

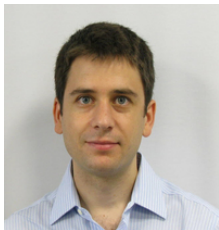
Walks in the quadrant: Tutte's invariant method

Mireille Bousquet-Mélou, LaBRI, CNRS, Université de Bordeaux

with

Olivier Bernardi, Brandeis University, Boston

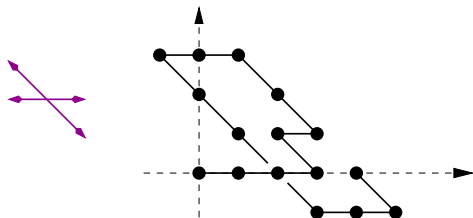
Kilian Raschel, CNRS, Université de Tours



Counting quadrant walks

Let \mathcal{S} be a finite subset of \mathbb{Z}^2 (set of **steps**) and $p_0 \in \mathbb{N}^2$ (starting point).

Example. $\mathcal{S} = \{10, \bar{1}0, 1\bar{1}, \bar{1}\bar{1}\}$, $p_0 = (0, 0)$

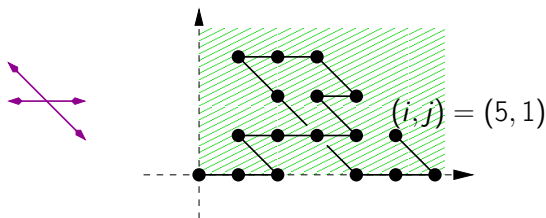


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- What is the number $q(n)$ of n -step walks starting at p_0 and contained in \mathbb{N}^2 ?
- For $(i, j) \in \mathbb{N}^2$, what is the number $q(i, j; n)$ of such walks that end at (i, j) ?

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The associated generating function:

$$Q(x, y; t) = \sum_{n \geq 0} \sum_{i, j \geq 0} q(i, j; n) x^i y^j t^n.$$

What is the **nature** of this series?

A hierarchy of formal power series

- Rational series

$$A(t) = \frac{P(t)}{Q(t)}$$

- Algebraic series

$$P(t, A(t)) = 0$$

- Differentially finite series (D-finite)

$$\sum_{i=0}^d P_i(t) A^{(i)}(t) = 0$$

- D-algebraic series

$$P(t, A(t), A'(t), \dots, A^{(d)}(t)) = 0$$

Multi-variate series: one DE per variable



The case of half-plane walks

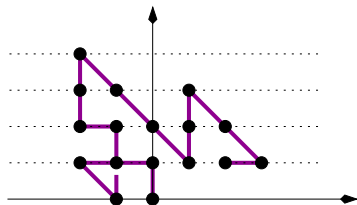


Example: steps $\mathcal{S} = \{01, \bar{1}0, 1\bar{1}\}$, upper half plane

- Generating function:

$$H(y; t) \equiv H(y) = \sum_{n \geq 0} \sum_{j \geq 0} h(j; n) y^j t^n$$

$h(j; n)$: number of n -setp walks in the half plane ending at ordinate j



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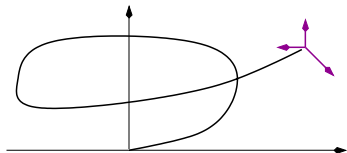
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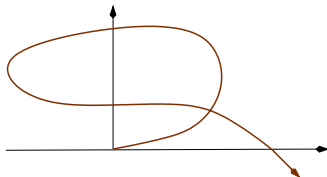
- Step-by-step construction:

$$H(y) = 1 + t(y + 1 + \bar{y})H(y) - t\bar{y}H(0)$$

with $\bar{y} = 1/y$.



