

# Spanning forests in regular planar maps

Mireille Bousquet-Mélou

joint work with Julien Courtiel

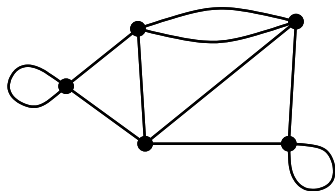
CNRS, LaBRI, Université de Bordeaux



# Planar maps

## Definition

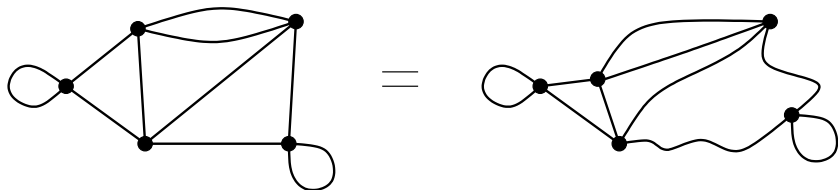
Planar map = connected planar graph  
+ embedding of this graph in the plane,  
taken up to continuous deformation



# Planar maps

## Definition

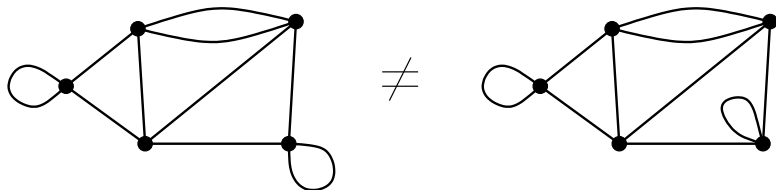
Planar map = connected planar graph  
+ embedding of this graph in the plane,  
taken up to continuous deformation



# Planar maps

## Definition

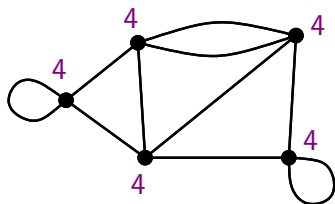
Planar map = connected planar graph  
+ embedding of this graph in the plane,  
taken up to continuous deformation



# Planar maps

## Definition

Planar map = connected planar graph  
+ embedding of this graph in the plane,  
taken up to continuous deformation



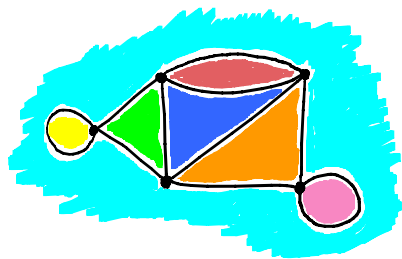
A **regular** (4-valent) map

# Planar maps

## Definition

Planar map = connected planar graph  
+ embedding of this graph in the plane,  
taken up to continuous deformation

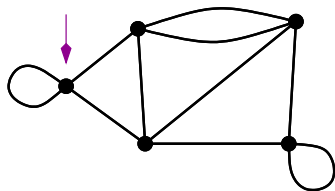
faces



# Planar maps

## Definition

Planar map = connected planar graph  
+ embedding of this graph in the plane,  
taken up to continuous deformation

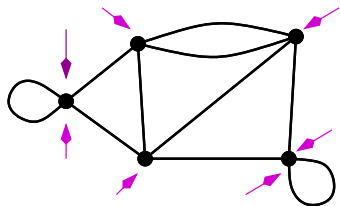


Maps are **rooted** at an external corner

# Planar maps

## Definition

Planar map = connected planar graph  
+ embedding of this graph in the plane,  
taken up to continuous deformation



Maps are **rooted** at an external corner



# Map enumeration: In combinatorics and statistical physics

- A typical question: how many rooted 4-valent maps with  $n$  vertices?
  - Tutte and his descendents (1960  $\rightarrow \dots$ )
  - Brézin-Itzykson-Parisi-Zuber and their descendents (1978  $\rightarrow \dots$ )

# Map enumeration: In combinatorics and statistical physics

- A typical question: how many rooted 4-valent maps with  $n$  vertices?
  - Tutte and his descendents (1960  $\rightarrow \dots$ )
  - Brézin-Itzykson-Parisi-Zuber and their descendents (1978  $\rightarrow \dots$ )
- Key object: the generating function of 4-valent maps, counted by vertices:

$$M(t) := \sum_M t^{\text{vertices}(M)} = 1 + 2t + 9t^2 + O(t^3)$$

# Map enumeration: In combinatorics and statistical physics

- A typical question: how many rooted 4-valent maps with  $n$  vertices?
  - Tutte and his descendents (1960  $\rightarrow \dots$ )
  - Brézin-Itzykson-Parisi-Zuber and their descendents (1978  $\rightarrow \dots$ )
- Key object: the generating function of 4-valent maps, counted by vertices:

$$\begin{aligned}M(t) &:= \sum_M t^{\text{vertices}(M)} = 1 + 2t + 9t^2 + O(t^3) \\ &= \sum_{n \geq 0} \frac{2 \cdot 3^n}{(n+1)(n+2)} \binom{2n}{n} t^n \\ &= \frac{(1-12t)^{3/2} - 1 + 18t}{54t^2}\end{aligned}$$

Arquès Bauer Bédard Bender Bernardi Bessis Bodirsky  
Bousquet-Mélou Boulatov Bouttier Brézin Brown Canfield Chauve  
Cori Di Francesco Duplantier Eynard Fusy Gao Goupil Goulden  
Gitter t'Hooft Itzykson Jackson Jacquard Kazakov Kostov Krikun  
Labelle Lehman Leroux Liskovets Liu Machì Mehta Mullin Parisi  
Poulalhon Richmond Robinson Schaeffer Schellenberg Sportiello  
Strehl **Tutte** Vainshtein Vauquelin Visentin Walsh Wanless  
Wormald Zinn-Justin Zuber Zvonkine...

