An introduction to

asymptotic enumeration

Part 1. Why are asymptotic results interesting?

Part 2. Asymptotics of sequences $a(n)$

Part 3. Limit laws
From exact to asymptotic enumeration

**Assumption:** we have an exact description of the sequence $a(n)$, like

- recurrence relation
- formula
- expression of the ordinary or exponential g.f. $A(t)$ or $\tilde{A}(t)$
- functional equation defining $A(t)$ or $\tilde{A}(t)$...

**Wanted**

$$a(n) \sim s(n) \cdots \text{ or } a(n) = s(n) + O(r(n))$$
Overview

Three rather general techniques:

- when \( a(n) \) is given explicitly as a sum
- when \( A(t) \) has dominant singularities of an algebraic-logarithmic type
- when \( A(t) \) is entire or diverges violently near its dominant singularity(-ies)

What you (really) need to remember from these techniques:

- their existence
- their range of applicability (how they work)
- where to find them
**Three rather general techniques**

- **Bare-hand asymptotics** on sums

\[
a(n) = \sum_{j=1}^{n} \frac{3^j (n-1)! \ n! \ (2j)! (24 + 18n + (5 + 13n)j)}{(j - 1)! \ j! \ (j + 1)! \ (j + 4)! \ (n-j)! \ (n-j+2)!}
\]

- **Singularity analysis**: when \( A(t) \) has dominant singularities of an algebraic-logarithmic type

\[
A(t) = \frac{1}{2} \left( 1 - \sqrt{1 - 4 \log \frac{1}{1 - \log \frac{1}{1-t}}} \right)
\]

- **Saddle-point asymptotics**: when \( A(t) \) is entire or diverges violently near its dominant singularity(-ies)

\[
A(t) = \exp \left( \frac{t}{1-t} \right)
\]
Three rather general techniques... and their applications

● **Software:** When \( A(t) \) is explicitly given

\[
A(t) = \frac{1}{2} \left( 1 - \sqrt{1 - 4 \log \frac{1}{1 - \log \frac{1}{1-t}}} \right), \quad A(t) = \exp \left( \frac{t}{1-t} \right)
\]

● **Almost automatic** asymptotic analysis for \( A(t) \) defined *implicitly* by

- an algebraic equation

\[
t^3 A(t)^4 + t^2 (3 + 4 t) A(t)^3 + t \left( 3 - 29 t + 6 t^2 \right) A(t)^2 \\
+ (1 - 7 t + 29 t^2 + 4 t^3) A(t) - (1 - t)^3 = 0
\]

- a linear differential equation

\[
2 + 2 (-1 + 6 t) A(t) + 4 t (-1 + 12 t) \frac{d}{dt} A(t) + t^2 (16 t - 1) \frac{d^2}{dt^2} A(t) = 0
\]
First method: Asymptotics of sums
Bare-hand asymptotics on sums

Input

\[ a(n) = \sum_{k} b(n, k) \]

Output

\[ a(n) \sim s(n) \cdots \text{ or } a(n) = s(n) + O(r(n)) \]
Bare-hand asymptotics on sums

Input

$$a(n) = \sum_{k} b(n, k)$$

Output

$$a(n) \sim s(n) \ldots \text{ or } a(n) = s(n) + 0(r(n))$$

Example

$$a(n) = \sum_{j=1}^{n} \frac{4(n-1)! \ n! \ (2j)!}{(j-1)! \ j! \ (j+1)! \ (j+4)! \ (n-j)! \ (n-j+2)!} P(j, n)$$

with

$$P(j, n) = 24+18n+(5-13n) \ j+(11n+20) \ j^2+(10n-2) \ j^3+(4n-11) \ j^4-6 \ j^5.$$  

$$a(n) \sim \frac{3^9 \sqrt{3} \ 9^n}{2^5 \ \pi \ n^7}$$
The method on a toy example

Input

\[ a(n) = \sum_{k=0}^{n} b(n, k) \quad \text{with} \quad b(n, k) = \binom{n}{k} \]

