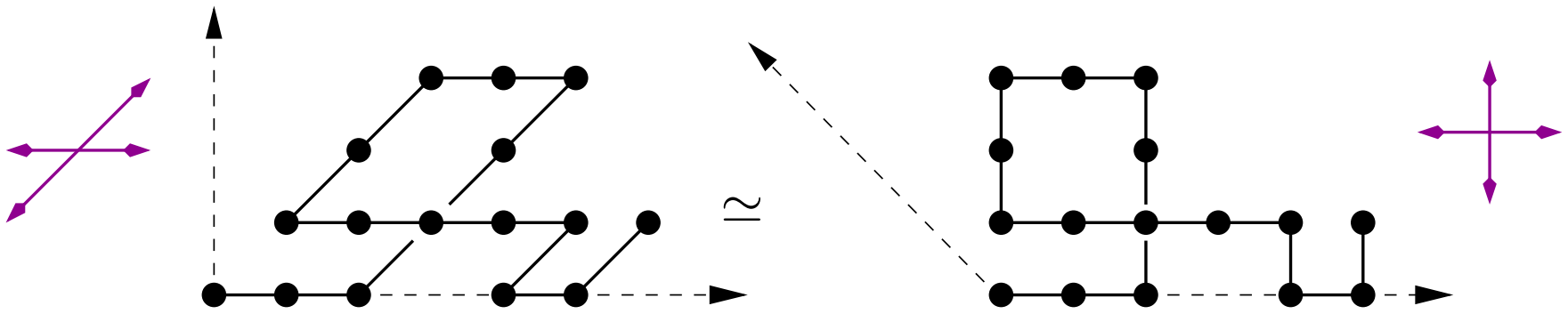
An elementary solution of Gessel's walks in the quadrant

Mireille Bousquet-Mélou, CNRS, LaBRI, Bordeaux



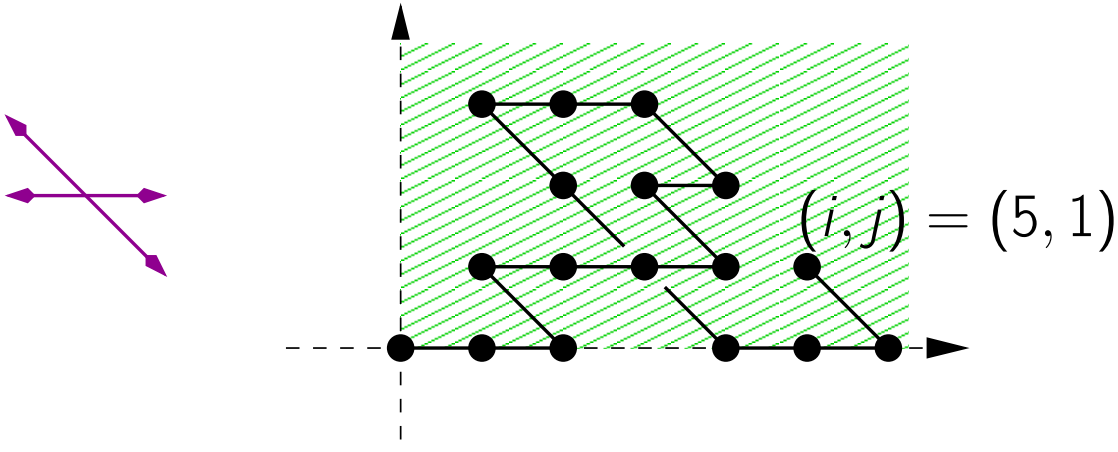
Walks in the quadrant

Let $\mathcal{S}$ be a finite subset of $\mathbb{Z}^2$ (set of steps) and $p_0 \in \mathbb{N}^2$ (starting point).

Questions

- What is the number $a(n)$ of n-step walks starting at p_0 and contained in the non-negative quadrant $\mathbb{N}^2$?
- For $(i,j) \in \mathbb{N}^2$, what is the number $a(i,j;n)$ of such walks that end at (i,j)?

Example. $\mathcal{S} = \{10, \bar{1}0, 1\bar{1}, \bar{1}1\}$, $p_0 = (0,0)$.



Example [Gouyou-Beauchamps 86], [mbm-Mishna 10]

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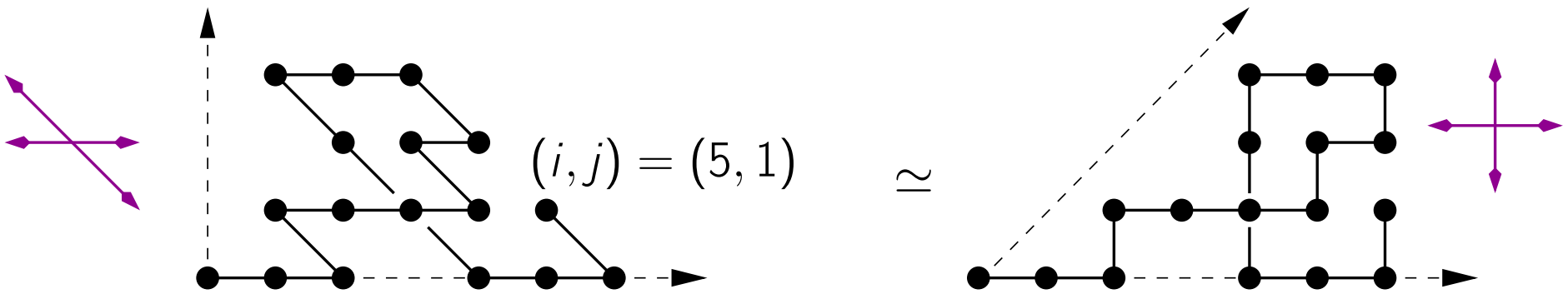

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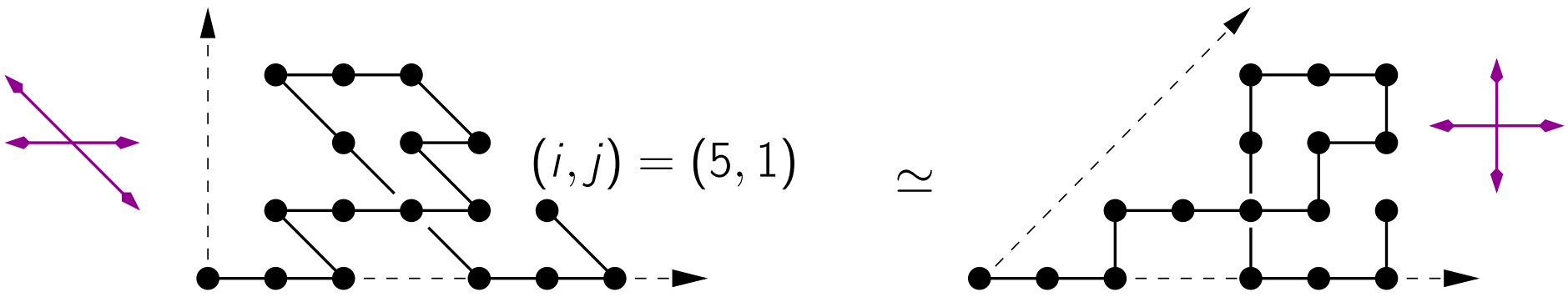

Nice numbers

If $n = 2m + \delta$, with $\delta \in \{0, 1\}$,

$$a(n) = \frac{n!(n+1)!}{m!(m+1)!(m+\delta)!(m+\delta+1)!}.$$

Moreover, if $n = 2m + i$,

$$a(i, j; n) = \frac{(i+1)(j+1)(i+j+2)(i+2j+3)n!(n+2)!}{(m-j)!(m+1)!(m+i+2)!(m+i+j+3)!}.$$

Many contributions

Adan, Banderier, Bernardi, Bostan, Cori, Denisov, Duchon, Dulucq, Fayolle, Gessel, Fisher, Flajolet, Gouyou-Beauchamps, Guttmann, Guy, Janse van Rensburg, Johnson, Kauers, Koutschan, Krattenthaler, Kurkova, Kreweras, van Leeuwarden, MacMahon, Melczer, Mishna, Niederhausen, Petkovšek, Prellberg, Raschel, Rechnitzer, Sagan, Salvy, Viennot, Wachtel, Wilf, Yeats, Zeilberger...

etc.

Generating functions

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$$a(n) = ? \qquad a(i, j; n) = ?$$

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Can one express these series? Are they rational? algebraic? D-finite?

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A hierarchy of formal power series

- The formal power series $A(t)$ is rational if it can be written

$$A(t) = P(t)/Q(t)$$

where $P(t)$ and $Q(t)$ are polynomials in t.