⊖



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$$H(y) = 1 + t(y + 1 + \bar{y})H(y) - t\bar{y}H(0)$$

or

$$(1 - t(y + 1 + \bar{y}))H(y) = 1 - t\bar{y}H(0),$$

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$$(1 - t(y + 1 + \bar{y}))y H(y) = y - tH(0).$$

- The polynomial $1 - t(y + \bar{x} + x\bar{y})$ is the **kernel** of this equation
- The equation is linear, with **one catalytic variable** y (tautological at $y = 0$) [Zeilberger 00]

The case of half-plane walks: the kernel method

- The equation (with $\bar{y} = 1/y$):

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- The equation (with $\bar{y} = 1/y$):

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- **Cancel the kernel** by an appropriate choice of y :

$$Y_0 = \frac{1 - t - \sqrt{(1 - t)^2 - 4t^2}}{2t} = t + t^2 + 2t^3 + \mathcal{O}(t^4)$$

The right-hand side also vanishes:

$$H(0) = H(0; t) = \frac{Y_0}{t}.$$

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Theorem

The generating function of walks in the upper half-plane is an algebraic series (for any step set and any starting point).

[Gessel 80, Duchon 00, mbm-Petkovšek 00, Banderier & Flajolet 02...]

Polynomial equations with **one** catalytic variable

Theorem [mbm-Jehanne 06]

Let $P(t, y, S(y; t), A_1(t), \dots, A_k(t))$ be a proper polynomial equation in one catalytic variable y (it defines uniquely $S(y; t), A_1(t), \dots, A_k(t)$ as formal power series in t). Then each of these series is algebraic.

The proof is constructive.

Example 1: for $S(y; t) = H(y; t)$, and $A_1 = S(0; t) = H(0; t)$,

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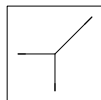
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Example 2: for $S(y; t) = Q(0, y; t)$ and $A_1(t) = Q(0, 0; t)$,

$$\frac{t}{y^2} - \frac{1}{y} - ty = t \left(tyS(y; t) + \frac{1}{y} \right)^2 - \left(tyS(y; t) + \frac{1}{y} \right) - 2t^2A_1(t).$$



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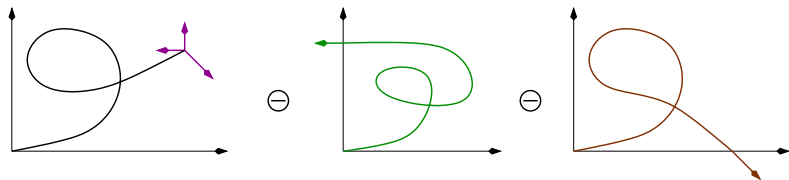
\Rightarrow Algebraicity follows from (a special case of) an Artin approximation theorem with “nested” conditions [Popescu 86, Swan 98]



Example: $\mathcal{S} = \{01, \bar{1}0, 1\bar{1}\}$

$$Q(x, y; t) = 1 + t(y + \bar{x} + x\bar{y})Q(x, y) - t\bar{x}Q(0, y) - tx\bar{y}Q(x, 0)$$

with $\bar{x} = 1/x$ and $\bar{y} = 1/y$.



$$Q(x, y; t) \equiv Q(x, y) = \sum_{n \geq 0} \sum_{i, j \geq 0} q(i, j; n) x^i y^j t^n$$



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- The equation is linear, with **two catalytic variables** x and y (tautological at $x = 0$ or $y = 0$)

Equations with **two** catalytic variables are harder...



Algebraic

[Kreweras 65, Gessel 86]

$$(1 - t(\bar{x} + \bar{y} + xy))_{xy}Q(x, y) = xy - tyQ(0, y) - txQ(x, 0)$$



D-finite, but transcendental

[Gessel 90]

$$(1 - t(y + \bar{x} + x\bar{y}))_{xy}Q(x, y) = xy - tyQ(0, y) - tx^2Q(x, 0)$$



Not D-finite, but D-algebraic

[Bernardi, mbm & Raschel 17]

$$(1 - t(x + \bar{x} + y + x\bar{y}))_{xy}Q(x, y) = xy - tyQ(0, y) - tx^2Q(x, 0)$$



Not D-algebraic (in y)

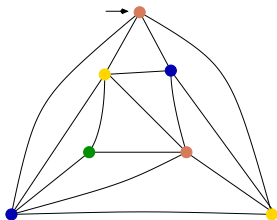
[Dreyfus, Hardouin & Singer 17]

$$(1 - t(x\bar{y} + \bar{x} + \bar{y} + y))_{xy}Q(x, y) = xy - tyQ(0, y) - tx(1 + x)Q(x, 0)$$

An old equation [Tutte 73]