Output

\[ a(n) \sim 2^n \]
The method on a toy example

Input

\[ a(n) = \sum_{k=0}^{n} b(n, k) \quad \text{with} \quad b(n, k) = \binom{n}{k} \]

Output

\[ a(n) \sim 2^n \]

The method:

1. Locate and estimate the largest summand(s) \((k \sim n/2)\)
2. Recenter the problem in the vicinity of the “heavy” \(k\)'s \((k = n/2 + r, \text{ with } r \text{ small})\).
   Estimate \(b(n, k)\) for those \(k\)'s (Stirling)
3. Determine a window on which to focus \((|k - n/2| < n^{2/3})\). Prove that the other summands are negligible
4. Replace the sum in the window by an integral
The method on a toy example

Input

\[ a(n) = \sum_{k=0}^{n} b(n, k) \quad \text{with} \quad b(n, k) = \binom{n}{k} \]

1. Locate and estimate the largest summand(s)

The sequence \( b(n, k) \) is unimodal, with max. at \( k = \lfloor n/2 \rfloor \). At this point,

\[ b(n, k) = \binom{n}{\lfloor n/2 \rfloor} = \sqrt{2} \frac{2^n}{\sqrt{\pi n}} (1 + O(1/n)). \]
The method on a toy example

Input

\[ a(n) = \sum_{k=0}^{n} b(n, k) \quad \text{with} \quad b(n, k) = \binom{n}{k} \]

1. Locate and estimate the largest summand(s)
   The sequence \( b(n, k) \) is unimodal, with max. at \( k = \lfloor n/2 \rfloor \). At this point,
   \[ b(n, k) = \binom{n}{\lfloor n/2 \rfloor} = \sqrt{2} \frac{2^n}{\sqrt{\pi n}} (1 + O(1/n)). \]

2. Recenter the problem in the vicinity of the “heavy” \( k \)'s.
   Estimate \( b(n, k) \) for those \( k \)'s (Stirling).
   Let \( k = n/2 + r \), with \( r = o(n^{3/4}) \). Then
   \[ b(n, n/2 + r) = \sqrt{2} \frac{2^n}{\sqrt{\pi n}} \exp \left( -\frac{2r^2}{n} \right) (1 + O(1/n) + O(r^4/n^3)) . \]
   \[ \Rightarrow \text{The significant summands are found among the terms } b(n, n/2 + r) \text{ for } r = O(n^{1/2+\epsilon}) \]
3. Determine a window on which to focus.
Prove that the other summands are negligible.
For \( |r| \geq n^{2/3} \),

\[
b(n, n/2 + r) \leq b(n, n/2 + n^{2/3}) = O \left( \frac{2^n}{\sqrt{n}} \exp(-2n^{1/3}) \right)
\]

\[
\implies \sum_{|r| \geq n^{2/3}} b(n, n/2 + r) = O \left( 2^n \sqrt{n} \exp(-2n^{1/3}) \right) = o(2^n/\sqrt{n}).
\]

4. Replace the sum in the window by an integral [Riemann]

For \( |r| < n^{2/3} \), the following holds uniformly in \( r \):

\[
b(n, n/2 + r) = \sqrt{2} \frac{2^n}{\sqrt{\pi n}} \exp \left( -\frac{2r^2}{n} \right) \left( 1 + O(n^{-1/3}) \right).
\]

Hence

\[
\sum_{r = -n^{2/3}}^{n^{2/3}} b(n, n/2 + r) = \sqrt{2} \frac{2^n}{\sqrt{\pi n}} \left( \sum_{r = -n^{2/3}}^{n^{2/3}} \exp \left( -\frac{2r^2}{n} \right) \right) \left( 1 + O(n^{-1/3}) \right)
\]

\[
= \sqrt{2} \frac{2^n}{\sqrt{\pi n}} \sqrt{n} \left( \int_{\mathbb{R}} \exp(-2x^2)dx \right) \left( 1 + o(1) \right)
\]

\[
= 2^n (1 + o(1)).
\]
References: Asymptotics of sums


Asymptotics and generating functions:

general principles

• Ph. Flajolet and R. Sedgewick, Analytic Combinatorics
http://algo.inria.fr/flajolet/Publications/books.html
An elementary example: Rational generating functions