- The formal power series $A(t)$ is algebraic (over $\mathbb{Q}(t)$) if it satisfies a (non-trivial) polynomial equation:

$$P(t, A(t)) = 0.$$

- The formal power series $A(t)$ is D-finite (holonomic) if it satisfies a (non-trivial) linear differential equation with polynomial coefficients:

$$P_k(t)A^{(k)}(t) + \cdots + P_0(t)A(t) = 0.$$

- Nice closure properties + asymptotics of the coefficients
- Extension to several variables (D-finite: one DE per variable)

Walks in the full space

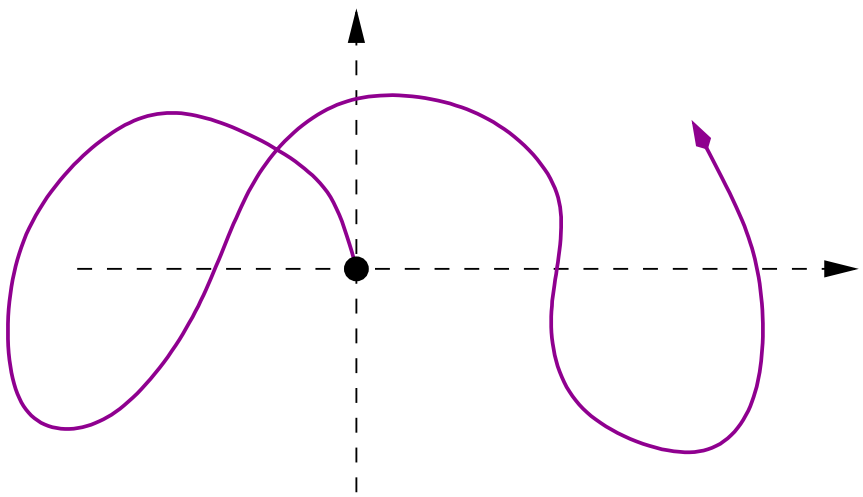
Rational series

A rational generating function:

$$a(n) = |\mathcal{S}|^n \quad \Leftrightarrow \quad A(t) = \sum_{n \geq 0} a(n) t^n = \frac{1}{1 - |\mathcal{S}|\, t}$$

More generally:

$$A(x, y; t) = \frac{1}{1 - t \sum_{(i,j) \in \mathcal{S}} x^i y^j}.$$

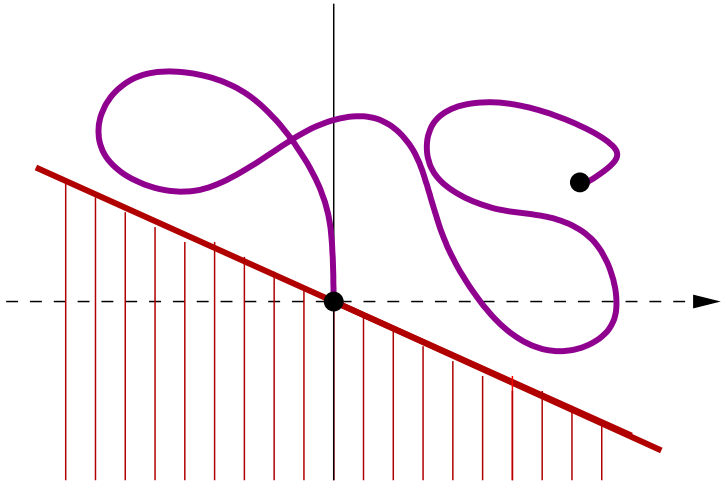


Walks in a rational half-space

Algebraic series

The associated generating function is algebraic, given by an explicit system of polynomial equations.

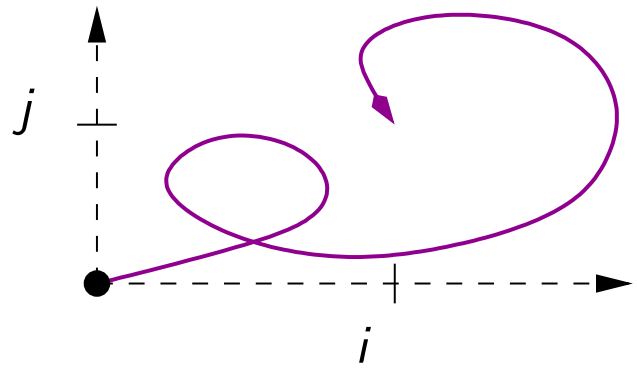
[Gessel 80], [Duchon 00], [mbm-Petkovšek 00]...



Walks in the quadrant

- Start from $p_0 = (0, 0)$

$$A(x, y; t) = \sum_{i,j,n \geq 0} a(i, j; n) x^i y^j t^n = ?$$



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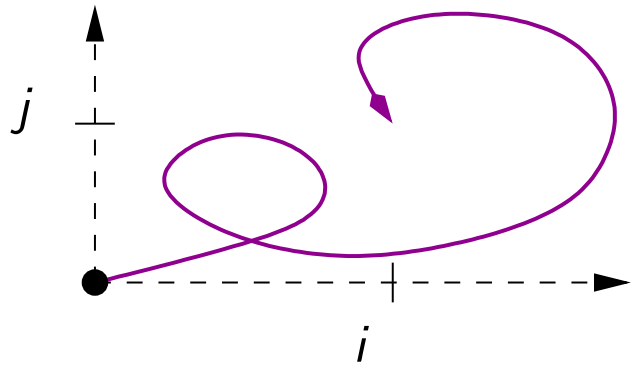
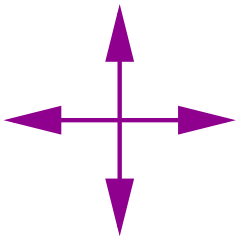
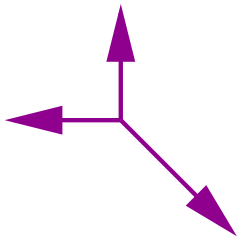




D–finite



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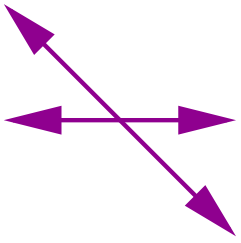

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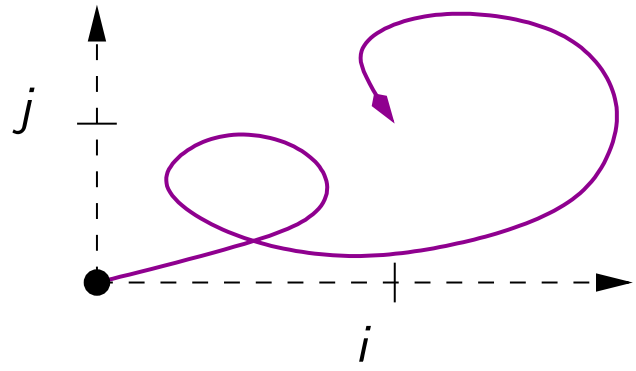

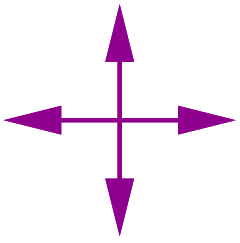

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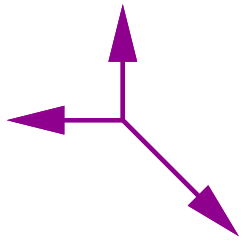

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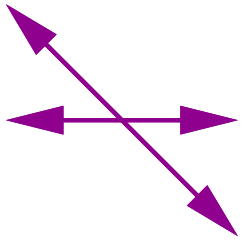

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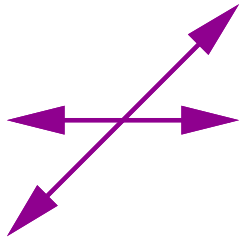

Algebraic

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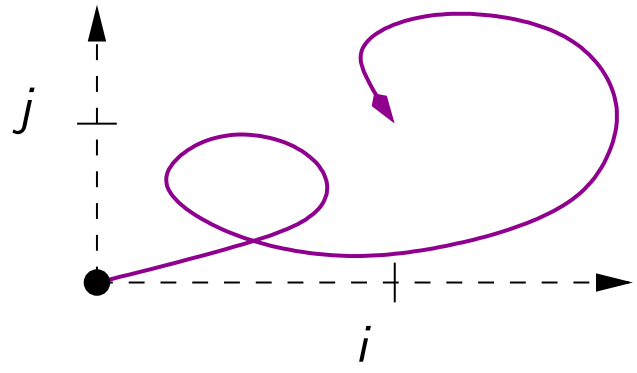

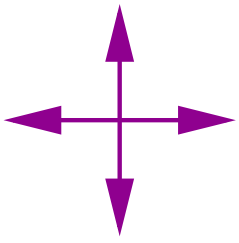

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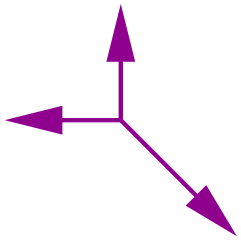

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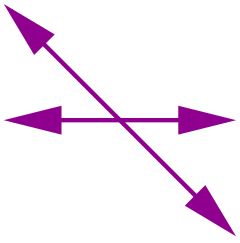

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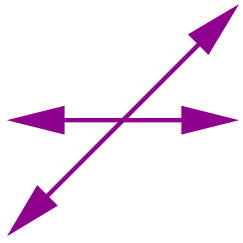

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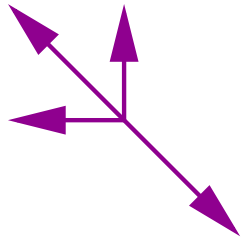

Non–D–finite

Walks in the quadrant

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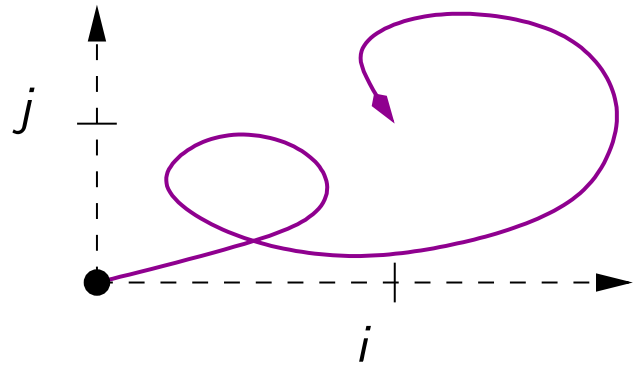

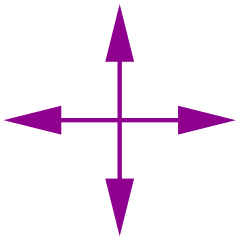

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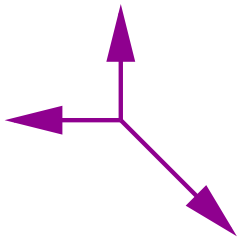