# Planar maps (tend to) have **algebraic** generating functions

$M(t_1, \dots, t_k)$  map GF: there exists a polynomial  $Q$  such that

$$Q(t_1, \dots, t_k; M(t_1, \dots, t_k)) = 0.$$

Arquès Bauer Bédard Bender Bernardi Bessis Bodirsky  
Bousquet-Mélou Boulatov Bouttier Brézin Brown Canfield Chauve  
Cori Di Francesco Duplantier Eynard Fusy Gao Goupil Goulden  
Guitter t'Hooft Itzykson Jackson Jacquard Kazakov Kostov Krikun  
Labelle Lehman Leroux Liskovets Liu Machì Mehta Mullin Parisi  
Poulalhon Richmond Robinson Schaeffer Schellenberg Sportiello  
Strehl **Tutte** Vainshtein Vauquelin Visentin Walsh Wanless  
Wormald Zinn-Justin Zuber Zvonkine...

# Planar maps (tend to) have **algebraic** generating functions

$M(t_1, \dots, t_k)$  map GF: there exists a polynomial  $Q$  such that

$$Q(t_1, \dots, t_k; M(t_1, \dots, t_k)) = 0.$$

Example:

$$M(t) = \frac{(1 - 12t)^{3/2} - 1 + 18t}{54t^2} \Rightarrow (54t^2 M(t) + 1 - 18t)^2 = (1 - 12t)^3$$

Arquès Bauer Bédard Bender Bernardi Bessis Bodirsky  
Bousquet-Mélou Boulatov Bouttier Brézin Brown Canfield Chauve  
Cori Di Francesco Duplantier Eynard Fusy Gao Goupil Goulden  
Guitter t'Hooft Itzykson Jackson Jacquard Kazakov Kostov Krikun  
Labelle Lehman Leroux Liskovets Liu Machì Mehta Mullin Parisi  
Poulalhon Richmond Robinson Schaeffer Schellenberg Sportiello  
Strehl **Tutte** Vainshtein Vauquelin Visentin Walsh Wanless  
Wormald Zinn-Justin Zuber Zvonkine...

# Large random maps

Maps in probability theory:

- the diameter behaves like  $n^{1/4}$
- rescaled by  $n^{1/4}$ , a planar map converges to a (universal) limiting object, the **Brownian map**

And much more!

[Angel, Curien, Marckert, Le Gall, Miermont...]

# Maps equipped with an additional structure

In combinatorics, but mostly in statistical physics

- **How many maps equipped with...**

- a spanning tree?  
[Mullin 67]
- a spanning forest?  
[Bouttier et al., Sportiello et al.]
- a self-avoiding walk?  
[Duplantier-Kostov 88]
- a proper  $q$ -colouring?  
[Tutte 74-83, Bouttier et al. 02]

- **What is the expected partition function of...**

- the Ising model? [Boulatov, [Kazakov, MBM, Schaeffer, Bouttier et al.]
- the hard-particle model? [MBM, Schaeffer, Jehanne, Bouttier et al. 02, 07]
- the Potts model? [Eynard-Bonnet 99, Baxter 01, MBM-Bernardi 09, Guionnet et al., Borot et al. 12]



# Maps equipped with an additional structure

In combinatorics, but mostly in statistical physics

- **How many maps equipped with...**

- a spanning tree?  
[Mullin 67]
- a spanning forest?  
[Bouttier et al., Sportiello et al.]
- ... and this talk
- a self-avoiding walk?  
[Duplantier-Kostov 88]
- a proper  $q$ -colouring?  
[Tutte 74-83, Bouttier et al. 02]

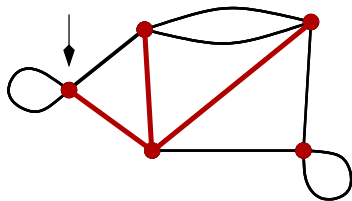
- **What is the expected partition function of...**

- the Ising model? [Boulatov, Kazakov, MBM, Schaeffer, Bouttier et al.]
- the hard-particle model? [MBM, Schaeffer, Jehanne, Bouttier et al. 02, 07]
- the Potts model? [Eynard-Bonnet 99, Baxter 01, MBM-Bernardi 09, Guionnet et al., Borot et al. 12]