- Properly coloured triangulations (q colours):

$$T(x, y; t) \equiv T(x, y) = x(q-1) + xyt T(x, y) T(1, y) \\ + xt \frac{T(x, y) - T(x, 0)}{y} - x^2 yt \frac{T(x, y) - T(1, y)}{x-1}$$



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Isn't this reminiscent of quadrant equations?

$$Q(x, y; t) \equiv Q(x, y) = 1 + txy Q(x, y) \\ + t \frac{Q(x, y) - Q(0, y)}{x} + t \frac{Q(x, y) - Q(x, 0)}{y}$$



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Theorem [Tutte 73-84]

- For $q = 4 \cos^2 \frac{\pi}{m}$, $q \neq 0, 4$, the series $T(1, y)$ satisfies an equation with one catalytic variable y .

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- For $q = 4 \cos^2 \frac{\pi}{m}$, $q \neq 0, 4$, the series $T(1, y)$ satisfies an equation with one catalytic variable y . This implies that it is algebraic [mbm-Jehanne 06].
- For any q , the generating function of properly q -coloured planar triangulations is differentially algebraic:

$$2(1-q)w + (w + 10H - 6wH')H'' + (4-q)(20H - 18wH' + 9w^2H'') = 0$$

with $H(w) = wT(1, 0; \sqrt{w})$.

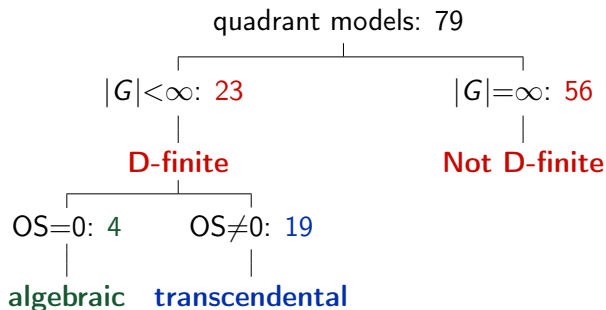
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- I. Adapt Tutte's method to quadrant walks: new and uniform proofs of algebraicity.
- II. Extension to an analytic context: some walks with an infinite group (hence not D-finite) are still D-algebraic.

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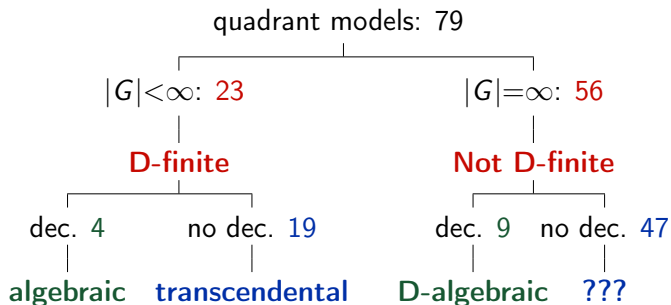
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II. Extension to an analytic context: some walks with an infinite group (hence not D-finite) are still D-algebraic.



I. New proofs for algebraic models





- The equation (with $\bar{x} = 1/x$ and $\bar{y} = 1/y$):

$$\begin{aligned}(1 - t(\bar{x} + \bar{y} + xy))xyQ(x, y) &= xy - txQ(x, 0) - tyQ(0, y) \\ &= xy - R(x) - S(y)\end{aligned}$$



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- If we take $x = t + ut^2$, both roots of the kernel

$$Y_{0,1} = \frac{x - t \pm \sqrt{(x - t)^2 - 4t^2x^3}}{2tx^2}$$

are (Laurent) series in t with rational coefficients in u , and can be legally substituted for y in $Q(x, y)$.



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are (Laurent) series in t with rational coefficients in u , and can be legally substituted for y in $Q(x, y)$. This gives four equations:

Kernel	RHS
$xY_0 = t(Y_0 + x + x^2Y_0^2)$	$xY_0 = R(x) + S(Y_0)$
$xY_1 = t(Y_1 + x + x^2Y_1^2)$	$xY_1 = R(x) + S(Y_1)$



- Four equations relating x , $R(x)$, Y_0 , Y_1 , $S(Y_0)$, $S(Y_1)$ and t :

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- Eliminate x from the kernel equations:

$$\frac{t}{Y_0^2} - \frac{1}{Y_0} - tY_0 = \frac{t}{Y_1^2} - \frac{1}{Y_1} - tY_1$$