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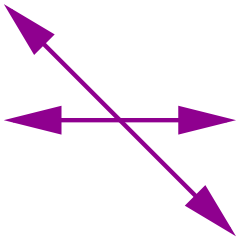

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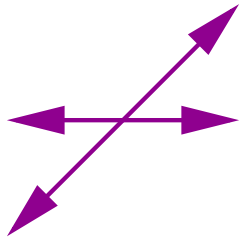

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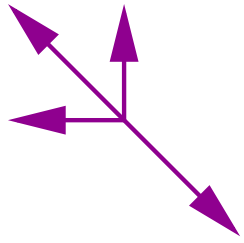
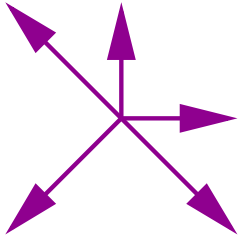
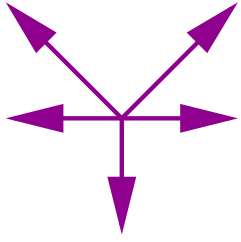


Non–D–finite



Non–D–finite



D–finite

Quadrant walks with *small* steps: classification

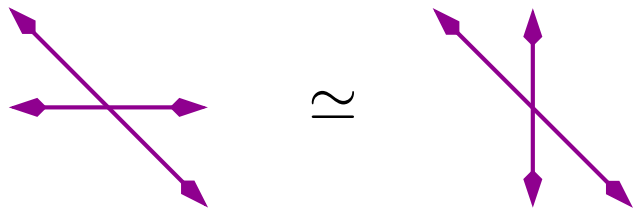
- $\mathcal{S} \subset \{\bar{1}, 0, 1\} \setminus \{00\} \Rightarrow 2^8 = 256$ step sets (or: models)

Quadrant walks with *small* steps: classification

- $\mathcal{S} \subset \{\bar{1}, 0, 1\} \setminus \{00\} \Rightarrow 2^8 = 256$ step sets (or: models)

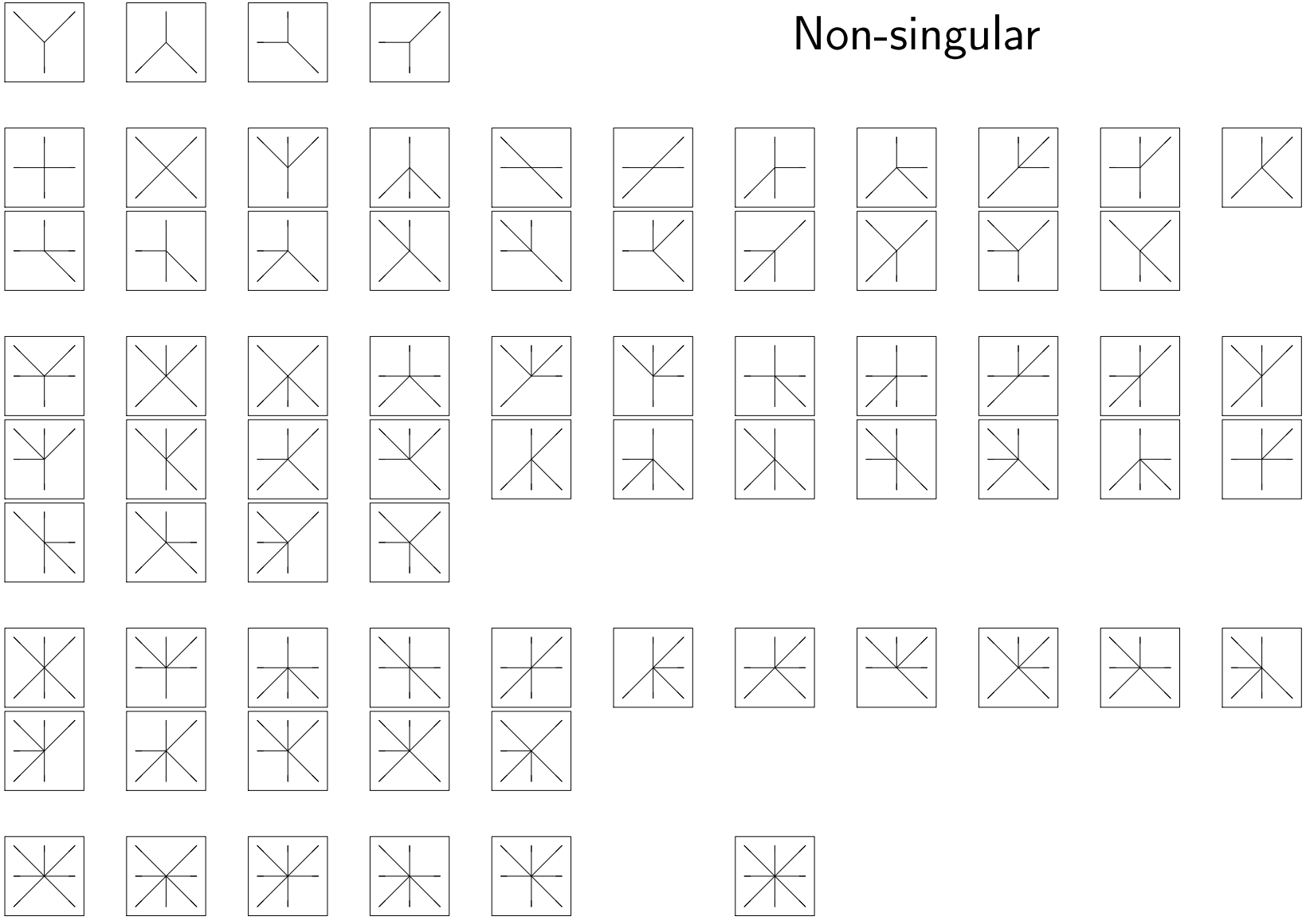
- However, some models are equivalent:
  - to a model of walks in the full or half-plane ($\Rightarrow$ algebraic)

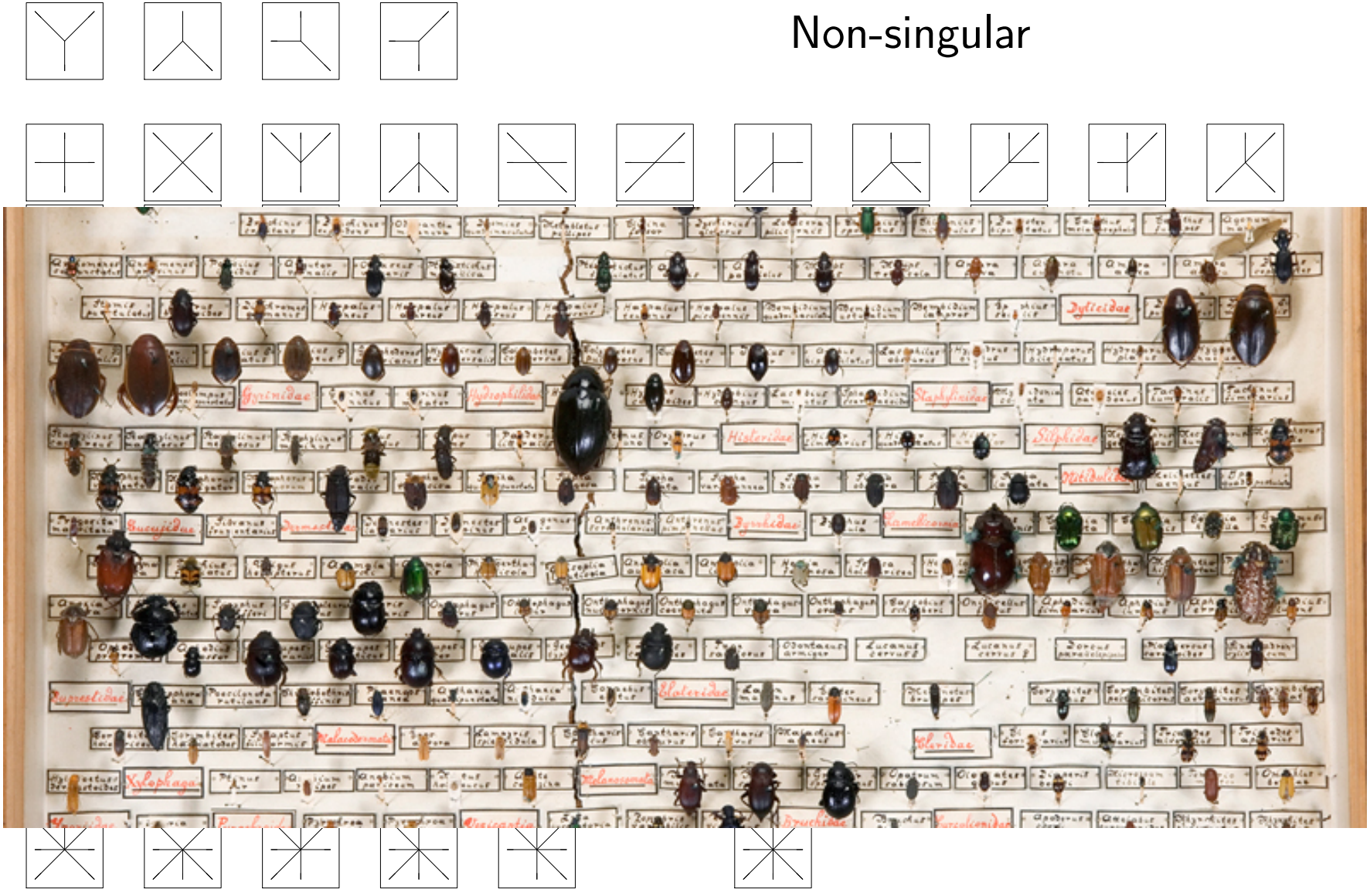
  - to another model in the collection (diagonal symmetry)



- One is left with 79 interesting distinct models.

