An elementary solution of Gessel’s walks in the quadrant

Mireille Bousquet-Mélou, CNRS, LaBRI, Bordeaux
Walks in the quadrant

Let $S$ be a finite subset of $\mathbb{Z}^2$ (set of steps) and $p_0 \in \mathbb{N}^2$ (starting point).

Questions

- What is the number $a(n)$ of $n$-step walks starting at $p_0$ and contained in the non-negative quadrant $\mathbb{N}^2$?
- For $(i, j) \in \mathbb{N}^2$, what is the number $a(i, j; n)$ of such walks that end at $(i, j)$?

Example. $S = \{10, \overline{10}, 1\overline{1}, \overline{11}\}$, $p_0 = (0, 0)$. 

![Diagram of walks in the quadrant](image)
Example [Gouyou-Beauchamps 86], [mbm-Mishna 10]

Take $\mathcal{S} = \{1\bar{0}, \bar{1}0, \bar{1}\bar{1}, \bar{1}1\}$ and $p_0 = (0,0)$
Example [Gouyou-Beauchamps 86], [mbm-Mishna 10]

Take $S = \{10, \bar{1}0, 1\bar{1}, \bar{1}1\}$ and $p_0 = (0, 0)$

\[
S = \{10, \bar{1}0, 1\bar{1}, \bar{1}1\}
\]

\[
p_0 = (0, 0)
\]

\[
(i, j) = (5, 1)
\]

Nice numbers

If $n = 2m + \delta$, with $\delta \in \{0, 1\}$,

\[
a(n) = \frac{n!(n+1)!}{m!(m+1)!(m+\delta)!(m+\delta+1)!}.
\]

Moreover, if $n = 2m + i$,

\[
a(i, j; n) = \frac{(i+1)(j+1)(i+j+2)(i+2j+3)n!(n+2)!}{(m-j)!(m+1)!(m+i+2)!(m+i+j+3)!}.
\]
Many contributions

Adan, Banderier, Bernardi, Bostan, Cori, Denisov, Duchon, Dulucq, Fayolle, Gessel, Fisher, Flajolet, Gouyou-Beauchamps, Guttmann, Guy, Janse van Rensburg, Johnson, Kauers, Koutschan, Krattenthaler, Kurkova, Kreweras, van Leeuwarden, MacMahon, Melczer, Mishna, Niederhausen, Petkovšek, Prellberg, Raschel, Rechnitzer, Sagan, Salvy, Viennot, Wachtel, Wilf, Yeats, Zeilberger...

dec.
Generating functions

- Our original question:

\[ a(n) = ? \quad a(i, j; n) = ? \]
Generating functions

- Our original question:

\[
a(n) = ? \quad a(i, j; n) = ?
\]

- Generating functions and their nature

\[
A(t) = \sum_{n \geq 0} a(n) t^n, \quad A(x, y; t) = \sum_{i, j, n} a(i, j; n) x^i y^j t^n
\]
Generating functions

• Our original question:

\[ a(n) = ? \quad a(i, j; n) = ? \]

• Generating functions and their nature

\[ A(t) = \sum_{n \geq 0} a(n) t^n, \quad A(x, y; t) = \sum_{i, j, n} a(i, j; n) x^i y^j t^n \]

Remarks

- \( A(1, 1; t) = A(t) \)
- \( A(0, 0; t) \) counts walks ending at (0, 0)
- \( A(x, 0; t) \) counts walks ending on the x-axis
Generating functions

- Our original question:

\[ a(n) = ? \quad a(i, j; n) = ? \]

- Generating functions and their nature

\[
A(t) = \sum_{n \geq 0} a(n) t^n, \quad A(x, y; t) = \sum_{i,j,n} a(i, j; n) x^i y^j t^n
\]

Can one express these series? Are they rational? algebraic? D-finite?

Remarks

- \( A(1, 1; t) = A(t) \)
- \( A(0, 0; t) \) counts walks ending at \((0, 0)\)
- \( A(x, 0; t) \) counts walks ending on the \(x\)-axis
A hierarchy of formal power series

- The formal power series $A(t)$ is **rational** if it can be written

$$A(t) = \frac{P(t)}{Q(t)}$$

where $P(t)$ and $Q(t)$ are polynomials in $t$.