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- Eliminate $R(x)$ from the RHS equations, and then x :

$$S(Y_0) + \frac{1}{Y_0} = S(Y_1) + \frac{1}{Y_1}$$



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$$S(Y_0) + \frac{1}{Y_0} = S(Y_1) + \frac{1}{Y_1}$$

The two following functions are **invariants**, in the sense of Tutte:

$$I(y) = \frac{t}{y^2} - \frac{1}{y} - ty \quad \text{and} \quad J(y) = S(y) + \frac{1}{y}.$$

The invariant lemma



We have

$$I(Y_0) = I(Y_1) \quad \text{and} \quad J(Y_0) = J(Y_1)$$

with

$$I(y) = \frac{t}{y^2} - \frac{1}{y} - ty \quad \text{and} \quad J(y) = tyQ(0, y) + \frac{1}{y}.$$

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Expanding at $y = 0$ gives the value of c .

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



Expanding at $y = 0$ gives the value of c .

Polynomial equation with one catalytic variable $\Rightarrow Q(0, y; t)$ is algebraic

Other set steps: are there invariants?

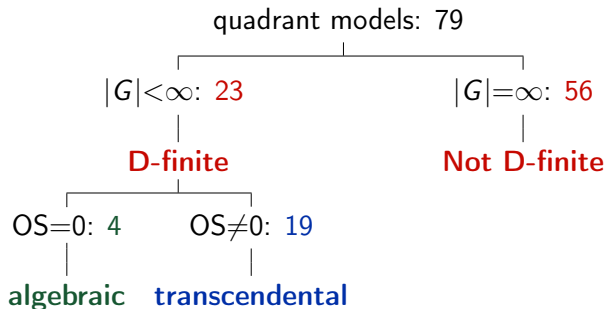
	from kernel	from RHS
	×	×
	×	—
	—	×
	—	—

Other set steps: are there invariants?

	from kernel	from RHS
	×	×
	×	—
	—	×
	—	—

High time for the complete picture!

Classification of quadrant walks with small steps



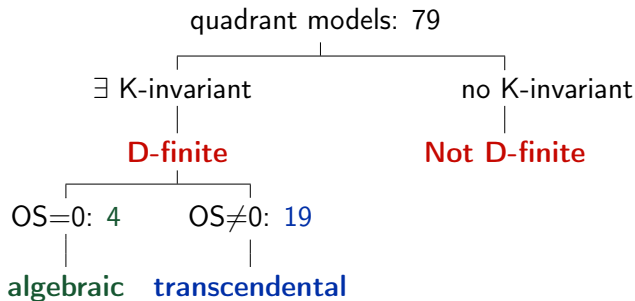
Theorem

The series $Q(x, y; t)$ is D-finite iff the group G is finite.
It is algebraic iff, in addition, the orbit sum is zero.

[mbm-Mishna 10], [Bostan-Kauers 10]
[Kurkova-Raschel 12]
[Mishna-Rechnitzer 07], [Melczer-Mishna 13]

D-finite
non-singular non-D-finite
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Classification of quadrant walks with small steps



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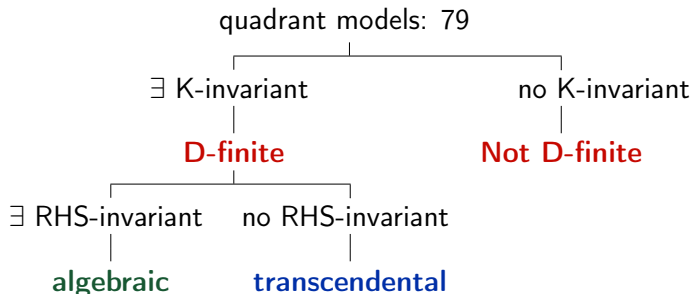
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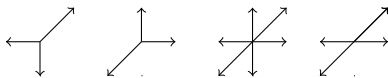
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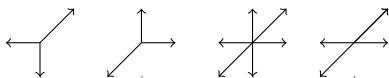
Algebraic models: a uniform approach

All models with a finite group and a zero orbit sum have two invariants \Rightarrow uniform solution via the solution of an equation with one catalytic variable