[mbm-Mishna 10]


Non-singular
Singular

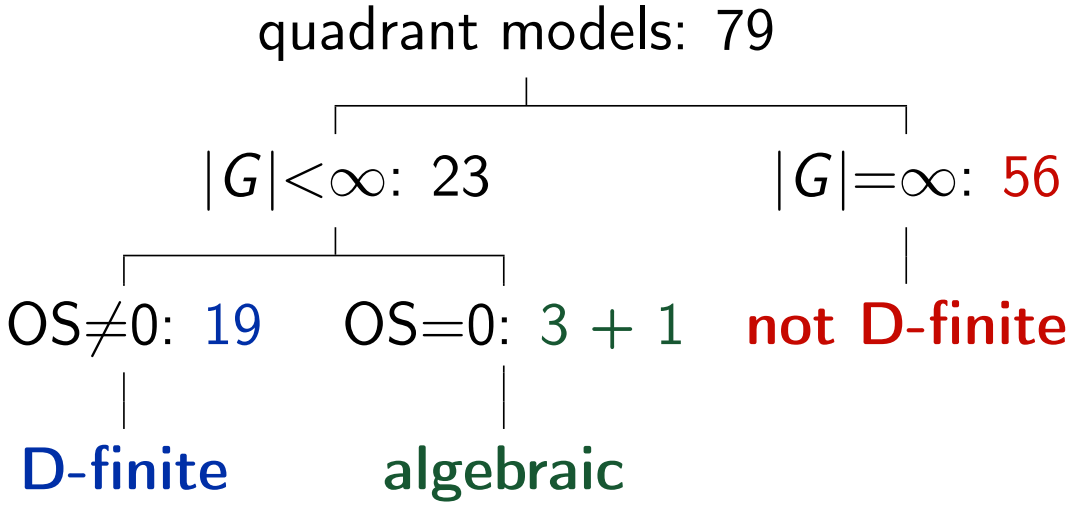

Non-singular
Singular

Classification of quadrant walks with small steps

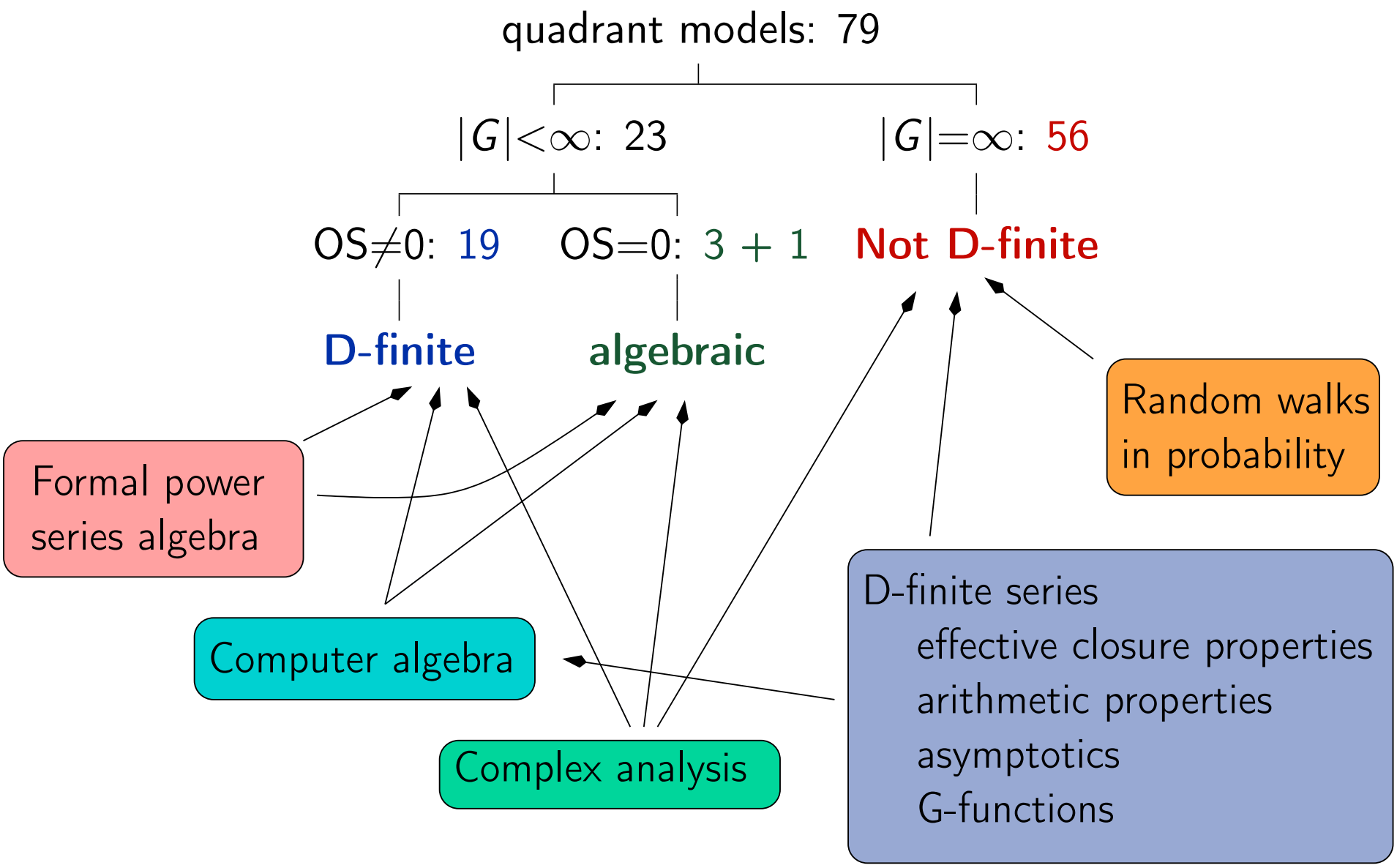
Theorem

The series $A(x, y; t)$ is D-finite iff the associated group G is finite.
It is algebraic iff, in addition, the orbit sum is zero.

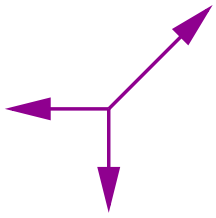
[mbm-Mishna 10], [Bostan-Kauers 10] — D-finite
[Kurkova-Raschel 12] — non-singular non-D-finite
[Mishna-Rechnitzer 07], [Melczer-Mishna 13] — singular non-D-finite



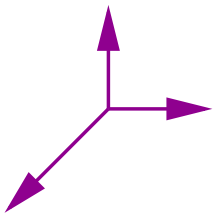
Classification of quadrant walks with small steps



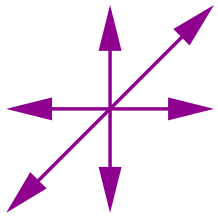
When the orbit sum is zero: the four algebraic models



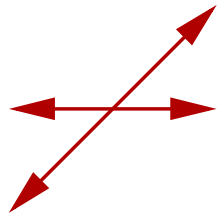
Kreweras



reverse Kreweras



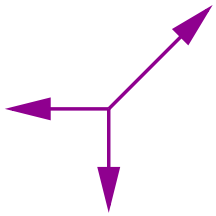
double Kreweras



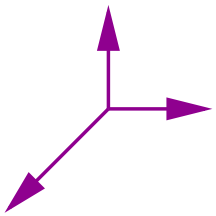
Gessel

- Kreweras walks
  - expression for $a(i,0;n)$ [Kreweras 65], simplified by [Niederhausen 83]
  - $A(x,y;t)$ is algebraic [Gessel 86, mbm 02/05]
- Unified approach for the Kreweras trilogy [mbm-Mishna 10]

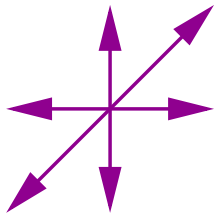
When the orbit sum is zero: the four algebraic models



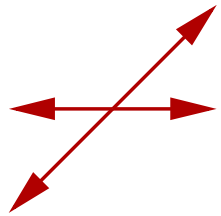
Kreweras



reverse Kreweras



double Kreweras



Gessel

- Kreweras walks
  - expression for $a(i,0;n)$ [Kreweras 65], simplified by [Niederhausen 83]
  - $A(x,y;t)$ is algebraic [Gessel 86, mbm 02/05]
- Unified approach for the Kreweras trilogy [mbm-Mishna 10]
- Gessel walks:
  - conjecture for $a(0,0;n)$ [Gessel $\simeq$ 00]
  - proof of this conjecture [Kauers, Koutschan & Zeilberger 08]
  - $A(t)$ and $A(x,y;t)$ are algebraic! [Bostan & Kauers 09a]
  - new proof via complex analysis [Bostan, Kurkova & Raschel 13]
  - an elementary and constructive proof [mbm 15]

Kreweras' walks

Theorem

- The number of walks of length $n = 3m + 2i$ ending at $(i, 0)$ is

$$a(i, 0; n) = \frac{4^m(2i+1)}{(m+i+1)(2m+2i+1)} \binom{2i}{i} \binom{3m+2i}{m}$$

[Kreweras 65]

- The generating function $A(x, y; t)$ is algebraic [Gessel 86].