- The formal power series $A(t)$ is **algebraic** (over $\mathbb{Q}(t)$) if it satisfies a (non-trivial) polynomial equation:

$$P(t, A(t)) = 0.$$  

- The formal power series $A(t)$ is **D-finite (holonomic)** if it satisfies a (non-trivial) linear differential equation with polynomial coefficients:

$$P_k(t)A^{(k)}(t) + \cdots + P_0(t)A(t) = 0.$$  

○ Nice closure properties + asymptotics of the coefficients
○ Extension to several variables (D-finite: one DE per variable)
Walks in the full space

Rational series

A rational generating function:

\[ a(n) = |S|^n \iff A(t) = \sum_{n \geq 0} a(n) t^n = \frac{1}{1 - |S| t} \]

More generally:

\[ A(x, y; t) = \frac{1}{1 - t \sum_{(i,j) \in S} x^i y^j}. \]
Walks in a rational half-space

Algebraic series

The associated generating function is algebraic, given by an explicit system of polynomial equations.

[Gessel 80], [Duchon 00], [mbm-Petkovšek 00]...
Walks in the quadrant

- Start from $p_0 = (0, 0)$

$$A(x, y; t) = \sum_{i, j, n \geq 0} a(i, j; n)x^i y^j t^n = ?$$
Walks in the quadrant

- Start from \( p_0 = (0, 0) \)

\[
A(x, y; t) = \sum_{i,j,n \geq 0} a(i, j; n)x^i y^j t^n = ?
\]
Walks in the quadrant

- Start from $p_0 = (0, 0)$

$$A(x, y; t) = \sum_{i,j,n \geq 0} a(i, j; n)x^i y^j t^n = ?$$
Walks in the quadrant

- Start from \( p_0 = (0, 0) \)

\[
A(x, y; t) = \sum_{i,j,n \geq 0} a(i, j; n)x^i y^j t^n = ?
\]
Walks in the quadrant

- Start from $p_0 = (0, 0)$

$A(x, y; t) = \sum_{i,j,n \geq 0} a(i, j; n)x^i y^j t^n = ?$
Quadrant walks with small steps: classification

- $S \subset \{\bar{1}, 0, 1\} \setminus \{00\} \Rightarrow 2^8 = 256$ step sets (or: models)
Quadrant walks with small steps: classification

• $S \subset \{-1, 0, 1\} \setminus \{00\} \Rightarrow 2^8 = 256$ step sets (or: models)

• However, some models are equivalent:
  – to a model of walks in the full or half-plane (⇒ algebraic)
  – to another model in the collection (diagonal symmetry)

• One is left with 79 interesting distinct models.

[mbm-Mishna 10]
Non-singular

Singular
<table>
<thead>
<tr>
<th>Non-singular</th>
</tr>
</thead>
<tbody>
<tr>
<td>Singular</td>
</tr>
</tbody>
</table>
Classification of quadrant walks with small steps

**Theorem**

The series \( A(x, y; t) \) is D-finite iff the associated group \( G \) is finite. It is algebraic iff, in addition, the orbit sum is zero.

[mbm-Mishna 10], [Bostan-Kauers 10]  
[Kurkova-Raschel 12]  
[Mishna-Rechnitzer 07], [Melczer-Mishna 13]  

D-finite

non-singular non-D-finite

singular non-D-finite

quadrant models: 79

\(|G| < \infty: 23\)  
\(|G| = \infty: 56\)  

\(\text{OS} \neq 0: 19\)  
\(\text{OS} = 0: 3 + 1\)  

D-finite  
not D-finite  
algebraic
Classification of quadrant walks with small steps

quadrant models: 79

\[ |G| < \infty: 23 \quad |G| = \infty: 56 \]

\[ \text{OS} \neq 0: 19 \quad \text{OS} = 0: 3 + 1 \quad \text{Not D-finite} \]

- D-finite
- Algebraic

- Formal power series algebra
- Computer algebra
- Complex analysis

- Random walks in probability
  - D-finite series
  - Effective closure properties
  - Arithmetic properties
  - Asymptotics
  - G-functions
When the orbit sum is zero: the four algebraic models

- Kreweras walks
  - expression for \( a(i, 0; n) \) [Kreweras 65], simplified by [Niederhausen 83]
  - \( A(x, y; t) \) is algebraic [Gessel 86, mbm 02/05]

- Unified approach for the Kreweras trilogy [mbm-Mishna 10]
When the orbit sum is zero: the four algebraic models

- **Kreweras walks**
  - expression for \( a(i, 0; n) \) [Kreweras 65], simplified by [Niederhausen 83]
  - \( A(x, y; t) \) is algebraic [Gessel 86, mbm 02/05]

- **Unified approach for the Kreweras trilogy** [mbm-Mishna 10]