# Forested maps

## Definition

Forested map = planar map  $M$  + spanning forest  $F$

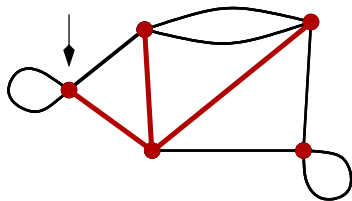


A spanning forest with **two** connected components

# Forested maps

## Definition

Forested map = planar map  $M$  + spanning forest  $F$



A spanning forest with **two** connected components

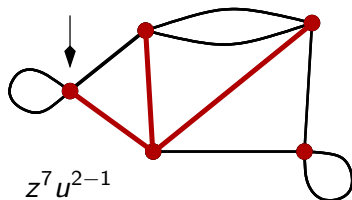
## Generating function

$$F(z, u) = \sum_{M, F} z^{\text{faces}(M)} u^{\text{components}(F)-1}$$

# Forested maps

## Definition

Forested map = planar map  $M$  + spanning forest  $F$



A spanning forest with **two** connected components

## Generating function

$$F(z, u) = \sum_{M, F} z^{\text{faces}(M)} u^{\text{components}(F)-1}$$

# Special values of $u$

## Generating function

$$F(z, u) = \sum_{M, F} z^{\text{faces}(M)} u^{\text{components}(F)-1}$$

- $u = 1$ : unweighted spanning forests

# Special values of $u$

## Generating function

$$F(z, u) = \sum_{M, F} z^{\text{faces}(M)} u^{\text{components}(F)-1}$$

- $u = 1$ : unweighted spanning forests
- $u = 0$ : unweighted spanning trees [Mullin 67]

# Special values of $u$

## Generating function

$$F(z, u) = \sum_{M, F} z^{\text{faces}(M)} u^{\text{components}(F)-1}$$

- $u = 1$ : unweighted spanning forests
- $u = 0$ : unweighted spanning trees [Mullin 67]
- $u = -1$ : root-connected acyclic orientations on (dual) quadrangulations [Las Vergnas 84]

Connections with other models: “physical” meaning for

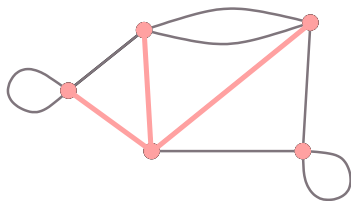
$$u \geq -1$$

- Connected subgraphs on quadrangulations (counted by cycles)



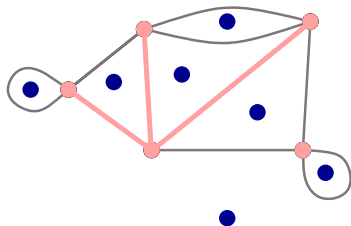
# Connections with other models: “physical” meaning for $u \geq -1$

- Connected subgraphs on quadrangulations (counted by cycles)



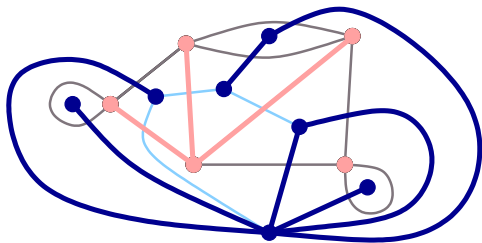
# Connections with other models: “physical” meaning for $u \geq -1$

- Connected subgraphs on quadrangulations (counted by cycles)



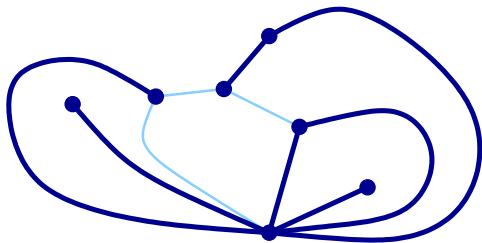
# Connections with other models: “physical” meaning for $u \geq -1$

- Connected subgraphs on quadrangulations (counted by cycles)



# Connections with other models: “physical” meaning for $u \geq -1$

- Connected subgraphs on quadrangulations (counted by cycles)



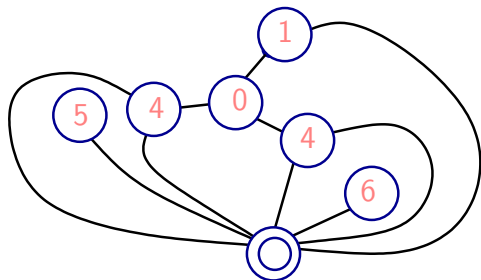
Connections with other models: “physical” meaning for

$$u \geq -1$$

- Connected subgraphs on quadrangulations (counted by cycles)
- Tutte polynomial  $T_M(u + 1, 1)$

# Connections with other models: “physical” meaning for $u \geq -1$

- Connected subgraphs on quadrangulations (counted by cycles)
- Tutte polynomial  $T_M(u+1, 1)$
- Sandpile model [Merino Lopez 97, Cori & Le Borgne 03]



$$T_M(u+1, 1) =$$

$$\sum_{c \text{ recurrent}} (u+1)^{\text{level}(c)}$$

# Connections with other models: “physical” meaning for $u \geq -1$

- Connected subgraphs on quadrangulations (counted by cycles)
- Tutte polynomial  $T_M(u+1, 1)$
- Sandpile model [Merino Lopez 97, Cori & Le Borgne 03]
- Limit  $q \rightarrow 0$  of the  $q$ -state Potts model:

$$\lim_{q \rightarrow 0} \frac{1}{q} Z_M(q, u+1) = u^{\text{vertices}(M)} \sum_{F \text{ forest}} u^{\text{components}(F)-1}$$

with  $Z_M(q, \nu)$  the partition function of the Potts model ( $q$  counts states,  $\nu$  monochromatic edges)

# Objectives

- Exact enumeration:
  - ▶ Generating function  $F(z, u)$  of forested maps
  - ▶ Nature of  $F$ :  
is it algebraic? D-finite? D-algebraic?