For instance, the series $Q(t) := A(0, 0; t^{1/3})$ has degree 3:

$$54t - 1 + (1 - 72t)Q(t) + 16tQ(t)^2 + 64t^2Q(t)^3 = 0$$

Gessel's walks

Theorem

- The number of walks of length $n = 2m$ ending at $(0, 0)$ is

$$a(0, 0; n) = 16^m \frac{(5/6)_m (1/2)_m}{(5/3)_m (2)_m},$$

where $(a)_m = a(a+1)\cdots(a+m-1)$ is the ascending factorial.
[Kauers, Koutschan, Zeilberger 08]

- The generating function $A(x, y; t)$ is algebraic.
[Bostan, Kauers 09]

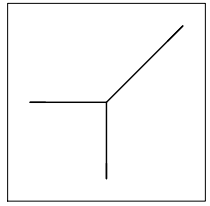
For instance, the series $Q(t) := A(0, 0; \sqrt{t})$ has degree 8:

$$\begin{aligned} &27t^7Q^8 + 108t^6Q^7 + 189t^5Q^6 + 189t^4Q^5 - 9t^3(32t^2 + 28t - 13)Q^4 \\ &\quad - 9t^2(64t^2 + 56t - 5)Q^3 - 2t(256t^3 - 312t^2 + 156t - 5)Q^2 \\ &\quad - (32t - 1)(16t^2 - 28t + 1)Q - 256t^3 - 576t^2 - 48t + 1 = 0. \end{aligned}$$

I. A solution of...

I. A solution of Kreweras' model

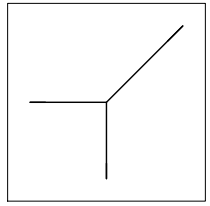
[mbm 02]



I. A solution of Kreweras' model

[mbm 02]

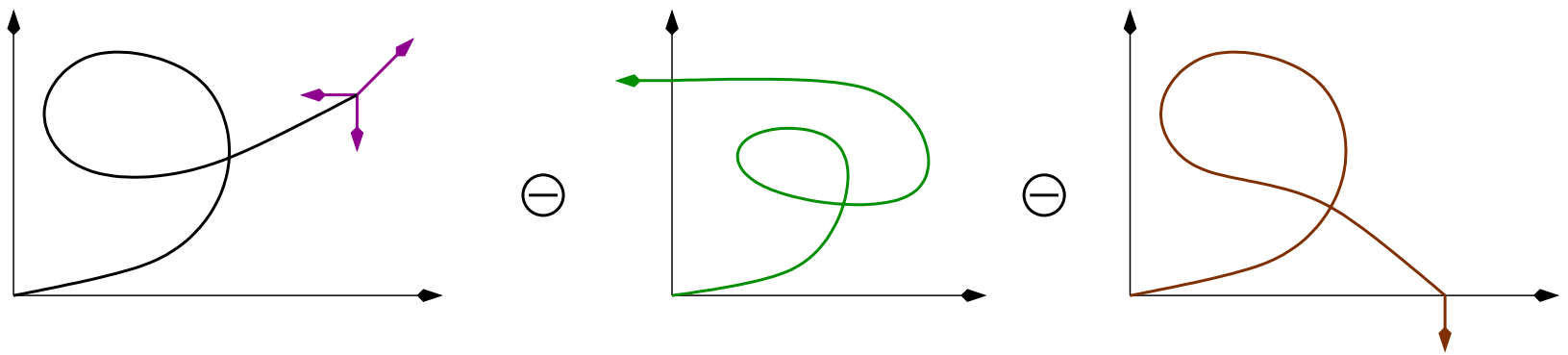
1. functional equation for $A(x, y; t)$
2. canceling the kernel: the roots Y_0 and Y_1
3. the group of the walk
4. symmetric functions of Y_0 and Y_1
5. Brown's quadratic method (1969)



1. A functional equation

Set step: $\mathcal{S} = \{11, \bar{1}0, 0\bar{1}\}$. Denote $\bar{x} = 1/x$ and $\bar{y} = 1/y$.

$$A(x, y; t) \equiv A(x, y) = 1 + t(xy + \bar{x} + \bar{y})A(x, y) - t\bar{x}A(0, y) - t\bar{y}A(x, 0)$$



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$$\boxed{\big(1 - t(xy + \bar{x} + \bar{y})\big)xyA(x, y) = xy - tyA(0, y) - txA(x, 0)}$$

- The polynomial $1 - t(xy + \bar{x} + \bar{y})$ is the kernel of this equation
- The equation is linear, with two catalytic variables x and y (tautological at $x = 0$ or $y = 0$)

2. Canceling the kernel

- The equation:

$$\big(1 - t(xy + \bar{x} + \bar{y})\big)xyA(x,y) = xy - tyA(0,y) - txA(x,0)$$

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$$Y_0(x) = \frac{1 - t\bar{x} - \sqrt{(1-t\bar{x})^2 - 4t^2x}}{2tx} = \qquad t + \bar{x}t^2 + O(t^3),$$

$$Y_1(x) = \frac{1 - t\bar{x} + \sqrt{(1-t\bar{x})^2 - 4t^2x}}{2tx} = \frac{\bar{x}}{t} - \bar{x}^2 \quad - t - \bar{x}t^2 + O(t^3).$$

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- Specializing y to Y_0 in the equation gives:

$$R(x) + R(Y_0) = xY_0$$

with $R(x) = txA(x,0) = txA(0,x)$ (symmetry).

This equation characterizes $R(x)$ but... why algebraicity?

3. The group of the model

The kernel

$$K(x,y) = 1 - t(xy + \bar{x} + \bar{y})$$

is left unchanged by the rational transformations

$$\Phi : (x,y) \mapsto (\bar{x}\bar{y}, y) \quad \text{and} \quad \Psi : (x,y) \mapsto (x, \bar{x}\bar{y})\,.$$

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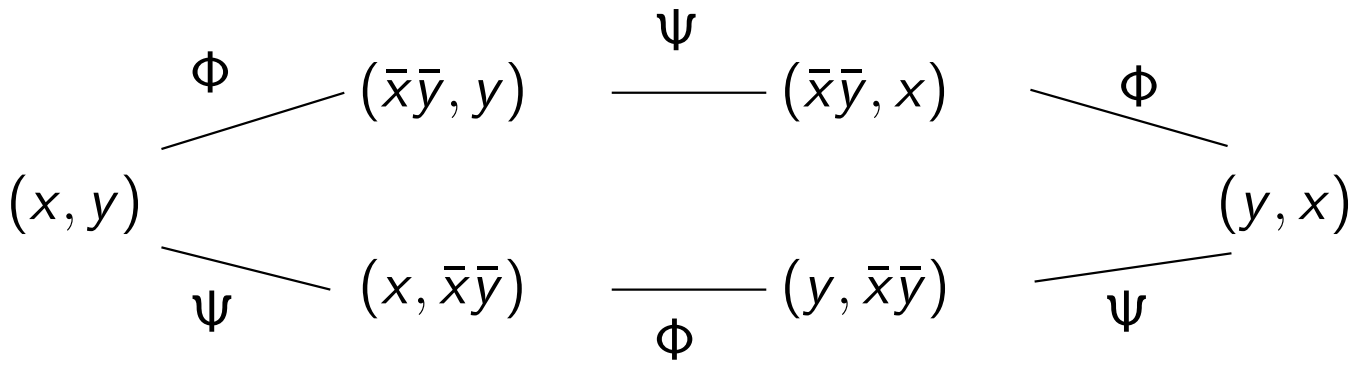
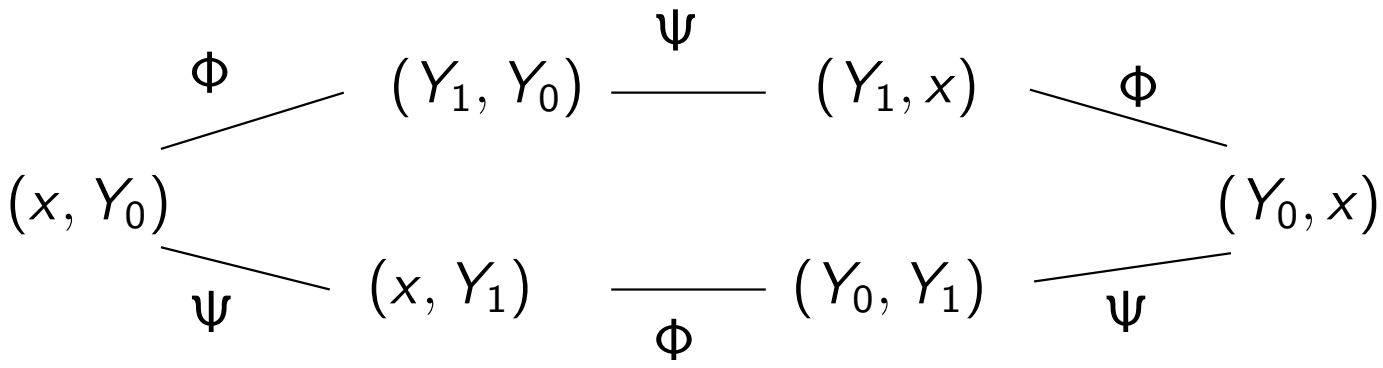
They are involutions, and generate a dihedral group G of order 6:



Build more pairs that cancel the kernel

• Since $K(x, Y_0) = 0$ and the transformations of G preserve the kernel, any element of the orbit of (x, Y_0) cancels the kernel.



