

Decreasing subsequences
and
Wilf-equivalence for involutions

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Two papers

- [BMS] M. Bousquet-Mélou, E. Steingrímsson, Decreasing subsequences and Wilf-equivalence for involutions, to appear in *JACO*.

- [BWX] J. Backelin, J. West, G. Xin, Wilf-equivalence for singleton classes, *FPSAC 2001*, Tempe, AZ. To appear in *Adv. Appl. Math.*

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Permutation patterns: an occurrence of the pattern $\tau = \tau_1 \cdots \tau_k$ in the permutation $\pi = \pi_1 \cdots \pi_n$ is a subsequence $\pi_{i_1} \pi_{i_2} \cdots \pi_{i_k}$ of π that is order isomorphic to τ .

Example: the permutation $\pi = 8 \ 6 \ 3 \ 5 \ 7 \ 1 \ 4 \ 2 \ 9$ contains the pattern $\tau = 4 \ 3 \ 2 \ 1$.

$S_n(\tau)$ is the set of permutations in S_n avoiding the pattern τ

The transformation ϕ_k on permutations ($k = 4$)

8 6 3 5 7 1 4 2 9

Find the lexicographically least occurrence of the pattern **4321**

The transformation ϕ_k on permutations ($k = 4$)

8 6 3 5 7 1 4 2 9

Find the lexicographically least occurrence of the pattern 4321 a.k.a. the A -sequence

The transformation ϕ_k on permutations ($k = 4$)

8 6 3 5 7 1 4 2 9

Shift it cyclically to the left

The transformation ϕ_k on permutations ($k = 4$)

6
8 3 5 7 1 4 2 9

The transformation ϕ_k on permutations ($k = 4$)

6
8 5 3 4 7 1 2 9

The transformation ϕ_k on permutations ($k = 4$)

8 5 3 4 7 1 2 6 9

new occurrence of 3214

The transformation ϕ_k on permutations ($k = 4$)

8 5 3 4 7 1 2 6 9

The transformation ϕ_k on permutations ($k = 4$)

8 5 3 4 7 1 2 6 9

The new \mathcal{A} -sequence

The transformation ϕ_k on permutations ($k = 4$)

8
5 3 4 7 1 2 6 9

The transformation ϕ_k on permutations ($k = 4$)

8
5 3 1 4 7 2 6 9

The transformation ϕ_k on permutations ($k = 4$)

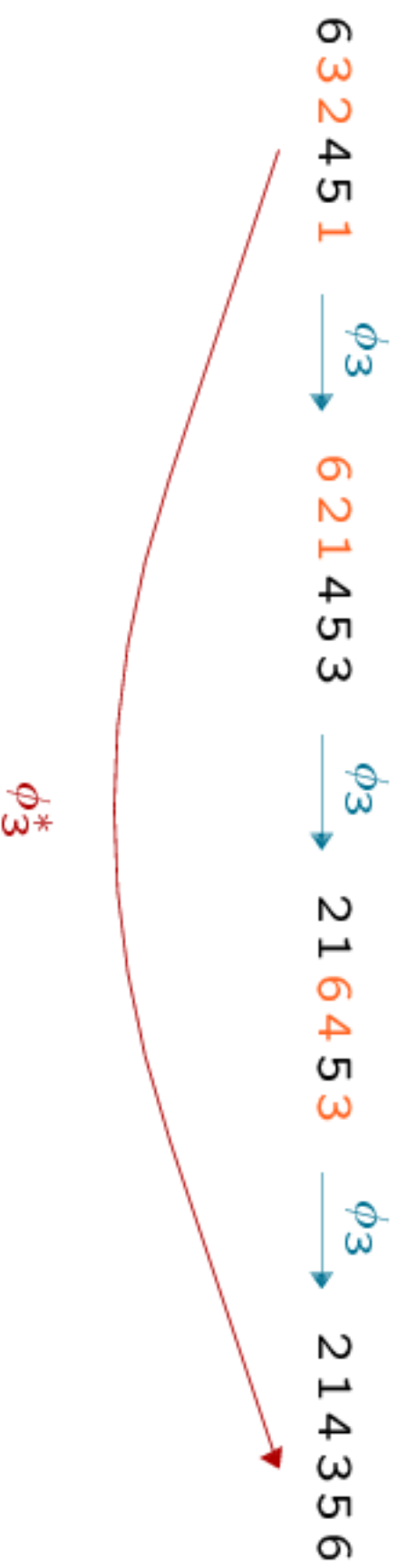
5 3 1 4 7 8 2 6 9

The transformation ϕ_k on permutations ($k = 4$)