# Objectives

- Exact enumeration:

- ▶ Generating function  $F(z, u)$  of forested maps
- ▶ Nature of  $F$ :  
is it algebraic? D-finite? D-algebraic?

i.e., does it satisfy a polynomial

equation with coefficients in  $\mathbb{Q}(z, u)$ ?

# Objectives

- Exact enumeration:

- ▶ Generating function  $F(z, u)$  of forested maps
- ▶ Nature of  $F$ :  
is it algebraic? D-finite? D-algebraic?

i.e., does it satisfy a polynomial / linear differential equation with coefficients in  $\mathbb{Q}(z, u)$ ?

# Objectives

- Exact enumeration:

- ▶ Generating function  $F(z, u)$  of forested maps
- ▶ Nature of  $F$ :  
is it algebraic? D-finite? D-algebraic?

i.e., does it satisfy a polynomial / linear differential / polynomial differential/ equation with coefficients in  $\mathbb{Q}(z, u)$ ?

# Objectives

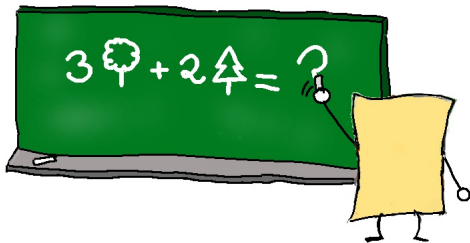
- Exact enumeration:
  - ▶ Generating function  $F(z, u)$  of forested maps
  - ▶ Nature of  $F$ :  
is it algebraic? D-finite? D-algebraic?

i.e., does it satisfy a polynomial / linear differential / polynomial differential/ equation with coefficients in  $\mathbb{Q}(z, u)$ ?

- Asymptotic enumeration and phase transition(s):
  - ▶ For  $u \geq -1$ , asymptotic behaviour of

$$f_n(u) := [z^n]F(z, u)$$

## Exact enumeration



## Theorem [Mullin 67]

The generating function of 4-valent maps equipped with a spanning tree satisfies

$$F'(z, 0) = 4 \sum_{i \geq 2} \frac{(3i - 2)!}{(i - 2)!i!^2} z^i$$

A D-finite, but not algebraic series

# Exact enumeration

## Theorem [MBM & Courtiel 13]

The generating function of 4-valent maps equipped with a spanning forest satisfies

$$F'(z, u) = 4 \sum_{i \geq 2} \frac{(3i - 2)!}{(i - 2)! i!^2} R^i$$

where  $R \equiv R(z, u)$  is the only series in  $z$  with constant term 0 such that

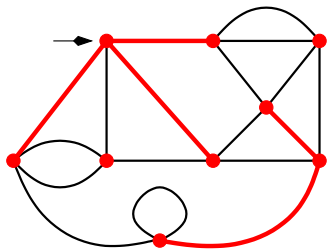
$$R = z + u \sum_{i \geq 2} \frac{(3i - 3)!}{(i - 1)!^2 i!} R^i.$$

# A combinatorial proof

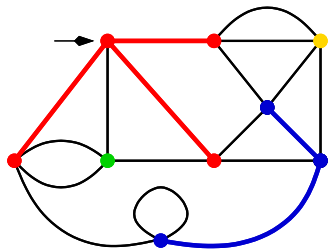
*Blocked edges on Eulerian maps and mobiles: application to spanning trees, hard particles and the Ising model,*  
Bouttier, Di Francesco & Guitter (2008)



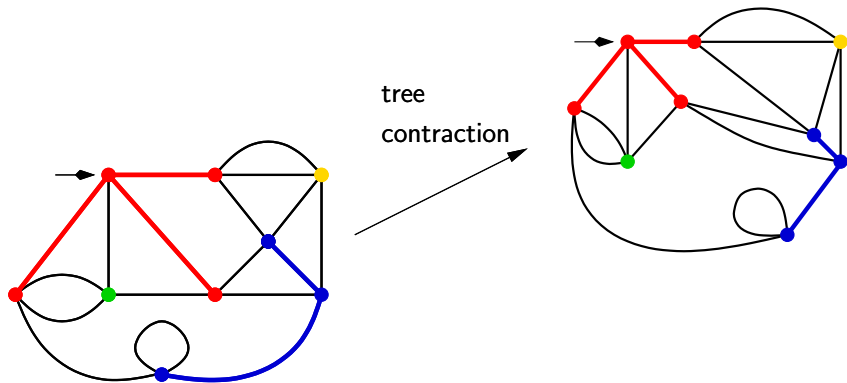
## Forested maps as decorated Eulerian maps



# Forested maps as decorated Eulerian maps

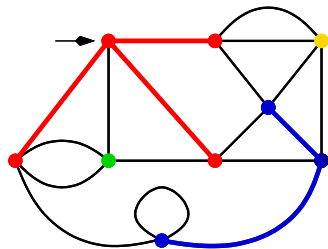


# Forested maps as decorated Eulerian maps

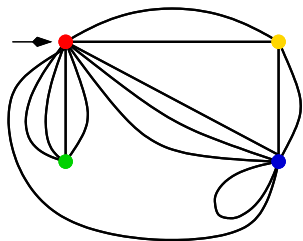


# Forested maps as decorated Eulerian maps

Eulerian map (even degrees)