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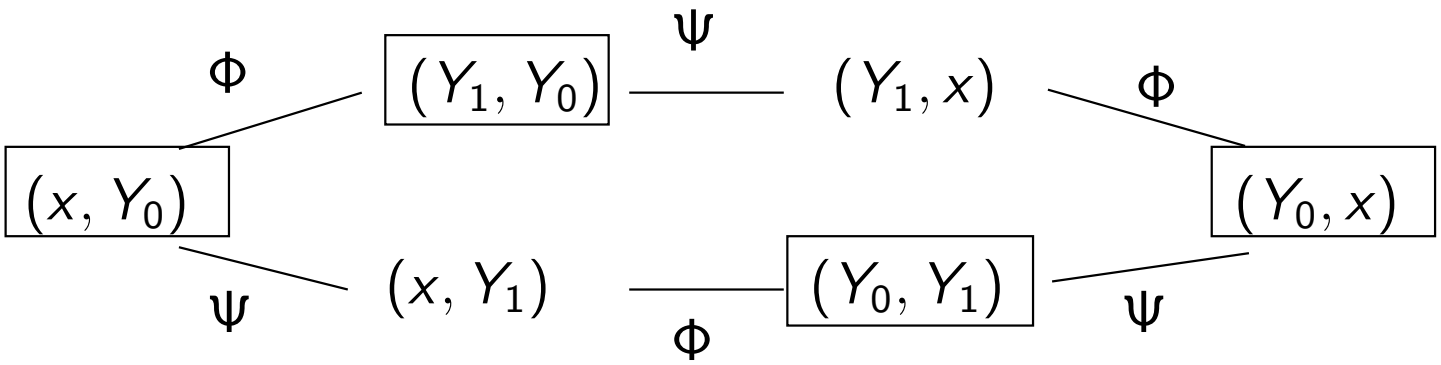

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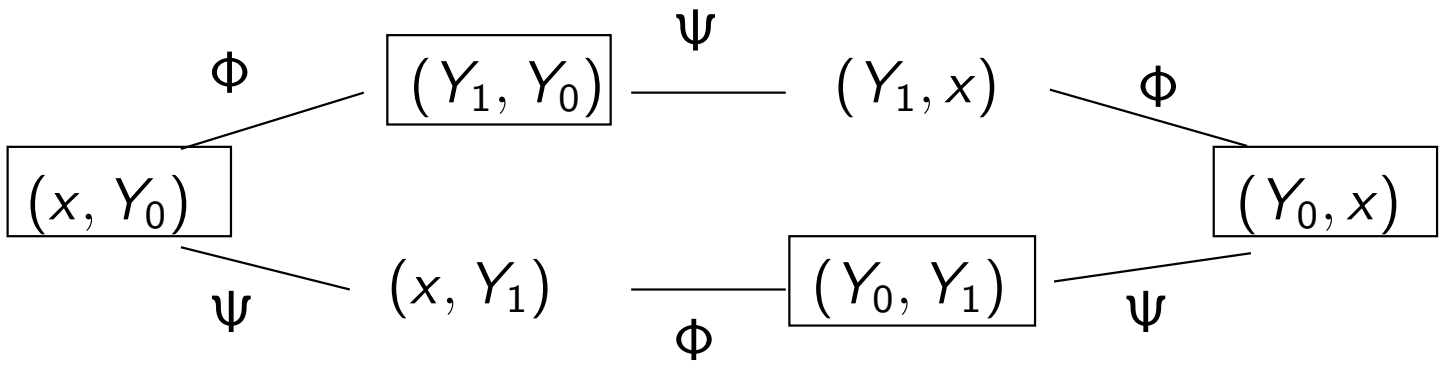

Gives two equations for R:

$$R(x) + R(Y_0) = xY_0$$
$$R(Y_0) + R(Y_1) = Y_0 Y_1 = \bar{x}$$

with $R(x) = txA(x, 0)$.

4. Symmetric functions of Y_0 and Y_1

- The elementary symmetric functions of the Y_i are polynomials in $\bar{x}$:

$$Y_0 + Y_1 = \frac{\bar{x}}{t} - \bar{x}^2 \quad \text{and} \quad Y_0 Y_1 = \bar{x}.$$

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- Rewrite the equations for R as

$$R(Y_0) - xY_0 = -R(x)$$
$$R(Y_1) - xY_1 = R(x) + 2\bar{x} - 1/t$$

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$$(R(Y_0) - xY_0)(R(Y_1) - xY_1) = -R(x)\,(R(x) + 2\bar{x} - 1/t)$$

- Extracting the non-negative powers of x gives:

$$x - 2R'(0) = -R(x)(R(x) + 2\bar{x} - 1/t).$$

No more Y_i ! Equivalently,

$$xtR(x)^2 + (2t - x)R(x) + tx^2 - 2txR'(0) = 0.$$

5. Finish: the quadratic method

The series $R(x) = txA(x, 0)$ satisfies

$$xtR(x)^2 + (2t - x)R(x) + tx^2 - 2txR'(0) = 0.$$

or

$$t^2x^2A(x, 0)^2 + (2t - x)A(x, 0) - 2tA(0, 0) + x = 0.$$

Tautological at $x = 0$: a quadratic equation with one catalytic variable x, and one additional unknown one-variable series $A(0, 0)$
$\Rightarrow$ Brown's quadratic method [Brown 69]

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$\Rightarrow$ Brown's quadratic method [Brown 69]

Theorem

Let $X \equiv X(t)$ be the unique series in t defined by

$$X = t(2 + X^3).$$

Then the generating function of Kreweras' walks ending on the x-axis is

$$A(x,0;t) = \frac{1}{tx}\left(\frac{1}{2t} - \frac{1}{x} - \left(\frac{1}{X} - \frac{1}{x}\right)\sqrt{1 - xX^2}\right).$$

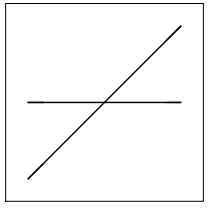
II. A solution of Gessel's model

[mbm 15]

II. A solution of Gessel's model

[mbm 15]

- functional equation for $A(x, y)$
- canceling the kernel: the roots Y_0 and Y_1
- the group of the walk
- symmetric functions of Y_0 and Y_1
- ...
- the *generalized* quadratic method [mbm-Jehanne 06]



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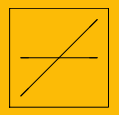
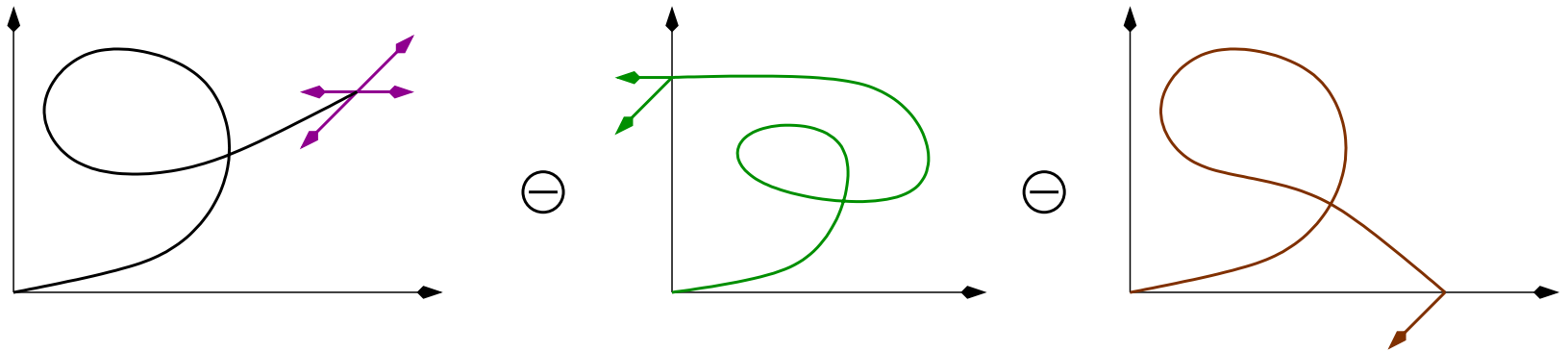


Set step: $\mathcal{S} = \{11, 10, \bar{1}\bar{1}, 0\bar{1}\}$

$$A(x,y) = 1 + t(xy + x + \bar{x}\bar{y} + \bar{x})A(x,y) - t\bar{x}(1+\bar{y})A(0,y) - t\bar{x}\bar{y}\,(A(x,0) - A(0,0))$$



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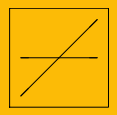

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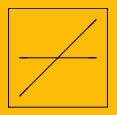
$$\boxed{(1 - t(xy + x + \bar{x}\bar{y} + \bar{x}))xyA(x,y) = xy - S(y) - R(x)}$$

with

$$S(y) = t(1+y)A(0,y) \quad \text{and} \quad R(x) = t\,(A(x,0) - A(0,0)).$$

• Main difference with K' walks: no x/y symmetry, two bivariate unknown series $R(x)$ and $S(y)$

2. Canceling the kernel



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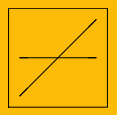

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- The kernel, as a polynomial in y, has two roots:

$$Y_0(x) = \frac{1 - t(x+\bar{x}) - \sqrt{(1-t(x+\bar{x}))^2 - 4t^2}}{2tx} = \qquad \bar{x}t + \cdots$$

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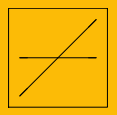

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Observe that $xY_i(x)$ is symmetric in x and $\bar{x}$.