Let

\[ A(t) = \frac{1}{(1 - \mu t)^k} \]

with \( k \in \mathbb{N} \) and \( \mu \in \mathbb{C} \). Then

\[ A(t) = \sum_{n \geq 0} \binom{n + k - 1}{k - 1} \mu^n t^n = \sum_{n \geq 0} \frac{(n + 1)(n + 2) \cdots (n + k - 1)}{(k - 1)!} \mu^n t^n \]

Then

\[ a(n) = \frac{n^{k-1}}{(k-1)!} \mu^n (1 + O(1/n)). \]

(Important) remark: As a function of the complex variable \( z \), the function \( A(z) \) has radius of convergence \( 1/|\mu| \), and a pole of multiplicity \( k \) at \( z_c = 1/\mu \).
An elementary example: Rational generating functions

Let

\[ A(t) = \frac{N(t)}{D(t)} \quad \text{with} \quad D(t) = \prod_{i=1}^{d} (1 - \mu_i t)^{k_i} \]

- Partial fraction expansion:

\[ A(t) = P(t) + \sum_{i=1}^{d} \sum_{k=1}^{k_i} \frac{c_{i,k}}{(1 - \mu_i t)^k} \]

- If \( \mu = |\mu_1| = \cdots = |\mu_j| > |\mu_{j+1}| \geq \cdots \geq |\mu_d| \) and \( e = \max(k_1, k_2, \cdots, k_j) \),

\[ a(n) = O(\mu^n n^{e-1}) \]

If moreover \( e = k_1 > k_2 \geq \cdots \geq k_j \),

\[ a(n) = c_{1,1} \frac{n^{e-1}}{(e-1)!} \mu_1^n (1 + O(1/n)) \]
Rational generating functions: some observations

Let $A(t) = \frac{N(t)}{D(t)}$ with $D(t) = \prod_{i=1}^{d} (1 - \mu_i t)^{k_i}$

If $\mu = |\mu_1| = \cdots = |\mu_j| > |\mu_{j+1}| \geq \cdots \geq |\mu_d|$ and $e = \max(k_1, k_2, \cdots, k_j)$,

$$a(n) = O(\mu^n n^{e-1})$$

As a function of a complex variable $z$, the function $A(z)$ has radius $\rho = 1/\mu$, and a pole of order $e$ at one of the $\mu_i$'s, $1 \leq i \leq j$.

"The location and nature of the dominant singularities of a function $A(z) = \sum a(n)z^n$, analytic around 0, are reflected in the asymptotic behaviour of its coefficients $a(n)$."


Analytic functions

Defn. The function \( A(z) \) is analytic near 0 if it is the sum of a convergent power series: for \( |z| < \rho \), with \( \rho > 0 \),

\[
A(z) = \sum_{n \geq 0} a(n)t^n
\]

The radius of convergence of \( A(z) \) is \( \sup\{r > 0 : \sum a(n)r^n \text{ converges} \} \)

Radius of cv. and exponential growth of coefficients
If the radius is \( \rho \), then for all \( \varepsilon > 0 \)

\[
|a(n)| \leq (\varepsilon + 1/\rho)^n \quad \text{for } n \text{ large enough}
\]

\[
(-\varepsilon + 1/\rho)^n \leq |a(n)| \quad \text{infinitely often}
\]

“The location of the dominant singularity (the radius) gives the exponential growth rate.”
Singularities

Let $A(z)$ be analytic in a domain $D$. A point $z_c$ of the border of $D$ is a singularity of $A$ if no analytic continuation is possible at $z_c$. That is, we cannot write

$$A(z) = \sum b(n)(z - z_c)^n \quad \text{for} \quad |z - z_c| \text{ small}.$$  

Examples

- log $z$ and $\sqrt{z}$ are analytic in $D = \mathbb{C} \setminus (-\infty, 0]$, with a unique singularity at 0.
- The Catalan g.f. $A(z) = (1 - \sqrt{1 - 4z})/(2z)$ is analytic in $\mathbb{C} \setminus [1/4, \infty)$, with a unique singularity at $1/4$. 