- **Gessel walks**:
  - conjecture for \( a(0, 0; n) \) [Gessel \( \simeq \) 00]
  - proof of this conjecture [Kauers, Koutschan & Zeilberger 08]
  - \( A(t) \) and \( A(x, y; t) \) are algebraic! [Bostan & Kauers 09a]
  - new proof via complex analysis [Bostan, Kurkova & Raschel 13]
  - an elementary and constructive proof [mbm 15]
The number of walks of length $n = 3m + 2i$ ending at $(i, 0)$ is

$$a(i, 0; n) = \frac{4^m(2i + 1)}{(m + i + 1)(2m + 2i + 1)} \binom{2i}{i} \binom{3m + 2i}{m}$$

[Kreweras 65]

The generating function $A(x, y; t)$ is algebraic [Gessel 86].

For instance, the series $Q(t) := A(0, 0; t^{1/3})$ has degree 3:

$$54t - 1 + (1 - 72t)Q(t) + 16tQ(t)^2 + 64t^2 Q(t)^3 = 0$$
Gessel’s walks

**Theorem**

- The number of walks of length $n = 2m$ ending at $(0, 0)$ is
  
  $$a(0, 0; n) = 16^m \frac{(5/6)_m(1/2)_m}{(5/3)_m(2)_m},$$

where $(a)_m = a(a + 1) \cdots (a + m - 1)$ is the ascending factorial.

[Kauers, Koutschan, Zeilberger 08]

- The generating function $A(x, y; t)$ is algebraic.
  [Bostan, Kauers 09]

For instance, the series $Q(t) := A(0, 0; \sqrt{t})$ has degree 8:

$$27t^7Q^8 + 108t^6Q^7 + 189t^5Q^6 + 189t^4Q^5 - 9t^3(32t^2 + 28t - 13)Q^4$$
$$- 9t^2(64t^2 + 56t - 5)Q^3 - 2t(256t^3 - 312t^2 + 156t - 5)Q^2$$
$$- (32t - 1)(16t^2 - 28t + 1)Q - 256t^3 - 576t^2 - 48t + 1 = 0.$$
I. A solution of...
I. A solution of Kreheras’ model

[mbm 02]
I. A solution of Kreheras’ model

1. functional equation for \( A(x, y; t) \)
2. canceling the kernel: the roots \( Y_0 \) and \( Y_1 \)
3. the group of the walk
4. symmetric functions of \( Y_0 \) and \( Y_1 \)
1. A functional equation

Set step: \( S = \{\overline{11}, \overline{10}, 0\overline{1}\} \). Denote \( \overline{x} = 1/x \) and \( \overline{y} = 1/y \).

\[
A(x, y; t) \equiv A(x, y) = 1 + t(xy + \overline{x} + \overline{y})A(x, y) - t\overline{x}A(0, y) - t\overline{y}A(x, 0)
\]
1. A functional equation

Set step: \( S = \{11, 10, 01\} \). Denote \( \bar{x} = 1/x \) and \( \bar{y} = 1/y \).

\[
A(x, y; t) \equiv A(x, y) = 1 + t(xy + \bar{x} + \bar{y})A(x, y) - t\bar{x}A(0, y) - t\bar{y}A(x, 0)
\]

or

\[
(1 - t(xy + \bar{x} + \bar{y}))A(x, y) = 1 - t\bar{x}A(0, y) - t\bar{y}A(x, 0),
\]
1. A functional equation

Set step: \( S = \{11, \bar{10}, 0\bar{1}\} \). Denote \( \bar{x} = 1/x \) and \( \bar{y} = 1/y \).

\[ A(x, y; t) \equiv A(x, y) = 1 + t(xy + \bar{x} + \bar{y})A(x, y) - t\bar{x}A(0, y) - t\bar{y}A(x, 0) \]

or

\[ (1 - t(xy + \bar{x} + \bar{y}))A(x, y) = 1 - t\bar{x}A(0, y) - t\bar{y}A(x, 0), \]

or

\[ (1 - t(xy + \bar{x} + \bar{y}))xyA(x, y) = xy - tyA(0, y) - txA(x, 0) \]

• The polynomial \( 1 - t(xy + \bar{x} + \bar{y}) \) is the kernel of this equation

• The equation is linear, with two catalytic variables \( x \) and \( y \) (tautological at \( x = 0 \) or \( y = 0 \))
2. Canceling the kernel