- Specializing y to Y_0 in the equation gives:

$$R(x) + S(Y_0) = xY_0$$

3. The group of the model

The kernel

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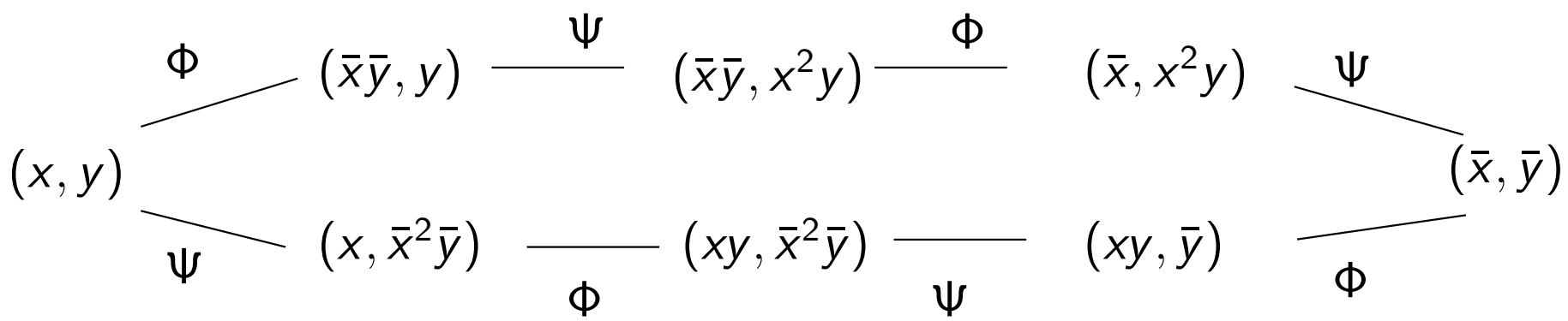
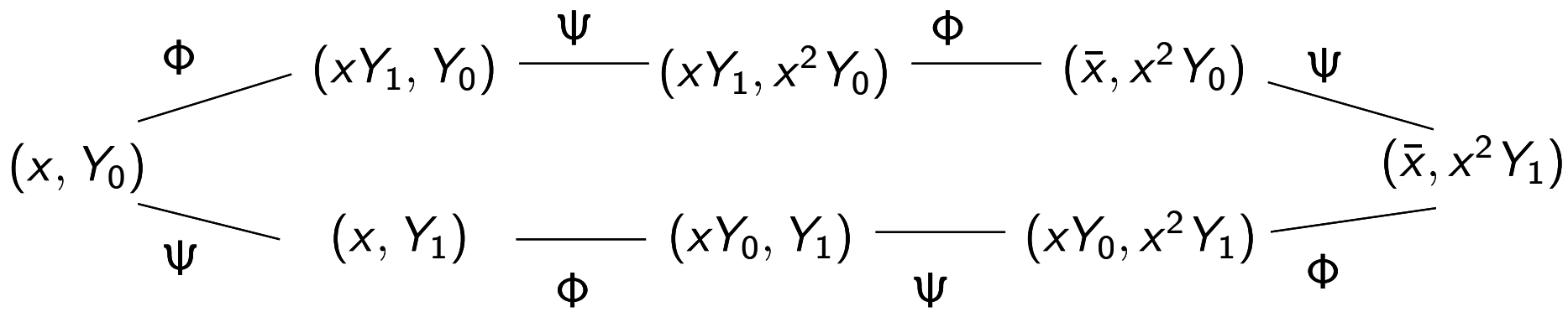


Build more pairs that cancel the kernel

Since $K(x, Y_0) = 0$ and the transformations of G preserve the kernel, any element of the orbit of (x, Y_0) cancels the kernel.



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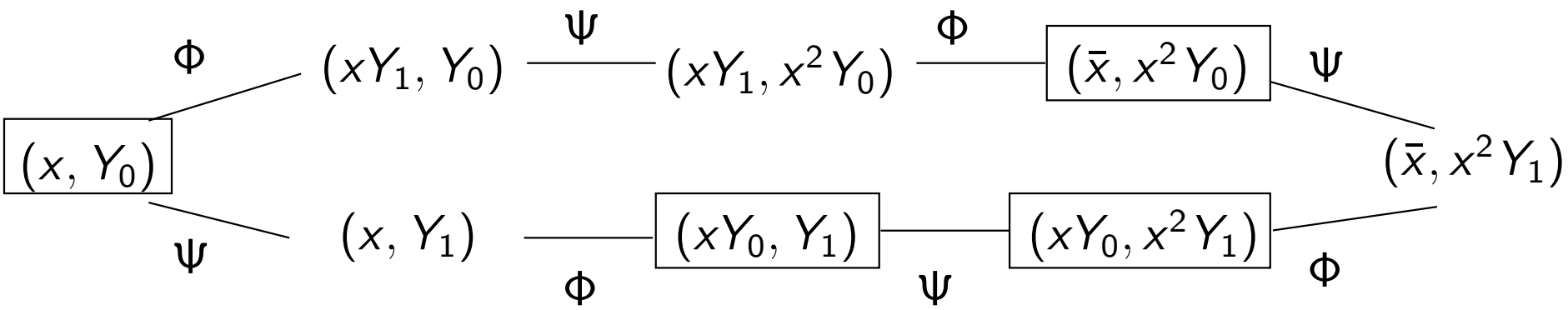

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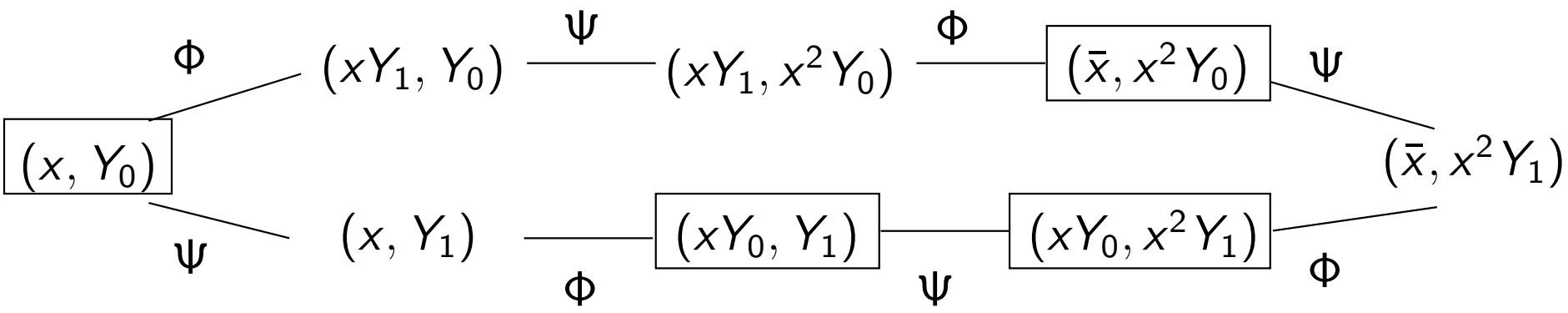

Gives four equations for R and S:

$$R(x) + S(Y_0) = xY_0$$
$$R(Y_0) + S(Y_1) = Y_0Y_1 = \bar{x}$$
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4. Symmetric functions of Y_0 and Y_1

- The elementary symmetric functions of the Y_i are polynomials in $\bar{x}$:

$$Y_0 + Y_1 = -1 + \frac{\bar{x}}{t} - \bar{x}^2 \quad \text{and} \quad Y_0 Y_1 = \bar{x}^2.$$

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- From the four equations:

$$\begin{aligned} R(x) + S(Y_0) &= xY_0 \\ R(xY_0) + S(Y_1) &= Y_0 Y_1 = \bar{x} \\ R(\bar{x}) + S(x^2 Y_0) &= xY_0 \\ R(xY_0) + S(x^2 Y_1) &= x^3 Y_0 Y_1 = x, \end{aligned}$$

one constructs two symmetric functions of Y_0 and Y_1, and extracts non-negative powers of x.
This gives an equation without any Y_i... but with $R(x)$ and $R(\bar{x})$.

4. Symmetric functions of Y_0 and Y_1 (cont'd)

This gives:

$$R(x)^2 + R(x)R(\bar{x}) + R(\bar{x})^2 + (2\bar{x} - 1/t + x + S(0))R(x) \\ +(2x - 1/t + \bar{x} + S(0))R(\bar{x}) = 2R'(0) - (\bar{x} + x - 1/t)S(0) - 1$$

while for Kreweras' walks:

$$xtR(x)^2 + (2t - x)R(x) + tx^2 - 2txR'(0) = 0.$$

- Main difference: the terms $R(\bar{x})$
- Main difficulty: the hybrid term $R(x)R(\bar{x})$

5. Les identités remarquables à notre secours

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$$R(x)^2 + R(x)R(\bar{x}) + R(\bar{x})^2 + (2\bar{x} - 1/t + x + S(0))R(x) \\ + (2x - 1/t + \bar{x} + S(0))R(\bar{x}) = 2R'(0) - (\bar{x} + x - 1/t)S(0) - 1.$$

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$\Rightarrow$ Multiply by $R(x) - R(\bar{x}) + \bar{x} - x$! This gives $P(x) = P(\bar{x})$, with

$$P(x) = R(x)^3 + (S(0) + 3\bar{x} - 1/t)R(x)^2 \\ + \left(2\bar{x}^2 - \bar{x}/t + x/t - x^2 - 2R'(0) + (2\bar{x} - 1/t)S(0)\right) R(x) \\ -x^2 S(0) + x\left(2R'(0) + S(0)/t - 1\right).$$

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We have obtained $P(x) = P(\bar{x})$, with

$$P(x) = R(x)^3 + (S(0) + 3\bar{x} - 1/t)R(x)^2 + \left(2\bar{x}^2 - \bar{x}/t + x/t - x^2 - 2R'(0) + (2\bar{x} - 1/t)S(0)\right) R(x) - x^2 S(0) + x\left(2R'(0) + S(0)/t - 1\right).$$

Extract nonnegative powers of x:

$$R(x)^3 + (S(0) + 3\bar{x} - 1/t)R(x)^2 + \left(2\bar{x}^2 - \bar{x}/t + x/t - x^2 - 2R'(0) + (2\bar{x} - 1/t)S(0)\right) R(x) = R''(0) + R'(0)(2S(0) + 2\bar{x} - 1/t) + xS(0)(x - 1/t) + x.$$

A cubic equation in one catalytic variable x, with three additional unknown one-variable series $S(0), R'(0)$ and $R''(0)$.