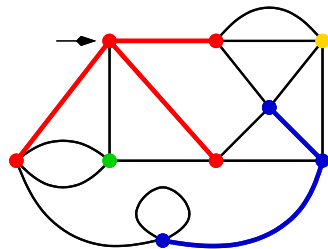


tree  
contraction

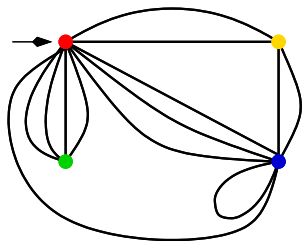


# Forested maps as decorated Eulerian maps

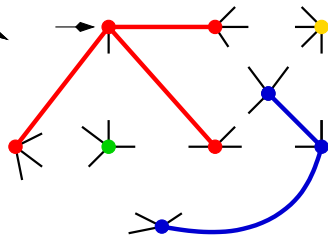
Eulerian map (even degrees)



tree  
contraction

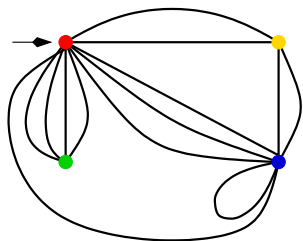


tree  
extraction



# Forested maps as decorated Eulerian maps

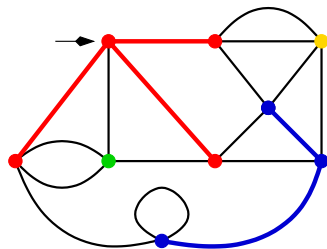
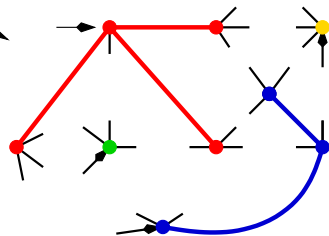
Eulerian map (even degrees)



tree  
contraction

BIJECTION

tree  
extraction



# Translation into generating functions

Let  $M(z; g_1, g_2, \dots; h_1, h_2, \dots)$  be the generating function of rooted Eulerian maps, with weight

- $z$  per face,
- $g_i$  per non-root vertex of degree  $2i$ ,
- $h_i$  per root-vertex (!) of degree  $2i$ .

# Translation into generating functions

Let  $M(z; g_1, g_2, \dots; h_1, h_2, \dots)$  be the generating function of rooted Eulerian maps, with weight

- $z$  per face,
- $g_i$  per non-root vertex of degree  $2i$ ,
- $h_i$  per root-vertex (!) of degree  $2i$ .

The generating function of forested maps is:

$$F(z, u) = M(z; ut_1^\ell, ut_2^\ell, \dots; t_1^c, t_2^c, \dots)$$

with  $t_i^\ell$  (resp.  $t_i^c$ ) the number of 4-valent trees rooted at a leaf (resp. a corner) having  $2i$  leaves:

$$t_i^\ell = \frac{(3i-3)!}{(i-1)!(2i-1)!}, \quad t_i^c = 4 \frac{(3i-3)!}{(i-2)!(2i)!}$$



# The generating function of Eulerian maps is known

## Theorem [Bouttier & Guitter 12]

The generating function of Eulerian maps satisfies

$$M'(z; g_1, g_2, \dots; h_1, h_2, \dots) = \sum_{i \geq 1} h_i \binom{2i}{i} R^i$$

where  $R$  is the unique series satisfying

$$R = z + \sum_{i \geq 1} g_i \binom{2i-1}{i} R^i.$$

[Tutte 62, Schaeffer 97, Bouttier-Di Francesco-Guitter 02]

# Exact enumeration

## Theorem [MBM & Courtiel 13]

The generating function of 4-valent maps equipped with a spanning forest satisfies

$$F'(z, u) = 4 \sum_{i \geq 2} \frac{(3i-2)!}{(i-2)!i!^2} R^i = \theta(R)$$

where

$$R = z + u \sum_{i \geq 2} \frac{(3i-3)!}{(i-1)!^2 i!} R^i = z + u \Phi(R).$$

# Exact enumeration

## Theorem [MBM & Courtiel 13]

The generating function of 4-valent maps equipped with a spanning forest satisfies

$$F'(z, u) = 4 \sum_{i \geq 2} \frac{(3i - 2)!}{(i - 2)! i!^2} R^i = \theta(R)$$

where

$$R = z + u \sum_{i \geq 2} \frac{(3i - 3)!}{(i - 1)!^2 i!} R^i = z + u \Phi(R).$$

## Corollary: nature of $F(z, u)$

The series  $F(z, u)$  is D-algebraic (in  $z$ ).

Proof:  $\theta$  and  $\Phi$  are D-algebraic (in fact, D-finite).

## Want to see the equation?

- The series  $F'(z, u)$  satisfies a differential equation of order 2.