- The equation:

\[(1 - t(xy + \bar{x} + \bar{y})) \cdot xyA(x, y) = xy - tyA(0, y) - txA(x, 0)\]
2. Canceling the kernel

- The equation:

\[(1 - t(xy + \bar{x} + \bar{y}))xyA(x, y) = xy - tyA(0, y) - txA(x, 0)\]

- The kernel \((1 - t(xy + \bar{x} + \bar{y}))\), as a polynomial in \(y\), has two roots:

\[
Y_0(x) = \frac{1 - t\bar{x} - \sqrt{(1 - t\bar{x})^2 - 4t^2x}}{2tx} = t + \bar{x}t^2 + O(t^3),
\]

\[
Y_1(x) = \frac{1 - t\bar{x} + \sqrt{(1 - t\bar{x})^2 - 4t^2x}}{2tx} = \frac{\bar{x}}{t} - \bar{x}^2 - t - \bar{x}t^2 + O(t^3).
\]
2. Canceling the kernel

- The equation:

\[(1 - t(xy + \bar{x} + \bar{y}))xyA(x, y) = xy - tyA(0, y) - txA(x, 0)\]

- The kernel \((1 - t(xy + \bar{x} + \bar{y}))\), as a polynomial in \(y\), has two roots:

\[Y_0(x) = \frac{1 - t\bar{x} - \sqrt{(1 - t\bar{x})^2 - 4t^2x}}{2tx} = t + \bar{x}t^2 + O(t^3),\]

\[Y_1(x) = \frac{1 - t\bar{x} + \sqrt{(1 - t\bar{x})^2 - 4t^2x}}{2tx} = \frac{\bar{x}}{t} - \bar{x}^2 - t - \bar{x}t^2 + O(t^3).\]

- Specializing \(y\) to \(Y_0\) in the equation gives:

\[R(x) + R(Y_0) = xY_0\]

with \(R(x) = txA(x, 0) = txA(0, x)\) (symmetry).

This equation characterizes \(R(x)\) but... why algebraicity?
3. The group of the model

The kernel

\[ K(x, y) = 1 - t(xy + \bar{x} + \bar{y}) \]

is left unchanged by the rational transformations

\[ \Phi : (x, y) \mapsto (\bar{x}\bar{y}, y) \quad \text{and} \quad \Psi : (x, y) \mapsto (x, \bar{x}\bar{y}) . \]
The kernel

\[ K(x, y) = 1 - t(xy + \bar{x} + \bar{y}) \]

is left unchanged by the rational transformations

\[ \Phi: (x, y) \mapsto (\bar{x}\bar{y}, y) \quad \text{and} \quad \Psi: (x, y) \mapsto (x, \bar{x}\bar{y}). \]

They are involutions, and generate a dihedral group \( G \) of order 6:
Build more pairs that cancel the kernel

- Since $K(x, Y_0) = 0$ and the transformations of $G$ preserve the kernel, any element of the orbit of $(x, Y_0)$ cancels the kernel.

\[
\begin{array}{c}
\Phi \quad (Y_1, Y_0) \quad \Psi \quad (Y_1, x) \quad \Phi \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
(x, Y_0) \quad (x, Y_1) \quad (Y_0, Y_1) \quad (Y_0, x) \\
\Psi \quad \Phi \quad \Psi
\end{array}
\]
Build more pairs that cancel the kernel

- Since $K(x, Y_0) = 0$ and the transformations of $G$ preserve the kernel, any element of the orbit of $(x, Y_0)$ cancels the kernel.
Build more pairs that cancel the kernel

- Since \( K(x, Y_0) = 0 \) and the transformations of \( G \) preserve the kernel, any element of the orbit of \((x, Y_0)\) cancels the kernel.

\[
\begin{align*}
\phi &: (x, Y_0) \rightarrow (Y_1, Y_0) \\
\psi &: (Y_1, Y_0) \rightarrow (Y_1, x) \\
\phi &: (Y_1, x) \rightarrow (Y_0, x) \\
\psi &: (Y_0, x) \rightarrow (Y_0, Y_1) \\
\phi &: (Y_0, Y_1) \rightarrow (x, Y_1) \\
\psi &: (x, Y_1) \rightarrow (x, Y_0)
\end{align*}
\]

Gives two equations for \( R \):

\[
\begin{align*}
R(x) + R(Y_0) &= xY_0 \\
R(Y_0) + R(Y_1) &= Y_0Y_1 = \bar{x}
\end{align*}
\]

with \( R(x) = txA(x, 0) \).
4. Symmetric functions of $Y_0$ and $Y_1$

- The elementary symmetric functions of the $Y_i$ are polynomials in $\bar{x}$:

\[ Y_0 + Y_1 = \frac{\bar{x}}{t} - \bar{x}^2 \quad \text{and} \quad Y_0 \, Y_1 = \bar{x}. \]
4. Symmetric functions of $Y_0$ and $Y_1$