6. Finish: the generalized quadratic method

$$R(x)^3 + (S(0) + 3\bar{x} - 1/t)R(x)^2 + \left(2\bar{x}^2 - \bar{x}/t + x/t - x^2 - 2R'(0) + (2\bar{x} - 1/t)S(0)\right) R(x) = R''(0) + R'(0)(2S(0) + 2\bar{x} - 1/t) + xS(0)(x - 1/t) + x.$$

Theorem

Every series $R(x;t) \equiv R(x)$ solution of a (proper) polynomial equation with one catalytic variable x is algebraic. That is, if

$$P(t, x, R(x), A_1, \ldots, A_k) = 0$$

for some polynomial P with coefficients in $\mathbb{Q}$, then $R(x)$ is algebraic over $\mathbb{Q}(t, x)$, and each A_i is algebraic over $\mathbb{Q}(t)$.

[mbm-Jehanne 2006] (after [Knuth 72] and [Brown 65])

6. Finish: the generalized quadratic method

Theorem

The generating function $A(x, y; t)$ is algebraic of degree 72 (explicit equations and rational parametrizations for $A(x, 0)$ and $A(0, y)$)

- In particular, the series $Q(t) := A(0, 0; \sqrt{t})$ has degree 8:

$$\begin{aligned} &27t^7Q^8 + 108t^6Q^7 + 189t^5Q^6 + 189t^4Q^5 - 9t^3(32t^2 + 28t - 13)Q^4 \\ &\quad - 9t^2(64t^2 + 56t - 5)Q^3 - 2t(256t^3 - 312t^2 + 156t - 5)Q^2 \\ &\quad - (32t - 1)(16t^2 - 28t + 1)Q - 256t^3 - 576t^2 - 48t + 1 = 0. \end{aligned}$$

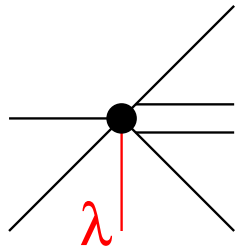
- The number of walks of length $n = 2m$ ending at $(0, 0)$ is

$$a(0, 0; 2m) = 16^m \frac{(5/6)_m (1/2)_m}{(5/3)_m (2)_m},$$

where $(a)_m = a(a + 1) \cdots (a + m - 1)$ is the ascending factorial.

Comments

- other elementary proofs will come for sure
- 3D Kreweras' walks remain mysterious
- works for other algebraic models with repeated steps, proved so far by computer algebra [Kauers, Yatchak 14(a)]



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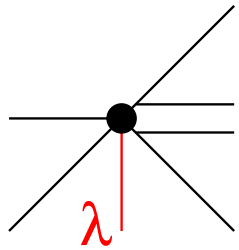
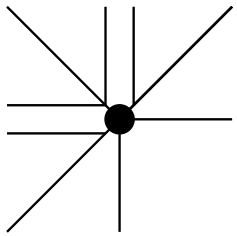


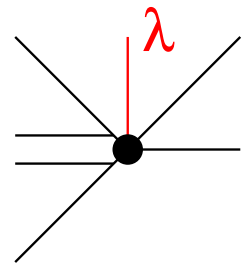
- Open: Find elementary proofs for



Algebraic

Conj. [Kauers, Yatchak 14(a)]

Proof via complex analysis [R. et al.]



D-finite

Proof via computer algebra

[Kauers, Yatchak 14(a)]

Polynomial equations with one catalytic variable

- Assume

$$P(F(x), A_1, \ldots, A_k, t, x) = 0$$

where $P(x_0, x_1, \ldots, x_k, t, x)$ is a polynomial with coefficients in $\mathbb{K}$, $F(x) \equiv F(x; t) \in \mathbb{K}[x][[t]]$, and $A_i \in \mathbb{K}[[t]]$ for all i.

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- For any series $X \equiv X(t)$ such that
  - the series $F(X) \equiv F(X; t)$ is well-defined
  - $\frac{\partial P}{\partial x_0}(F(X), A_1, \ldots, A_k, t, X) = 0$,

one has

$$\frac{\partial P}{\partial x}(F(X), A_1, \ldots, A_k, t, X) = 0.$$

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$$\frac{\partial P}{\partial x}(F(X), A_1, \ldots, A_k, t, X) = 0.$$

Proof: differentiate (1) with respect to x

$$F'(x)\,\frac{\partial P}{\partial x_0}(F(x), A_1, \ldots, A_k, t, x) + \frac{\partial P}{\partial x}(F(x), A_1, \ldots, A_k, t, x) = 0.$$

Polynomial equations with one catalytic variable

- Assume there exist k series $X_1, \ldots, X_k$ such that

$$\frac{\partial P}{\partial x_0}(F(X_i), A_1, \ldots, A_k, t, X_i) = 0.$$

In this case, for each X_i,

$$\frac{\partial P}{\partial x}(F(X_i), A_1, \ldots, A_k, t, X_i) = 0$$

and

$$P(F(X_i), A_1, \ldots, A_k, t, X_i) = 0.$$

- This system of $3k$ polynomial equations in $3k$ unknowns $A_1, \ldots, A_k$, $X_1, \ldots, X_k$, $F(X_1), \ldots, F(X_k)$ may imply (together with the fact that the X_i are distinct) the algebraicity of the A_i.

[mbm-Jehanne 06]

A computer algebra approach

Example. When $\mathcal{S} = \{\bar{1}0, 0\bar{1}, 11\}$, the equation reads

$$(1 - t(\bar{x} + \bar{y} + xy))xyA(x, y; t) = xy - tyA(0, y; t) - txA(x, 0; t).$$

Naïve route: guess and check!

- Guess a polynomial equation Pol satisfied by $A(x, y; t)$ (degrees $[18, 18, 17, 12]$ in x, y, t, A)
- Let $F(x, y; t)$ be the solution of Pol that coincides with $A(x, y; t)$ up to high order (in t)
- Prove that $F(x, y; t)$ is a formal power series in t with polynomial coefficients in x and $y \Rightarrow F(x, 0; t)$ and $F(0, y; t)$ are well-defined
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- Then

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A computer algebra approach: climax

Algebraicity of Gessel's model [Bostan-Kauers 10]

- When $\mathcal{S} = \{10, \bar{1}0, 11, \bar{1}\bar{1}\}$, the series $A(x, y; t)$ is algebraic (degree 72).
- In particular, the series $A(0, 0; t)$, which counts loops, has degree 8, and the following expansion:

$$A(0,0;t) = \sum_{n \geq 0} 16^n \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} t^{2n}, \tag{1}$$

with $(i)_n = i(i+1)\cdots(i+n-1)$.

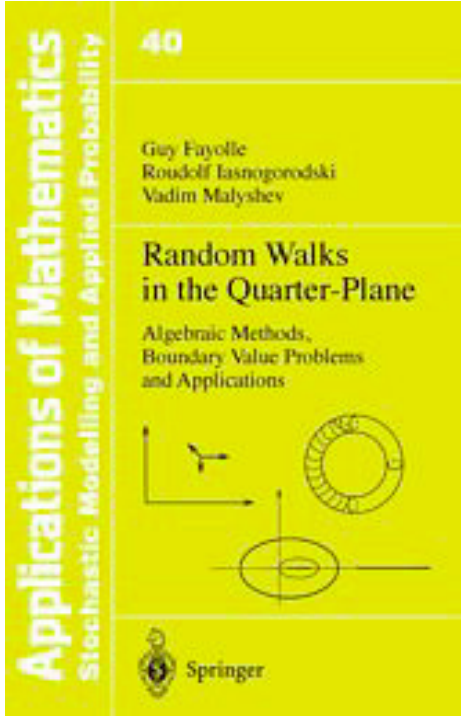
(1): Conjectured around 2001, first proof by [Kauers, Koutschan & Zeilberger 09] using computer algebra.

The algebraicity of $A(x, y; t)$ has now been re-proved using a complex analysis approach [Bostan, Kurkova & Raschel 13(a)]

A complex analysis approach

Markov chains with small steps in the quadrant: stationary distribution(s)
[Malyshev 71+]

Le petit livre jaune
[Fayolle, Iasnogorodski & Malyshev 99]



$\Rightarrow$ Reduction to a boundary value problem of the Riemann-Carleman type

A complex analysis approach

An expression of Q for any non-singular model $\mathcal{S}$

$$\tilde{K}(x,0;t)A(x,0;t) - \tilde{K}(0,0;t)A(0,0;t) = xY_0(x;t) +$$
$$\frac{1}{2i\pi}\int_{x_1(t)}^{x_2(t)} u\,[Y_0(u;t) - Y_1(u;t)]\left[\frac{\partial_u w(u;t)}{w(u;t)-w(x;t)} - \frac{\partial_u w(u;t)}{w(u;t)-w(0;t)}\right] \mathrm{d}u$$

where Y_0, Y_1, x_1 and x_2 are explicit algebraic series and w is explicit/ very well understood.

[Raschel 12] + Fayolle, Kurkova

+ other formulas that complete the characterization of $A(x,y;t)$

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where Y_0, Y_1, x_1 and x_2 are explicit algebraic series and w is explicit/ very well understood.
In particular, w is D-finite (in fact, algebraic!) iff the group is finite.

[Raschel 12] + Fayolle, Kurkova

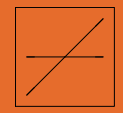
+ other formulas that complete the characterization of $A(x,y;t)$

A complex analysis approach: climax(es)

Theorems

- If $\mathcal{S}$ has an infinite group and is not singular, then $A(x, y; t)$ is not D-finite in x ($\equiv$ no differential equation with respect to x)

[Kurkova & Raschel 12]

- A new proof of the algebraicity of Gessel's model 

[Bostan, Kurkova & Raschel 13(a)]