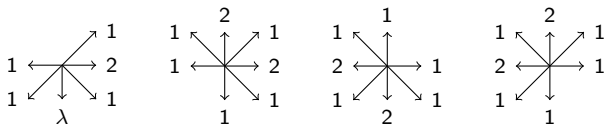


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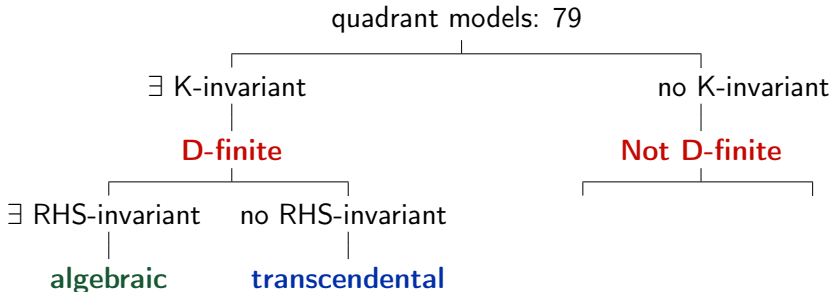
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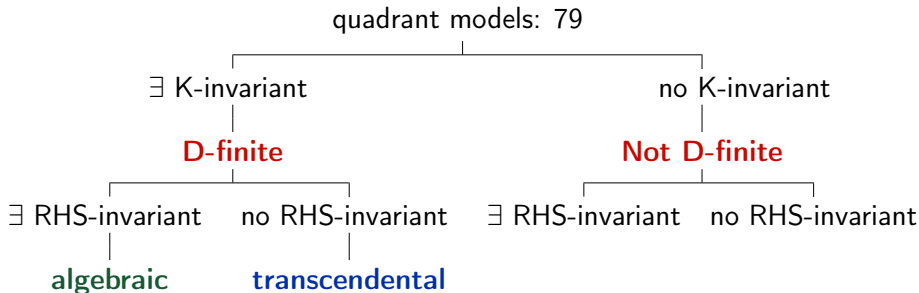
This applies as well to weighted algebraic models [Kauers, Yatchak 14(a)]:



II. The non-D-finite case



II. The non-D-finite case



A weaker (and analytic) notion of invariants

- Still require that $I(Y_0) = I(Y_1)$, where Y_0, Y_1 are the roots of the kernel

$$xyK(x, y) = xy - xyt \sum_{(i,j) \in \mathcal{S}} x^i y^j = a(x)y^2 + b(x)y + c(x)$$

... but only for **some** complex values of x (and t).

- meromorphicity condition in a domain

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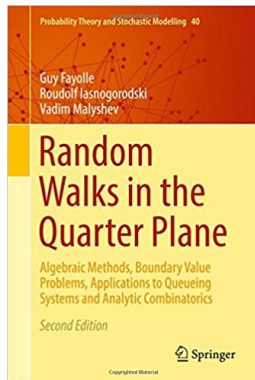
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Fayolle, Iasnogorodski, Malyshev [1999]



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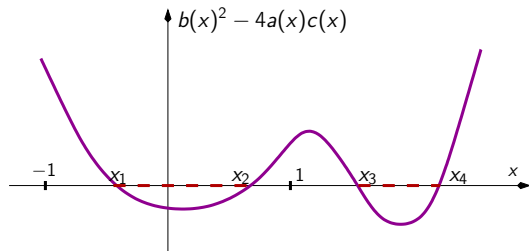
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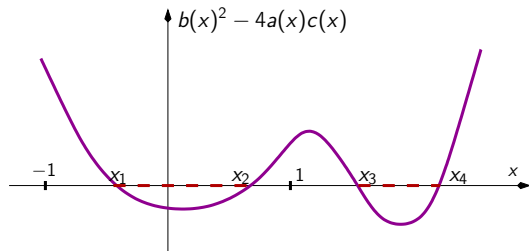


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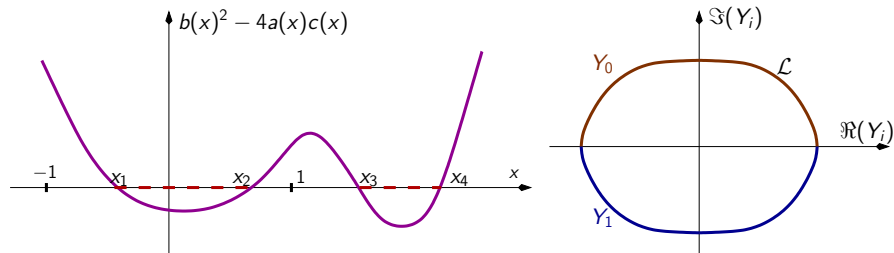
Def. A function $I(y; t)$ is a **weak invariant** if, for $x \in [x_1, x_2]$, $I(Y_0) = I(Y_1)$.