5 3 1 4 7 8 2 6 9

No more occurrences of 4321

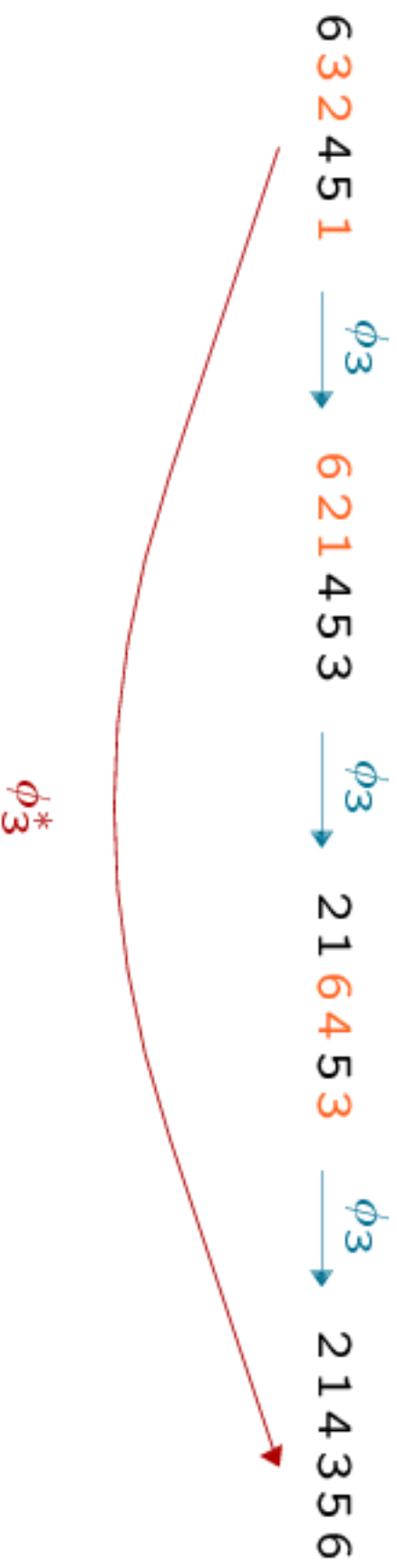
(But several occurrences of 3214)



- Let ϕ_k be the map that performs the cyclic shift of the *A*-sequence (the lexicographically least occurrence of $k \dots 21$)

- Let ϕ_k^* be the map that iterates ϕ_k until no occurrences of $k \dots 21$ are left:

$$\phi_k^* : S_n \rightarrow S_n(k \dots 21)$$



The BWX result (weak version): For all k and n ,

$$\phi_k^* : S_n((k-1) \cdots 21k) \longrightarrow S_n(k \cdots 21)$$

is a bijection.

The patterns $(k-1) \cdots 21k$ and $k \cdots 21$ are Wilf-equivalent.

The BWX result (weak version)

For any k and n ,

$$\phi_k^* : S_n((k-1) \cdots 21k) \longrightarrow S_n(k \cdots 21)$$

is a bijection.

Here, $S_n(\tau)$ denotes the number of permutations of length n avoiding the pattern τ .

The BWX result (strong version)

For any k and any Ferrers shape λ ,

$$\phi_k^* : S_\lambda((k-1) \cdots 21k) \longrightarrow S_\lambda(k \cdots 21)$$

is a bijection.

Here, $S_\lambda(\tau)$ denotes the number of full rook placements on the board λ avoiding the pattern τ .