# Want to see the equation?

- The series  $F'(z, u)$  satisfies a differential equation of order 2.

$$\begin{aligned} & 9 F'^2 F''^5 u^6 + 36 F'^2 F''^3 F''' u^5 z + 144 F'^2 F''^4 u^5 - 12 (21 z - 1) F' F''^5 u^5 + 432 F'^2 F''^2 F''' u^4 z \\ & \quad - 48 (24 z - 1) F' F''^3 F''' u^4 z + 864 F'^2 F''^3 u^4 - 96 (27 z - 2) F' F''^4 u^4 \\ & + 4 (27 z - 1) (15 z - 1) F''^5 u^4 + 1728 F'^2 F'' F''' u^3 z - 288 (21 z - 2) F' F''^2 F''' u^3 z \\ & \quad + 10368 F' F''^2 u^2 z^3 + 16 (27 z - 1) (21 z - 1) F''^3 F''' u^3 z + 2304 F'^2 F''^2 u^3 \\ & \quad - 288 (31 z - 4) F' F''^3 u^3 - 64 (6 uz - 162 z^2 + 33 z - 1) F''^4 u^3 + 2304 F'^2 F''' u^2 z \\ & \quad - 2304 (6 z - 1) F' F'' F''' u^2 z - 192 (8 uz - 54 z^2 + 29 z - 1) F''^2 F''' u^2 z \\ & \quad - 768 (2 u + 189 z - 7) F'''^2 uz^3 + 2304 F'^2 F'' u^2 - 3072 (3 z - 1) F' F''^2 u^2 \\ & \quad - 192 (24 uz - 27 z^2 + 55 z - 2) F''^3 u^2 - 1536 (21 z - 2) F' F''' uz \\ & \quad - 768 (12 uz + 81 z^2 + 24 z - 1) F'' F''' uz + 1536 (9 z + 2) F' F'' u \\ & - 512 (39 uz + 81 z^2 + 51 z - 2) F''^2 u + 36864 F' z - 1024 (12 uz - 162 z^2 + 33 z - 1) F''' z \\ & \quad - 1024 (36 uz + 27 z - 1) F'' - 24576 z = 0. \end{aligned}$$

# Want to see the equation?

- The series  $F'(z, u)$  satisfies a differential equation of order 2.

$$\begin{aligned} &9 F'^2 F''^5 u^6 + 36 F'^2 F''^3 F''' u^5 z + 144 F'^2 F''^4 u^5 - 12 (21 z - 1) F' F''^5 u^5 + 432 F'^2 F''^2 F''' u^4 z \\ &\quad - 48 (24 z - 1) F' F''^3 F''' u^4 z + 864 F'^2 F''^3 u^4 - 96 (27 z - 2) F' F''^4 u^4 \\ &+ 4 (27 z - 1) (15 z - 1) F''^5 u^4 + 1728 F'^2 F'' F''' u^3 z - 288 (21 z - 2) F' F''^2 F''' u^3 z \\ &\quad + 10368 F' F''^2 u^2 z^3 + 16 (27 z - 1) (21 z - 1) F''^3 F''' u^3 z + 2304 F'^2 F''^2 u^3 \\ &\quad - 288 (31 z - 4) F' F''^3 u^3 - 64 (6 uz - 162 z^2 + 33 z - 1) F''^4 u^3 + 2304 F'^2 F''' u^2 z \\ &\quad - 2304 (6 z - 1) F' F'' F''' u^2 z - 192 (8 uz - 54 z^2 + 29 z - 1) F''^2 F''' u^2 z \\ &\quad - 768 (2 u + 189 z - 7) F'''^2 u z^3 + 2304 F'^2 F'' u^2 - 3072 (3 z - 1) F' F''^2 u^2 \\ &\quad - 192 (24 uz - 27 z^2 + 55 z - 2) F''^3 u^2 - 1536 (21 z - 2) F' F''' uz \\ &\quad - 768 (12 uz + 81 z^2 + 24 z - 1) F'' F''' uz + 1536 (9 z + 2) F' F'' u \\ &- 512 (39 uz + 81 z^2 + 51 z - 2) F''^2 u + 36864 F' z - 1024 (12 uz - 162 z^2 + 33 z - 1) F''' z \\ &\quad - 1024 (36 uz + 27 z - 1) F'' - 24576 z = 0. \end{aligned}$$

- No DE of order 2 for  $F(z, u)$  itself [Bostan, Salvy, Singer]

# Want to see the equation?

- The series  $F'(z, u)$  satisfies a differential equation of order 2.

$$\begin{aligned} &9 F'^2 F''^5 u^6 + 36 F'^2 F''^3 F''' u^5 z + 144 F'^2 F''^4 u^5 - 12 (21 z - 1) F' F''^5 u^5 + 432 F'^2 F''^2 F''' u^4 z \\ &\quad - 48 (24 z - 1) F' F''^3 F''' u^4 z + 864 F'^2 F''^3 u^4 - 96 (27 z - 2) F' F''^4 u^4 \\ &+ 4 (27 z - 1) (15 z - 1) F''^5 u^4 + 1728 F'^2 F'' F''' u^3 z - 288 (21 z - 2) F' F''^2 F''' u^3 z \\ &\quad + 10368 F' F''^2 u^2 z^3 + 16 (27 z - 1) (21 z - 1) F''^3 F''' u^3 z + 2304 F'^2 F''^2 u^3 \\ &\quad - 288 (31 z - 4) F' F''^3 u^3 - 64 (6 uz - 162 z^2 + 33 z - 1) F''^4 u^3 + 2304 F'^2 F''' u^2 z \\ &\quad - 2304 (6 z - 1) F' F'' F''' u^2 z - 192 (8 uz - 54 z^2 + 29 z - 1) F''^2 F''' u^2 z \\ &\quad - 768 (2 u + 189 z - 7) F''^2 uz^3 + 2304 F'^2 F'' u^2 - 3072 (3 z - 1) F' F''^2 u^2 \\ &\quad - 192 (24 uz - 27 z^2 + 55 z - 2) F''^3 u^2 - 1536 (21 z - 2) F' F''' uz \\ &\quad - 768 (12 uz + 81 z^2 + 24 z - 1) F'' F''' uz + 1536 (9 z + 2) F' F'' u \\ &- 512 (39 uz + 81 z^2 + 51 z - 2) F''^2 u + 36864 F' z - 1024 (12 uz - 162 z^2 + 33 z - 1) F''' z \\ &\quad - 1024 (36 uz + 27 z - 1) F'' - 24576 z = 0. \end{aligned}$$