- The elementary symmetric functions of the $Y_i$ are polynomials in $\bar{x}$:
  \[ Y_0 + Y_1 = \frac{\bar{x}}{t} - \bar{x}^2 \quad \text{and} \quad Y_0 Y_1 = \bar{x}. \]

- Rewrite the equations for $R$ as
  \[
  R(Y_0) - xY_0 = -R(x) \\
  R(Y_1) - xY_1 = R(x) + 2\bar{x} - 1/t
  \]
4. Symmetric functions of $Y_0$ and $Y_1$

- The elementary symmetric functions of the $Y_i$ are polynomials in $\bar{x}$:
  \[ Y_0 + Y_1 = \frac{\bar{x}}{t} - \bar{x}^2 \quad \text{and} \quad Y_0 Y_1 = \bar{x}. \]

- Rewrite the equations for $R$ as
  \[
  R(Y_0) - xY_0 = -R(x) \\
  R(Y_1) - xY_1 = R(x) + 2\bar{x} - 1/t
  \]

- The product is symmetric in $Y_0$ and $Y_1$:
  \[
  (R(Y_0) - xY_0)(R(Y_1) - xY_1) = -R(x)(R(x) + 2\bar{x} - 1/t)
  \]
4. Symmetric functions of $Y_0$ and $Y_1$

• The elementary symmetric functions of the $Y_i$ are polynomials in $\bar{x}$:

$$Y_0 + Y_1 = \frac{\bar{x}}{t} - \bar{x}^2 \quad \text{and} \quad Y_0 Y_1 = \bar{x}.$$ 

• Rewrite the equations for $R$ as

$$R(Y_0) - xY_0 = -R(x)$$
$$R(Y_1) - xY_1 = R(x) + 2\bar{x} - 1/t$$

• The product is symmetric in $Y_0$ and $Y_1$:

$$(R(Y_0) - xY_0)(R(Y_1) - xY_1) = -R(x)(R(x) + 2\bar{x} - 1/t)$$

• Extracting the non-negative powers of $x$ gives:

$$x - 2R'(0) = -R(x)(R(x) + 2\bar{x} - 1/t).$$

No more $Y_i$! Equivalently,

$$xtR(x)^2 + (2t - x)R(x) + tx^2 - 2txR'(0) = 0.$$
The series \( R(x) = txA(x, 0) \) satisfies
\[
xtR(x)^2 + (2t - x)R(x) + tx^2 - 2txR'(0) = 0.
\]
or
\[
t^2x^2A(x, 0)^2 + (2t - x)A(x, 0) - 2tA(0, 0) + x = 0.
\]
Tautological at \( x = 0 \): a quadratic equation with one catalytic variable \( x \), and one additional unknown one-variable series \( A(0, 0) \)
⇒ Brown’s quadratic method [Brown 69]
5. Finish: the quadratic method

The series $R(x) = t x A(x, 0)$ satisfies

$$x t R(x)^2 + (2t - x) R(x) + t x^2 - 2 t x R'(0) = 0.$$ 

or

$$t^2 x^2 A(x, 0)^2 + (2t - x) A(x, 0) - 2 t A(0, 0) + x = 0.$$ 

Tautological at $x = 0$: a quadratic equation with one catalytic variable $x$, and one additional unknown one-variable series $A(0, 0)$

$\Rightarrow$ Brown’s quadratic method [Brown 69]

**Theorem**

Let $X \equiv X(t)$ be the unique series in $t$ defined by

$$X = t (2 + X^3).$$

Then the generating function of Kreweras’ walks ending on the $x$-axis is

$$A(x, 0; t) = \frac{1}{t x} \left( \frac{1}{2t} - \frac{1}{x} - \left( \frac{1}{X} - \frac{1}{x} \right) \sqrt{1 - x X^2} \right).$$
II. A solution of Gessel’s model

[mbm 15]
II. A solution of Gessel’s model

[mbm 15]

- functional equation for $A(x, y)$
- canceling the kernel: the roots $Y_0$ and $Y_1$
- the group of the walk
- symmetric functions of $Y_0$ and $Y_1$
- ...
- the *generalized* quadratic method [mbm-Jehanne 06]
1. A functional equation