The BWX result (strong version)

For any k and any Ferrers shape λ ,

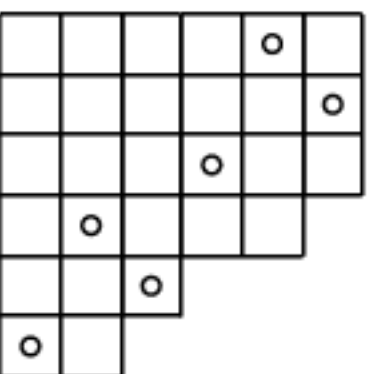
$$\phi_k^* : S_\lambda((k-1) \cdots 21k) \longrightarrow S_\lambda(k \cdots 21)$$

is a bijection.

Here, $S_\lambda(\tau)$ denotes the number of full rook placements on the board λ avoiding the pattern τ .

$$\triangleleft \triangleleft \diamond \triangleright \triangleright$$

A full rook placement on a Ferrers board λ
(one dot in each row and each column)



5 6 4 2 3 1

The BWX result (strong version)

For any k and any Ferrers shape λ ,

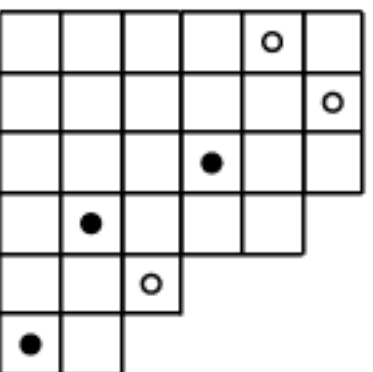
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The BWX result (strong version)

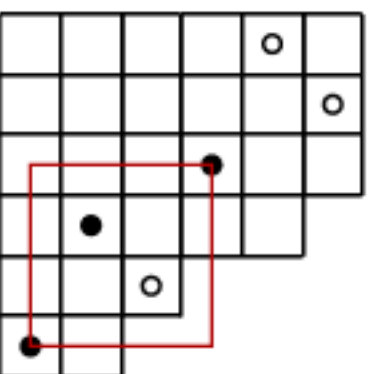
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5 6 4 2 3 1

Not an occurrence of 321
because rectangle is not
contained in board

The BWX result (strong version)

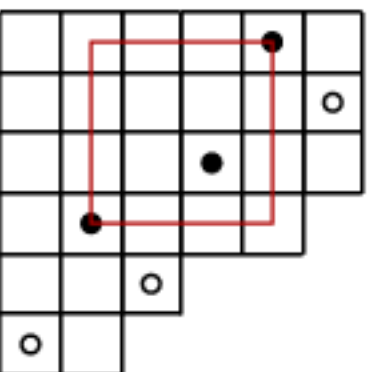
For any k and any Ferrers shape λ ,

$$\phi_k^* : S_\lambda((k-1)\cdots 21k) \longrightarrow S_\lambda(k\cdots 21)$$

is a bijection.

Here, $S_\lambda(\tau)$ denotes the number of full rook placements on the board λ avoiding the pattern τ .

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5 6 4 2 3 1

Lex-least occurrence
of 321 on this board
(\mathcal{A} -sequence)

The BWX result (strong version)

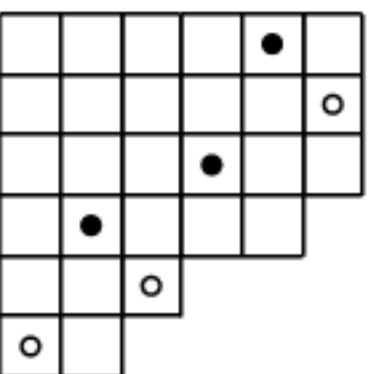
For any k and any Ferrers shape λ ,

$$\phi_k^* : S_\lambda((k-1) \cdots 21k) \longrightarrow S_\lambda(k \cdots 21)$$