- No DE of order 2 for  $F(z, u)$  itself [Bostan, Salvy, Singer]  
It could still be D-finite...

## A related D-algebraicity result for maps

Theorem [Bernardi & MBM 07–14]

The generating function of planar maps  $M$  weighted by their size (edge number) and their Potts partition function  $Z_M(q, \nu)$  is D-algebraic.

This is more general (for another family of maps). But:

- the proof is much harder
- little combinatorics, much algebra
- it is not clear how to derive asymptotic results and locate phase transitions.



## A related D-algebraicity result for maps

Theorem [Bernardi & MBM 07–14]

The generating function of planar maps  $M$  weighted by their size (edge number) and their Potts partition function  $Z_M(q, \nu)$  is D-algebraic.

This is more general (for another family of maps). But:

- the proof is much harder
- little combinatorics, much algebra
- it is not clear how to derive asymptotic results and locate phase transitions.

## A related D-algebraicity result for maps

Theorem [Bernardi & MBM 07–14]

The generating function of planar maps  $M$  weighted by their size (edge number) and their Potts partition function  $Z_M(q, \nu)$  is D-algebraic.

This is more general (for another family of maps). But:

- the proof is much harder
- little combinatorics, much algebra
- it is not clear how to derive asymptotic results and locate phase transitions.

# A related D-algebraicity result for maps

Theorem [Bernardi & MBM 07–14]

The generating function of planar maps  $M$  weighted by their size (edge number) and their Potts partition function  $Z_M(q, \nu)$  is D-algebraic.

This is more general (for another family of maps). But:

- the proof is much harder
- little combinatorics, much algebra
- it is not clear how to derive asymptotic results and locate phase transitions.

# Asymptotic results

# What we want to do

Take  $u \geq -1$ , and estimate

$$f_n(u) := [z^n]F(z, u)$$

- exponential growth  $\prec$  radius of convergence  $\rho_u$  of  $F(z, u)$
- correction terms  $\prec$  nature of the singularities of  $F(z, u)$  at  $\rho_u$  and more generally on the circle  $|z| = \rho_u$

## Singularity analysis

[Flajolet-Odlyzko 90], [Flajolet-Sedgewick 09]

## The radius of convergence of $F(z, u)$

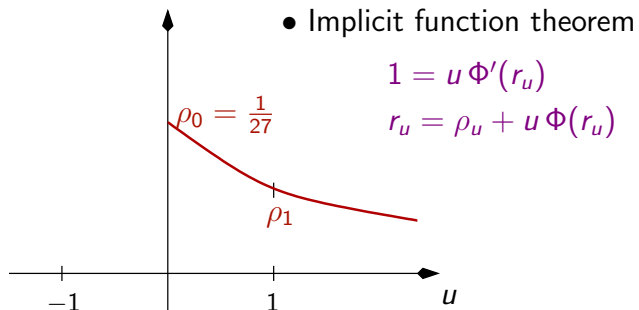
For  $u \geq -1$ , let  $\rho_u$  be the radius of convergence of  $F(z, u)$

$$F' = \theta(R) \quad \text{with} \quad R = z + u\Phi(R)$$

# The radius of convergence of $F(z, u)$

For  $u \geq -1$ , let  $\rho_u$  be the radius of convergence of  $F(z, u)$

$$F' = \theta(R) \quad \text{with} \quad R = z + u\Phi(R)$$

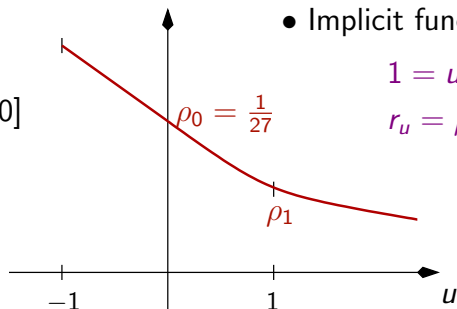


# The radius of convergence of $F(z, u)$

For  $u \geq -1$ , let  $\rho_u$  be the radius of convergence of  $F(z, u)$

$$F' = \theta(R) \quad \text{with} \quad R = z + u\Phi(R)$$

- $\rho$  is affine on  $[-1, 0]$



- Implicit function theorem

$$1 = u\Phi'(r_u)$$

$$r_u = \rho_u + u\Phi(r_u)$$



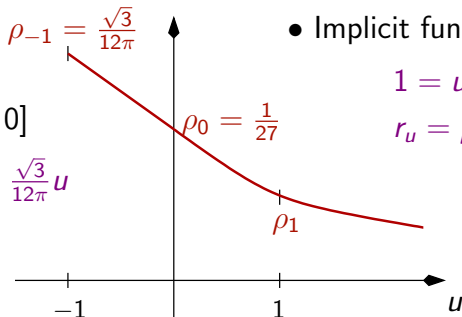
# The radius of convergence of $F(z, u)$