![Disk of convergence](image)

![3D graph](image)
The Catalan g.f.  \( A(z) = (1 - \sqrt{1 - 4z})/(2z) \) is analytic in \( \mathbb{C} \setminus [1/4, \infty) \), with a unique singularity at 1/4.
Dominant singularities of power series

Let $A(z) = \sum a(n)z^n$ be a series with positive radius of convergence $\rho$.

- Then $A(z)$ has at least one singularity on the circle $|z| = \rho$. Such singularities are called dominant.

- If $a(n) \geq 0$, one of the dominant singularities is $\rho$ itself [Pringsheim].
Extraction of coefficients: Cauchy’s formula

Let $A(z) = \sum a(n)z^n$ be a series with positive radius of cv, and analytic in a domain $D$. Then

$$a(n) = \frac{1}{2i\pi} \oint_{C} A(z) \frac{dz}{z^{n+1}}$$

for every contour $C \subset D$ encircling positively the origin.
Second method: singularity analysis

- Ph. Flajolet and R. Sedgewick, Analytic Combinatorics
  http://algo.inria.fr/flajolet/Publications/books.html
  Chapters VI, VII

- Ph. Flajolet and A. Odlyzko, Singularity analysis of generating functions,
Singularity analysis

- **Basic series:** We know the asymptotic behaviour of the coefficients of certain simple (algebraic-logarithmic) functions, typically

\[
S(z) = \frac{1}{(1 - t)^\alpha} \left( \frac{1}{t} \log \frac{1}{1 - t} \right)^\beta
\]

\[
\Rightarrow s(n) = \frac{n^{\alpha - 1}}{\Gamma(\alpha)} (\log n)^\beta \left( 1 + O\left(\frac{1}{\log n}\right) \right)
\]

- **Transfer theorems:** Under certain hypotheses, if, in the neighborhood of its dominant singularity $z_c$ (supposed to be unique),

\[
A(z) = S(z) + O(R(z)) \quad \text{as } z \to z_c,
\]

where $R(z)$ is **simple**, this estimate can be **transfered** to coefficients:

\[
a(n) = s(n) + O(r(n)) \quad \text{as } n \to \infty.
\]

- Similar statement with $o(R(z))$ and $o(r(n))$ instead of $O(R(z))$ and $O(r(n))$. 
Singularity analysis: blind application

- GF of Motzkin numbers:

\[ A(z) = \frac{1 - z - \sqrt{(1 + z)(1 - 3z)}}{2z^2}. \]

- Dominant singularity at \( z_c = 1/3 \). As \( z \to z_c \),

\[ A(z) = 3 - 3^{3/2} \sqrt{1 - 3z} + 15/2(1 - 3z) + O((1 - 3z)^{3/2}) + O(R(z)) \]

- “Hence”, for \( n \) large,

\[ a(n) = [z^n] \left( -3^{3/2} \sqrt{1 - 3z} \right) + O([z^n](1 - 3z)^{3/2}) \]

\[ = \frac{3^{3/2}}{\Gamma(-1/2)} 3^n n^{-3/2} + O(3^n n^{-5/2}) \]

which can also be painfully obtained by analysing the sum

\[ a(n) = \sum_k \frac{n!}{k!(k + 1)!(n - 2k)!}. \]
Singularity analysis: blind application

- GF of column-convex polygons counted by (half-)perimeter:

\[ A(z) = (1 - z) \left( 1 - \frac{2\sqrt{2}}{3\sqrt{2} - \sqrt{(1 + z)/(1 - z)(1 - z + \sqrt{1 - 6z + z^2})}} \right) \]