Set step: \( S = \{11, 10, \bar{11}, 0\bar{1}\} \)

\[
A(x, y) = 1 + t(xy + x + \bar{x}\bar{y} + \bar{x})A(x, y) \\
- t\bar{x}(1 + \bar{y})A(0, y) - t\bar{x}\bar{y}(A(x, 0) - A(0, 0))
\]
1. A functional equation

Set step: \( S = \{11, 10, \bar{1}1, 0\bar{1}\} \)

\[
A(x, y) = 1 + t(xy + x + \bar{x}\bar{y} + \bar{x})A(x, y) \\
- t\bar{x}(1 + \bar{y})A(0, y) - t\bar{x}\bar{y}(A(x, 0) - A(0, 0))
\]

or

\[
(1 - t(xy + x + \bar{x}\bar{y} + \bar{x}))xyA(x, y) = xy - S(y) - R(x)
\]

with

\[
S(y) = t(1 + y)A(0, y) \quad \text{and} \quad R(x) = t(A(x, 0) - A(0, 0)).
\]

- Main difference with K’ walks: no \( x/y \) symmetry, two bivariate unknown series \( R(x) \) and \( S(y) \)
2. Canceling the kernel

- The equation:

\[
(1 - t(xy + x + \bar{x}\bar{y} + \bar{x}))xyA(x, y) = xy - S(y) - R(x)
\]
2. Canceling the kernel

- The equation:

\[
(1 - t(xy + x + \bar{x}y + \bar{x}))xyA(x, y) = xy - S(y) - R(x)
\]

- The kernel, as a polynomial in \( y \), has two roots:

\[
Y_0(x) = \frac{1 - t(x + \bar{x}) - \sqrt{(1 - t(x + \bar{x}))^2 - 4t^2}}{2tx} = \bar{x}t + \cdots
\]

\[
Y_1(x) = \frac{1 - t(x + \bar{x}) + \sqrt{(1 - t(x + \bar{x}))^2 - 4t^2}}{2tx} = \frac{\bar{x}}{t} - (1 + \bar{x}^2) - \bar{x}t + \cdots
\]

Observe that \( xY_i(x) \) is symmetric in \( x \) and \( \bar{x} \).
2. Canceling the kernel

- The equation:

\[(1 - t(xy + x + \bar{x}y + \bar{x}))xyA(x, y) = xy - S(y) - R(x)\]

- The kernel, as a polynomial in \(y\), has two roots:

\[Y_0(x) = \frac{1 - t(x + \bar{x}) - \sqrt{(1 - t(x + \bar{x}))^2 - 4t^2}}{2tx} = \bar{x}t + \cdots\]

\[Y_1(x) = \frac{1 - t(x + \bar{x}) + \sqrt{(1 - t(x + \bar{x}))^2 - 4t^2}}{2tx} = \frac{\bar{x}}{t} - (1 + \bar{x}^2) - \bar{x}t + \cdots\]

Observe that \(xY_i(x)\) is symmetric in \(x\) and \(\bar{x}\).

- Specializing \(y\) to \(Y_0\) in the equation gives:

\[R(x) + S(Y_0) = xY_0\]
3. The group of the model

The kernel

\[ K(x, y) = 1 - t(xy + x + \bar{x}\bar{y} + \bar{x}) \]

is left unchanged by the rational transformations

\[ \Phi : (x, y) \mapsto (\bar{x}\bar{y}, y) \text{ and } \Psi : (x, y) \mapsto (x, \bar{x}^2\bar{y}) \].
3. The group of the model

The kernel

\[ K(x, y) = 1 - t(xy + x + \bar{x}\bar{y} + \bar{x}) \]

is left unchanged by the rational transformations

\[ \Phi : (x, y) \mapsto (\bar{x}\bar{y}, y) \quad \text{and} \quad \Psi : (x, y) \mapsto (x, \bar{x}^2\bar{y}) . \]

They are involutions, and generate a dihedral group of order 8:
Build more pairs that cancel the kernel

Since $K(x, Y_0) = 0$ and the transformations of $G$ preserve the kernel, any element of the orbit of $(x, Y_0)$ cancels the kernel.
Build more pairs that cancel the kernel

Since $K(x, Y_0) = 0$ and the transformations of $G$ preserve the kernel, any element of the orbit of $(x, Y_0)$ cancels the kernel.
Build more pairs that cancel the kernel

Since $K(x, Y_0) = 0$ and the transformations of $G$ preserve the kernel, any element of the orbit of $(x, Y_0)$ cancels the kernel.