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5

6 4 2 3 1

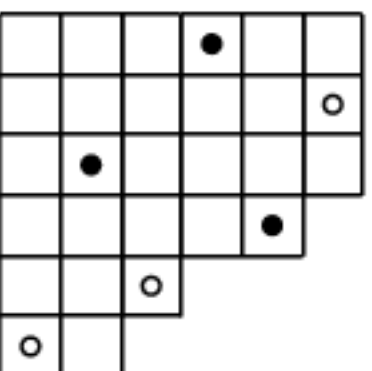
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For any k and any Ferrers shape λ ,

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4 6 2 5 3 1

The BWX result (strong version)

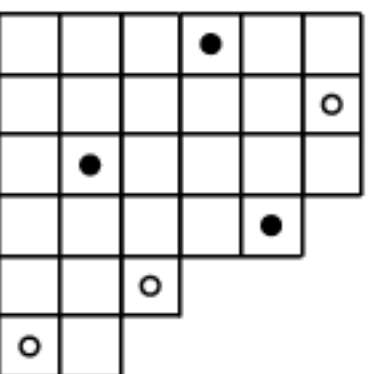
For any k and any Ferrers shape λ ,

$$\phi_k^* : S_\lambda((k-1) \cdots 21k) \longrightarrow S_\lambda(k \cdots 21)$$

is a bijection.

The patterns $(k-1) \cdots 21k$ and $k \cdots 21$ are shape-equivalent.

This is much stronger than the case where λ is a square!

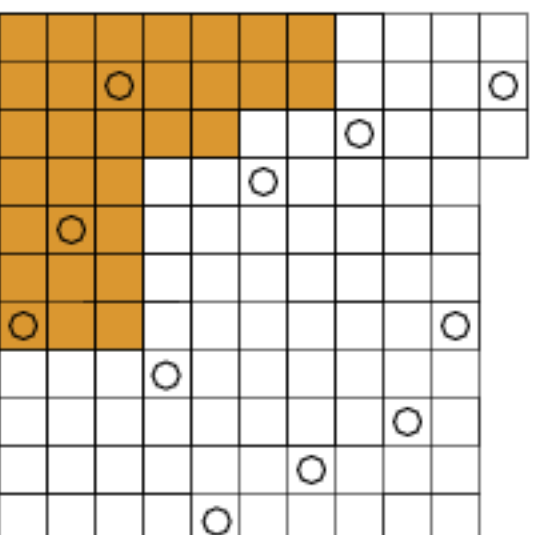


4 6 2 5 3 1

Why patterns on Ferrers boards?

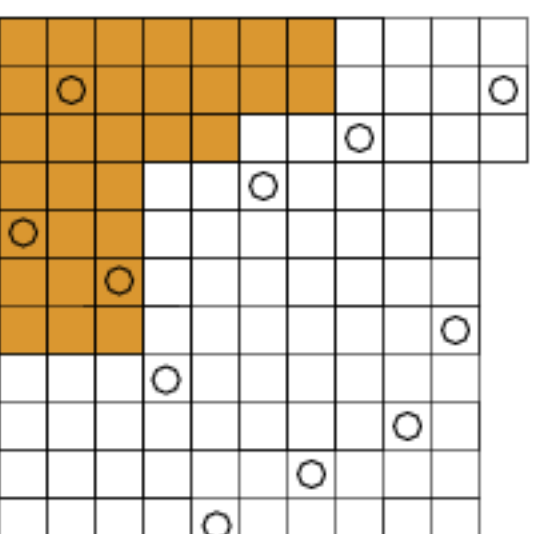
Let σ and σ' be two shape-equivalent permutations on $\{1, 2, \dots, k\}$.
Let τ be a permutation on $\{k+1, k+2, \dots, \ell\}$.

Then $\sigma\tau$ and $\sigma'\tau$ are also shape-equivalent.



π avoids **21345**

Apply ϕ^*
→
on the orange shape



π avoids **32145**

[BWX] A family of shape-equivalences

Since $k \cdots 21$ and $(k-1) \cdots 21k$ are shape-equivalent...

Coro: The patterns $k \cdots 21\tau$ and $(k-1) \cdots 21k\tau$ are shape-equivalent, for any pattern τ on the letters $k+1, \dots, \ell$.