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Def. A function $I(y; t)$ is a **weak invariant** if, for $x \in [x_1, x_2]$, $I(Y_0) = I(Y_1)$. Moreover, $I(y)$ must be meromorphic inside $\mathcal{L} = Y([x_1, x_2])$, with finite limits on \mathcal{L} .

Can we find weak invariants?

Theorem [Fayolle et al. 99, Raschel 12]

For each non-singular model, there exists an (explicit) weak invariant of the form

$$I(y; t) = \wp(\mathcal{R}(y; t), \omega_1(t), \omega_3(t))$$

where

- \wp is Weierstrass elliptic function
- its periods ω_1 and ω_3 are elliptic integrals
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$$\omega_1 = i \int_{x_1}^{x_2} \frac{dx}{\sqrt{-\delta(x)}}, \quad \omega_2 = \int_{x_2}^{x_3} \frac{dx}{\sqrt{\delta(x)}}, \quad \omega_3 = \int_{x(y_1)}^{x_1} \frac{dx}{\sqrt{\delta(x)}}.$$

$$\mathcal{R}(y; t) = \int_{f(y_2)}^{f(y)} \frac{dz}{\sqrt{4z^3 - g_2z - g_3}}$$

g_2, g_3 polynomials in t , $f(y)$ rational in y and algebraic in t .

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Proposition [Bernardi-mbm-Raschel 17]

$I(y; t)$ is D-algebraic in y and t .

The invariant lemma



For appropriate values of x , we have:

$$I(Y_0) = I(Y_1) \quad \text{and} \quad J(Y_0) = J(Y_1)$$

where $I(y)$ is the weak invariant and $J(y) = t(1 + y)Q(0, y) + 1/y$.

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Explicit expression of $Q(0,y;t)$ in terms of the (explicit) weak invariant $I(y)$

The invariant lemma



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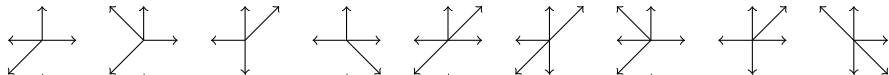
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Corollary

For the 9 models with an infinite group and a RHS-invariant, the series $Q(x, y; t)$ is D-algebraic.



Final comments

quadrant models: 79

\exists K-invariant: 23

no K-invariant: 56

D-finite

Not D-finite

\exists RHS-invariant

no RHS-invariant

\exists RHS-invariant

no RHS-invariant

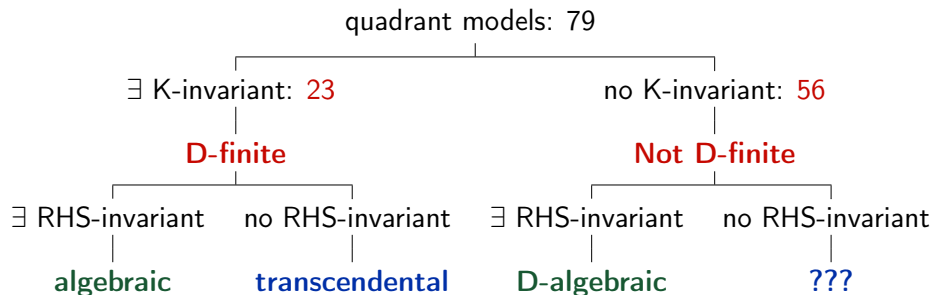
algebraic

transcendental

D-algebraic

???

Final comments



- Nature of $Q(x, y; t)$ when no decoupling function exists?
[Dreyfus, Hardouin, Roques, Singer 17(a)]: $Q(0, y)$ is not DA in y
- The existence of decoupling function depends much on the starting point.
- To do: find explicit DEs (done for y)