Gives four equations for $R$ and $S$:

\[
\begin{align*}
R(x) + S(Y_0) &= xY_0 \\
R(Y_0) + S(Y_1) &= Y_0 Y_1 = \bar{x} \\
R(\bar{x}) + S(x^2 Y_0) &= xY_0 \\
R(xY_0) + S(x^2 Y_1) &= x^3 Y_0 Y_1 = x.
\end{align*}
\]
4. Symmetric functions of $Y_0$ and $Y_1$

- The elementary symmetric functions of the $Y_i$ are polynomials in $\bar{x}$:

$$Y_0 + Y_1 = -1 + \frac{\bar{x}}{t} - \bar{x}^2 \quad \text{and} \quad Y_0 Y_1 = \bar{x}^2.$$
4. Symmetric functions of $Y_0$ and $Y_1$

- The elementary symmetric functions of the $Y_i$ are polynomials in $\bar{x}$:
  \[ Y_0 + Y_1 = -1 + \frac{\bar{x}}{t} - \bar{x}^2 \quad \text{and} \quad Y_0 Y_1 = \bar{x}^2. \]

- From the four equations:
  \[
  \begin{align*}
  R(x) + S(Y_0) &= xY_0 \\
  R(xY_0) + S(Y_1) &= Y_0 Y_1 = \bar{x} \\
  R(\bar{x}) + S(x^2 Y_0) &= xY_0 \\
  R(xY_0) + S(x^2 Y_1) &= x^3 Y_0 Y_1 = x,
  \end{align*}
  \]
  one constructs two symmetric functions of $Y_0$ and $Y_1$, and extracts non-negative powers of $x$.
  This gives an equation without any $Y_i$... but with $R(x)$ and $R(\bar{x})$. 
4. Symmetric functions of $Y_0$ and $Y_1$ (cont’d)

This gives:

$$R(x)^2 + R(x)R(\bar{x}) + R(\bar{x})^2 + (2\bar{x} - 1/t + x + S(0))R(x) + (2x - 1/t + \bar{x} + S(0))R(\bar{x}) = 2R'(0) - (\bar{x} + x - 1/t)S(0) - 1$$

while for Kreneras’ walks:

$$xtR(x)^2 + (2t - x)R(x) + tx^2 - 2txR'(0) = 0.$$ 

- Main difference: the terms $R(\bar{x})$
- Main difficulty: the hybrid term $R(x)R(\bar{x})$
5. Les identités remarquables à notre secours

The equation reads:

\[
R(x)^2 + R(x)R(\bar{x}) + R(\bar{x})^2 + (2\bar{x} - 1/t + x + S(0))R(x) \\
+ (2x - 1/t + \bar{x} + S(0))R(\bar{x}) = 2R'(0) - (\bar{x} + x - 1/t)S(0) - 1.
\]

But but but...

\[
(a^2 + ab + b^2)(a - b) = a^3 - b^3
\]
The equation reads:

\[
R(x)^2 + R(x)R(\bar{x}) + R(\bar{x})^2 + (2\bar{x} - 1/t + x + S(0))R(x)
+ (2x - 1/t + \bar{x} + S(0))R(\bar{x}) = 2R'(0) - (\bar{x} + x - 1/t)S(0) - 1.
\]

But but but...

\[
(a^2 + ab + b^2)(a - b) = a^3 - b^3
\]

\[\Rightarrow\text{ Multiply by } R(x) - R(\bar{x}) + \bar{x} - x \text{!} \]
5. Les identités remarquables à notre secours

The equation reads:

\[ R(x)^2 + R(x)R(\bar{x}) + R(\bar{x})^2 + (2\bar{x} - 1/t + x + S(0))R(x) \]
\[ + (2x - 1/t + \bar{x} + S(0))R(\bar{x}) = 2R'(0) - (\bar{x} + x - 1/t)S(0) - 1. \]

But but but...

\[(a^2 + ab + b^2)(a - b) = a^3 - b^3\]

⇒ Multiply by \( R(x) - R(\bar{x}) + \bar{x} - x \) ! This gives \( P(x) = P(\bar{x}) \), with

\[ P(x) = R(x)^3 + (S(0) + 3\bar{x} - 1/t)R(x)^2 \]
\[ + (2\bar{x}^2 - \bar{x}/t + x/t - x^2 - 2R'(0) + (2\bar{x} - 1/t)S(0)) \]
\[ - x^2S(0) + x \left( 2R'(0) + S(0)/t - 1 \right). \]
We have obtained \( P(x) = P(\bar{x}) \), with

\[
P(x) = R(x)^3 + (S(0) + 3\bar{x} - 1/t)R(x)^2 \\
+ (2\bar{x}^2 - \bar{x}/t + x/t - x^2 - 2R'(0) + (2\bar{x} - 1/t)S(0)) R(x) \\
- x^2S(0) + x(2R'(0) + S(0)/t - 1).
\]