That is, for any Ferrers shape λ ,

$$|S_\lambda(k \cdots 21\tau)| = |S_\lambda((k-1) \cdots 21k\tau)| = \dots = |S_\lambda(12 \cdots k\tau)|.$$

▷▷▷◇▷▷▷

“Reversing a small decreasing prefix gives a shape-equivalent permutation”.

Example: $|S_n(432168597)| = |S_n(123468597)|$

[BMS] A family of symmetric shape-equivalences

Analogous result for *involutions* and *symmetric rook placements*

Coro: The patterns $k \cdots 21\tau$ and $(k-1) \cdots 21k\tau$ are *symmetric shape-equivalent*, for any pattern τ on the letters $k+1, \dots, \ell$.

That is, for any *self-conjugate* shape λ ,

$$|\mathcal{I}_\lambda(k \cdots 21\tau)| = |\mathcal{I}_\lambda((k-1) \cdots 21k\tau)| = \dots = |\mathcal{I}_\lambda(12 \cdots k\tau)|,$$

where $\mathcal{I}_\lambda(\sigma)$ is the number of *diagonally symmetric rook placements* on the board λ that avoid σ .

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“Reversing a small decreasing prefix gives a symmetric shape-equivalent permutation”.

Example: $|\mathcal{I}_n(3214)| = |\mathcal{I}_n(1234)|$ (= Motzkin)

(conjectured by Guibert in 95, proved by Jaggard in 2003).

An unexpected property of the transformation ϕ_k^* [BMS]

- The map ϕ_k^* preserve involutions.

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BUT: The elementary step ϕ_k does not!

Example: try ϕ and ϕ^* on the involution 86435271, with $k=4$.

An unexpected property of the transformation ϕ_k^* [BMS]

- The map ϕ_k^* preserve involutions.

- **Even better**, it commutes with taking the inverse of a permutation:

$$\phi^*(\pi^{-1}) = (\phi^*(\pi))^{-1}.$$

- **Even better**, it commutes with taking the symmetric of a rook placement w.r.t. the main diagonal.

▷ ▷ ▷ ▷ ▷

BUT: The elementary step ϕ_k does not!

Example: try ϕ and ϕ^* on the involution 86435271, with $k=4$.

What does it mean?

“The map ϕ_k^* commutes with the taking the symmetric of placements”

Equivalently: Let ψ be the symmetric of the transformation ϕ :

$$\psi(\pi) := \left(\phi \left(\pi^{-1} \right) \right)^{-1} .$$

The map ϕ_k shifts to the left the “lowest” decreasing subsequence.

The map ψ_k shifts to the left the “leftmost” decreasing subsequence.

The transformations ϕ and ψ have similar definitions.

For all placements π , iterating ϕ_k or ψ_k results in the same $k \dots 21$ -avoiding placement:

$$\psi^*(\pi) = \phi^*(\pi).$$

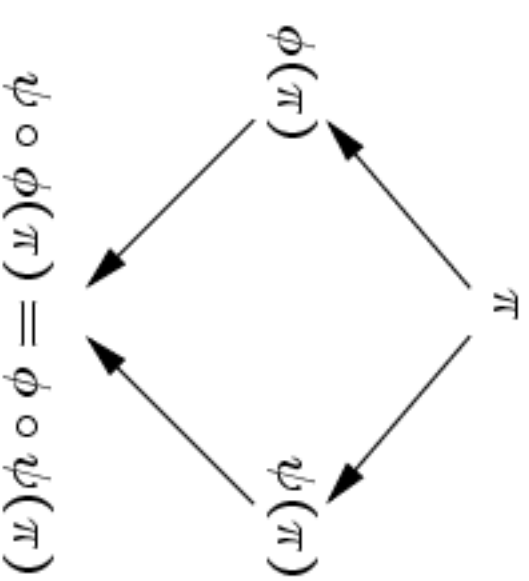
An even stronger property: local convergence

For any placement π ,

either

$$\begin{array}{c} \pi \\ \parallel \\ \parallel \\ \blacktriangleright \blacktriangleleft \\ \phi(\pi) = \psi(\pi) \end{array}$$

