.

Corollary

For the 9 models with an infinite group and a RHS-invariant, the series $Q(x, y; t)$ is D-algebraic.
Final comments

quadrant models: 79

exist K-invariant: 23
no K-invariant: 56

D-finite

exist RHS-invariant
no RHS-invariant

algebraic
transcendental

Not D-finite

exist RHS-invariant
no RHS-invariant

D-algebraic

???

Nature of $Q(x, y; t)$ when no decoupling function exists?

[Dreyfus, Hardouin, Roques, Singer 17(a)]: $Q(0, y)$ is not DA in $y$.

The existence of decoupling function depends much on the starting point.

To do: find explicit DEs (done for $y$).
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