- Dominant singularity at \( z_c = 3 - 2\sqrt{2} \). As \( z \to z_c \),

\[ A(z) = c_0 + c_1 (1 - z/z_c)^{1/2} + o((1 - z/z_c)^{1/2}) \]

- “Hence”

\[ a_n = \frac{c_1}{\Gamma(-1/2)} (3 + 2\sqrt{2})^n n^{-3/2} + o(z_c^{-n} n^{-3/2}) \]
Applying singularity analysis to Gilbert L.

• Permutations with \( k \) cycles (\( k \) fixed):

\[
S_k(z) = \frac{1}{k!} \left( \log \frac{1}{1 - z} \right)^k
\]

This is a simple algebraic-logarithmic function:

\[
\frac{s_k(n)}{n!} \sim \frac{1}{(k - 1)!} n^{-1} (\log n)^{k-1}
\]

• Derangements:

\[
D(z) = \frac{e^{-z}}{1 - z} \sim \frac{e^{-1}}{1 - z} \quad \Rightarrow \frac{d(n)}{n!} \sim e^{-1}
\]
1. Coefficients of simple algebraic-logarithmic series

The basic series:

\[ S(z) = \frac{1}{(1 - t)^\alpha} \left( \frac{1}{t} \log \frac{1}{1 - t} \right)^\beta \]

\[ \implies s(n) = \frac{n^{\alpha-1}}{\Gamma(\alpha)} (\log n)^\beta \left(1 + O(1/\log n)\right) \]

Remarks

- if \( \alpha = 0, -1, -2, -3, \ldots \), then \( 1/\Gamma(\alpha) = 0 \) and this should be read

\[ s(n) = O \left( n^{\alpha-1} (\log n)^{\beta-1} \right) \]

Actually

\[ [z^n](1 - z)^k \left( \frac{1}{t} \log \frac{1}{1 - t} \right)^\beta = (-1)^k \beta! n^{-k-1} (\log n)^{\beta-1} \left(1 + O(1/\log n)\right) \]

- The numbers \( s(n) \) can always be expanded further.
- Maple does this for you!
2. Transfer theorems: \( \triangle \)-domains

- If \( A(z) \) is analytic in a \( \triangle \)-domain, and

\[
A(z) = O(R(z)) \quad \text{as } z \to 1,
\]

where \( R(z) \) is a simple alg-log function, then

\[
a(n) = O(r(n)) \quad \text{as } n \to \infty.
\]

- Similarly:

\[
A(z) = o(R(z)) \quad \Rightarrow a(n) = o(r(n))
\]

\[
A(z) \sim R(z) \quad \Rightarrow a(n) \sim r(n)
\]

[Flajolet-Odlyzko 90]
Examples of series analytic in a $\Delta$-domain

$$D(z) = \frac{e^{-z}}{1 - z}$$

$$C(z) = (1-z) \left( 1 - \frac{2\sqrt{2}}{3\sqrt{2} - \sqrt{(1+z)/(1-z)(1-z + \sqrt{1-6z+z^2})}} \right)$$

Counter-examples:

$$A(z) = \frac{e^z}{\sqrt{1 - z^2}}$$  two dominant singularities

$$P(z) = \frac{1}{(1-z)(1-z^2)(1-z^3) \ldots}$$  natural boundary
Several dominant singularities

- If \( A(z) \) is analytic in a generalized \( \Delta \)-domain, and, in the neighborhood of each dominant singularity \( z_i \),

\[
A(z) = H_i(z) + O((1 - z/z_i)^\alpha) \quad \text{as } z \to z_i,
\]

where \( H_i \) is analytic around \( z_i \), then

\[
a(n) = O(n^{\alpha-1}) \quad \text{as } n \to \infty.
\]

The contributions of the dominant singularities add up
Several dominant singularities: examples

Let

\[ A(z) = \frac{e^z}{\sqrt{1 - z^2}} \]