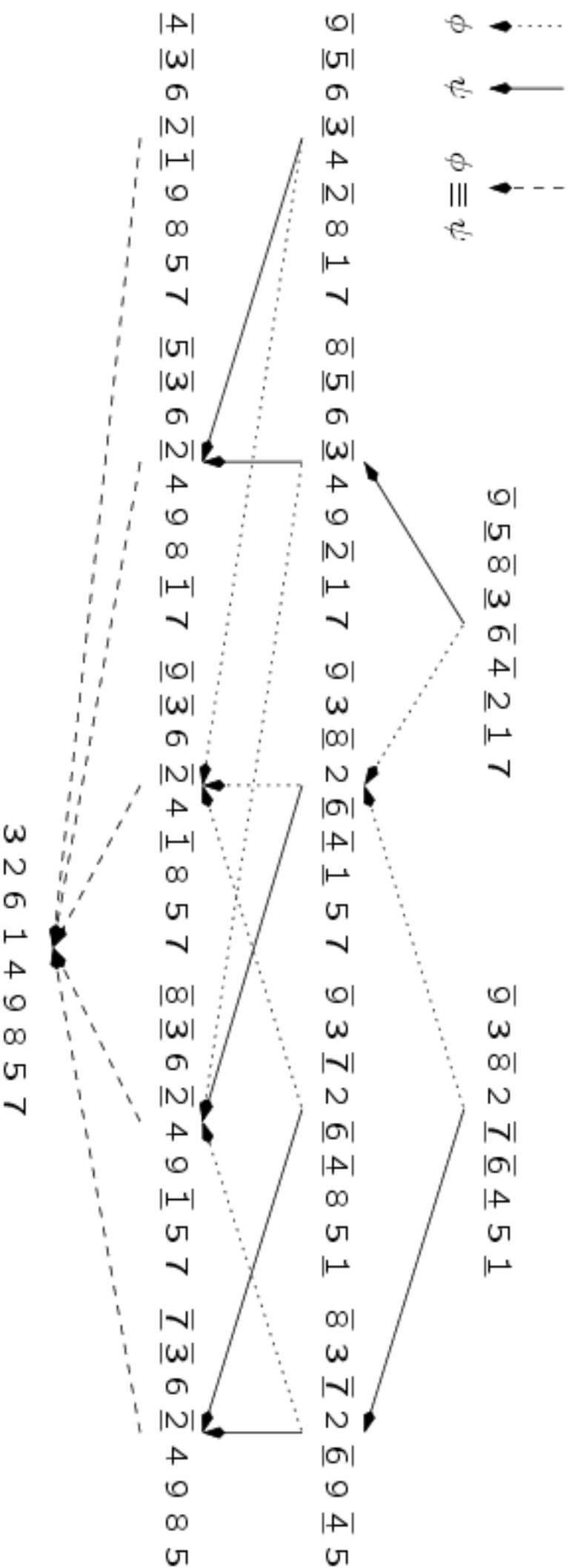
or



Coro [Global convergence]

Any iterated application of ϕ and ψ to a placement π yields ultimately to the same placement avoiding $k \dots 21$.

Example ($k = 4$)



Any iterated application of ϕ and ψ yields ultimately to the same placement avoiding $k \dots 21$.

Problem

- Give better explanations of the properties of ϕ_k^*
- Maybe via another description of ϕ_k^* (or ϕ_k)?
- Connection with the RSK correspondence?

$$T_n(12\dots k) = T_n(k\dots 21)$$

follows from the properties of the RSK correspondence by taking conjugates of standard tableaux.

Wilf-equivalence of permutations/involutions

12... $k\tau \equiv k\dots 21\tau$ (BWV 01)

231 $\tau \equiv 312\tau$ (Stankova-West 02)

4132 \equiv 3142 (Stankova 94)

12... $k\tau \equiv_I k\dots 21\tau$ (BMS 04)

54321 \equiv_I 45321 ?

54321 \equiv_I 45312 ?

Is this true? Is this all?