For  $u \geq -1$ , let  $\rho_u$  be the radius of convergence of  $F(z, u)$

$$F' = \theta(R) \quad \text{with} \quad R = z + u\Phi(R)$$

- $\rho$  is affine on  $[-1, 0]$

$$\rho_u = \frac{1}{27}(1+u) - \frac{\sqrt{3}}{12\pi}u$$



- Implicit function theorem

$$1 = u\Phi'(r_u)$$

$$r_u = \rho_u + u\Phi(r_u)$$

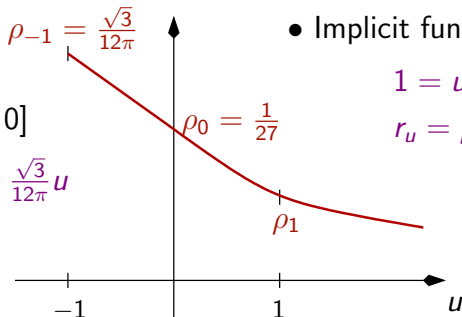
# The radius of convergence of $F(z, u)$

For  $u \geq -1$ , let  $\rho_u$  be the radius of convergence of  $F(z, u)$

$$F' = \theta(R) \quad \text{with} \quad R = z + u\Phi(R)$$

- $\rho$  is affine on  $[-1, 0]$

$$\rho_u = \frac{1}{27}(1+u) - \frac{\sqrt{3}}{12\pi}u$$



- Implicit function theorem

$$1 = u\Phi'(r_u)$$

$$r_u = \rho_u + u\Phi(r_u)$$

## Corollary

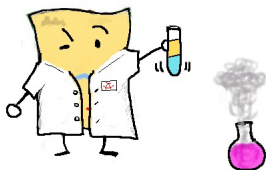
$F(z, -1)$  cannot be D-finite.

Proof: it has integer coefficients but a transcendental radius.

# The phase transition at $u = 0$

- $f_n(u) = [z^n]F(z, u)$

$-1 \leq u < 0$	$u = 0$	$u > 0$
$f_n(u) \sim \kappa_u \frac{\rho_u^{-n}}{n^3 \ln^2 n}$	$f_n(u) \sim \kappa_u \frac{\rho_u^{-n}}{n^3}$	$f_n(u) \sim \kappa_u \frac{\rho_u^{-n}}{n^{5/2}}$
new “universality class” for maps?	maps with a spanning tree	standard map behaviour



## Et encore...

- Generating function of  $p$ -valent forested maps, for any  $p \geq 3$
- Asymptotic behaviour in the cubic case  $\Rightarrow$  universality
- Large random forested maps:  $\simeq cn$  components, discrete limit law for the size of the root component
- Large random maps equipped with a sandpile configuration: phase transition for the *level* at  $u = 0$
- General forested planar maps (J. Courtiel, in preparation)

## Et encore...

- Generating function of  $p$ -valent forested maps, for any  $p \geq 3$
- Asymptotic behaviour in the cubic case  $\Rightarrow$  universality
- Large random forested maps:  $\simeq cn$  components, discrete limit law for the size of the root component
- Large random maps equipped with a sandpile configuration: phase transition for the *level* at  $u = 0$
- General forested planar maps (J. Courtiel, in preparation)

## Et encore...

- Generating function of  $p$ -valent forested maps, for any  $p \geq 3$
- Asymptotic behaviour in the cubic case  $\Rightarrow$  universality
- Large random forested maps:  $\simeq cn$  components, discrete limit law for the size of the root component
- Large random maps equipped with a sandpile configuration: phase transition for the level at  $u = 0$
- General forested planar maps (J. Courtiel, in preparation)

## Et encore...

- Generating function of  $p$ -valent forested maps, for any  $p \geq 3$
- Asymptotic behaviour in the cubic case  $\Rightarrow$  universality
- Large random forested maps:  $\simeq cn$  components, discrete limit law for the size of the root component
- Large random maps equipped with a sandpile configuration: phase transition for the level at  $u = 0$
- General forested planar maps (J. Courtiel, in preparation)

## Et encore...

- Generating function of  $p$ -valent forested maps, for any  $p \geq 3$
- Asymptotic behaviour in the cubic case  $\Rightarrow$  universality
- Large random forested maps:  $\simeq cn$  components, discrete limit law for the size of the root component
- Large random maps equipped with a sandpile configuration: phase transition for the level at  $u = 0$
- General forested planar maps (J. Courtiel, in preparation)



## Et encore...

- Generating function of  $p$ -valent forested maps, for any  $p \geq 3$
- Asymptotic behaviour in the cubic case  $\Rightarrow$  universality
- Large random forested maps:  $\simeq cn$  components, discrete limit law for the size of the root component
- Large random maps equipped with a sandpile configuration: phase transition for the level at  $u = 0$
- General forested planar maps (J. Courtiel, in preparation)