There are two dominant singularities, \( \pm 1 \).

\[ A(z) = \frac{e}{\sqrt{2(1 - z)}} + O(\sqrt{1 - z}) \quad \text{as } z \to 1, \]

\[ A(z) = \frac{e^{-1}}{\sqrt{2(1 + z)}} + O(\sqrt{1 + z}) \quad \text{as } z \to -1. \]

Hence

\[ a(n) = \frac{e}{\sqrt{2\pi n}} + \frac{(-1)^n e^{-1}}{\sqrt{2\pi n}} + O(n^{-3/2}). \]

Maple does this for you! [Algolib] http://algo.inria.fr/libraries/
**Implicit generating functions**

The generating function of labelled rooted trees is given by

\[ A(z) = z \exp(A(z)) \]

- The only dominant singularity is at \( z_c = 1/e \), and \( A(z_c) = 1 \).
- \( A(z) \) is analytically defined in a \( \Delta \)-domain
- Singular behaviour of \( A(z) \).

\[ A(z) = 1 - \sqrt{2}\sqrt{1 - ze} + o(\sqrt{1 - ze}) \quad \text{as } z \to 1/e \]

Hence

\[ \frac{a(n)}{n!} \sim \frac{-\sqrt{2}}{\Gamma(-1/2)} e^n n^{-3/2} \approx \frac{e^n n^{-3/2}}{\sqrt{2\pi}} \]
Proofs of the asymptotics of basic series:
The “right” integration contour

The basic series:

\[ S(z) = \frac{1}{(1-t)^\alpha} \left( \frac{1}{t} \log \frac{1}{1-t} \right)^\beta \quad \rightarrow \quad s(n) = \frac{1}{2i\pi} \int_C S(z) \frac{dz}{z^{n+1}} \]

\[ S(z) = (1-z)^{-2/3} \quad \text{and} \quad n = 4 \]

\[ r_n = 1 + \log^2 \frac{n}{n} \]
Why the Gamma function?

Hankel’s expression of the reciprocal of the Gamma function:

\[
\frac{1}{\Gamma(s)} = \frac{1}{2i\pi} \int_{\mathcal{H}} (-z)^{-s} e^{-z} dz
\]
Proof of the transfer theorems: The “right” integration contour
Third method: Saddle point asymptotics

- Ph. Flajolet and R. Sedgewick, Analytic Combinatorics
  http://algo.inria.fr/flajolet/Publications/books.html
  Chapter VIII
Saddle point bounds

• Let \( A(z) = \sum_n a(n)z^n \) be a power series with non-negative coefficients and radius of cv. \( \rho \). Then

\[
a(n) \leq \frac{A(x)}{x^n} \quad \text{for all } 0 < x < \rho
\]

• Let \( r = x \) be chosen so as to minimize \( A(x)/x^n \):

\[
\frac{rA'(r)}{A(r)} = n \quad \text{(saddle-point equation)}
\]

Then

\[
a(n) \leq \frac{A(r)}{r^n}.
\]
Saddle point bounds

- Let \( r = x \) be chosen so as to minimize \( A(x)/x^n \):

\[
\frac{rA'(r)}{A(r)} = n \quad \text{ (saddle-point equation)}
\]

Then

\[
a(n) \leq \frac{A(r)}{r^n}.
\]

Example: the exponential

Take \( A(z) = e^z = \sum z^n/n! \). Then \( A'(z) = e^z = A(z) \) and \( r = n \)

\[
\frac{1}{n!} \leq \left( \frac{e}{n} \right)^n
\]

Cf. Stirling’s formula

\[
\frac{1}{n!} = \left( \frac{e}{n} \right)^n \frac{1}{\sqrt{2\pi n}} (1 + O(1/n))
\]
Saddle point estimates

We have seen

\[ a(n) \leq \frac{A(r)}{r^n} \quad \text{with} \quad \frac{rA'(r)}{A(r)} = n. \]

**Theorem.** Under suitable conditions (…)

\[ a(n) \sim \frac{A(r)}{r^n \sqrt{2r^2 \pi G''(r)}} \]

with \( G(z) = \log \frac{A(z)}{z^n} \)
Proof: integration on a circle

\[
\frac{A(z)}{z^n} = \exp(G(z))
\]

\[
a(n) = \frac{1}{2i\pi} \int_C \frac{A(z) \, dz}{z^n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(G(re^{i\theta})) \, d\theta
\]

What are the suitable conditions?

- The part of integral before \(-\theta_n\) and after \(\theta_n\) is negligible

- For \(z = r_n e^{i\theta}\), with \(\theta \in [-\theta_n, \theta_n]\), the quadratic approximation holds:

\[
G(z) = G(r_n) - \frac{1}{2} r_n^2 \theta^2 G'''(r_n) + o(1)
\]

- \(\theta\) is large enough to complete the integral to a Gaussian integral

\[
r_n^2 \theta_n^2 G'''(r_n) \to \infty
\]
Examples

• Involution

\[ A(z) = \exp(z + z^2/2) \]

• Bell numbers (set partitions)

\[ A(z) = \exp(e^z - 1) \]

• A fast growing singular function

\[ A(z) = \exp\left(\frac{z}{1-z}\right) \]

• Integer partitions

\[ A(z) = \frac{1}{(1-z)(1-z^2)(1-z^3)\ldots} \]

Maple does this for you... [Algolib ]
Applications of singularity analysis:
Automated asymptotics for algebraic generating functions

Def. The series $A(t)$ is algebraic if there exists a non-trivial polynomial $P(\cdot, \cdot)$ such that $P(t, A(t)) = 0$

Example: Dyck paths: $A(t) = 1 + t^2 A(t)^2$

- Ph. Flajolet and R. Sedgewick, Analytic Combinatorics, Chapter VII
  http://algo.inria.fr/flajolet/Publications/books.html
Let $A(t)$ be a solution of $P(A(t)) = 0$.

- The singularities of $A(t)$ are found among:
  - the roots of the dominant coefficient of $P$
  - the roots of the discriminant of $P$

- In the neighborhood of a singularity $z_c$, $A(z)$ admits a local expansion of the form

$$A(z) = \sum_{k \geq k_0} b_k (1 - z/z_c)^{k/d}$$

with $k_0 \in \mathbb{Z}$ and $d \in \{1, 2, \ldots\}$

- Singularity analysis:

$$a(n) \sim b_{k_0} z_c^{-n} n^{k_0/d - 1} \Gamma(k_0/n_d)$$
Applications: (almost) automated asymptotics for D-finite generating functions

**Def.** The series $A(t)$ is **D-finite** if its satisfies a linear differential equation with polynomial coefficients:

$$P_e(t)A^{(e)}(t) + \cdots + P_1(t)A'(t) + P_0(t)A(t) = 0$$

Equivalently, the sequence $a(n)$ satisfies a linear recurrence relation with polynomial coefficients

$$Q_0(n)a(n) + Q_1(n)a(n - 1) + \cdots + Q_d(n)a(n - d) = 0$$

---

Singular behaviour of D-finite series

Let $A(t)$ be D-finite:

$$P_e(t)A^{(e)}(t) + \cdots + P_1(t)A'(t) + P_0(t)A(t) = 0$$

- The singularities of $A$ are found among the roots of $P_e(t)$

- In the neighborhood of a **regular root** $z_c$ of $P_e$, the solutions of the ODE have a **regular** local expansion formed of terms

$$
(1 - z/z_c)^{\sigma} \log \left( \frac{1}{1 - z/z_c} \right)^i
$$

- In the neighborhood of a **irregular root** $z_c$ of $P_e$, the expansions may involve terms of the form

$$
\exp(P(1/w))R(w)
$$

where $R$ is regular, $P$ is a polynomial and $w = (1 - z/z_c)^{1/d}$