Extract nonnegative powers of \( x \):

\[
R(x)^3 + (S(0) + 3\bar{x} - 1/t)R(x)^2 \\
+ (2\bar{x}^2 - \bar{x}/t + x/t - x^2 - 2R'(0) + (2\bar{x} - 1/t)S(0)) R(x) \\
= R''(0) + R'(0)(2S(0) + 2\bar{x} - 1/t) + xS(0)(x - 1/t) + x.
\]

A cubic equation in one catalytic variable \( x \), with three additional unknown one-variable series \( S(0), R'(0) \) and \( R''(0) \).
6. Finish: the generalized quadratic method

\[ R(x)^3 + (S(0) + 3\bar{x} - 1/t)R(x)^2 \]
\[ + (2\bar{x}^2 - \bar{x}/t + x/t - x^2 - 2R'(0) + (2\bar{x} - 1/t)S(0)) R(x) \]
\[ = R''(0) + R'(0)(2S(0) + 2\bar{x} - 1/t) + xS(0)(x - 1/t) + x. \]

**Theorem**

Every series \( R(x; t) \equiv R(x) \) solution of a (proper) polynomial equation with one catalytic variable \( x \) is algebraic. That is, if

\[ P(t, x, R(x), A_1, \ldots, A_k) = 0 \]

for some polynomial \( P \) with coefficients in \( \mathbb{Q} \), then \( R(x) \) is algebraic over \( \mathbb{Q}(t, x) \), and each \( A_i \) is algebraic over \( \mathbb{Q}(t) \).

[mbm-Jehanne 2006] (after [Knuth 72] and [Brown 65])
6. Finish: the generalized quadratic method

**Theorem**

The generating function $A(x, y; t)$ is algebraic of degree 72 (explicit equations and rational parametrizations for $A(x, 0)$ and $A(0, y)$)

- In particular, the series $Q(t) := A(0, 0; \sqrt{t})$ has degree 8:

  
  \[
  27t^7Q^8 + 108t^6Q^7 + 189t^5Q^6 + 189t^4Q^5 - 9t^3(32t^2 + 28t - 13)Q^4 \\
  - 9t^2(64t^2 + 56t - 5)Q^3 - 2t(256t^3 - 312t^2 + 156t - 5)Q^2 \\
  - (32t - 1)(16t^2 - 28t + 1)Q - 256t^3 - 576t^2 - 48t + 1 = 0.
  \]

- The number of walks of length $n = 2m$ ending at $(0, 0)$ is

  \[
  a(0, 0; 2m) = 16^m \frac{(5/6)_m(1/2)_m}{(5/3)_m(2)_m},
  \]

  where $(a)_m = a(a + 1) \cdots (a + m - 1)$ is the ascending factorial.
other elementary proofs will come for sure
3D Kreweras’ walks remain mysterious
works for other algebraic models with repeated steps, proved so far by computer algebra [Kauers, Yatchak 14(a)]
other elementary proofs will come for sure
3D Kreweras’ walks remain mysterious
works for other algebraic models with repeated steps, proved so far by computer algebra [Kauers, Yatchak 14(a)]

Open: Find elementary proofs for

- Algebraic
  - Conj. [Kauers, Yatchak 14(a)]
  - Proof via complex analysis [R. et al.]
- D-finite
  - Proof via computer algebra [Kauers, Yatchak 14(a)]
Polynomial equations with one catalytic variable

• Assume

\[ P(F(x), A_1, \ldots, A_k, t, x) = 0 \]

where \( P(x_0, x_1, \ldots, x_k, t, x) \) is a polynomial with coefficients in \( \mathbb{K} \), \( F(x) \equiv F(x; t) \in \mathbb{K}[x][[t]] \), and \( A_i \in \mathbb{K}[[t]] \) for all \( i \).
Assume

\[ P(F(x), A_1, \ldots, A_k, t, x) = 0 \]

where \( P(x_0, x_1, \ldots, x_k, t, x) \) is a polynomial with coefficients in \( K \), \( F(x) \equiv F(x; t) \in K[x][[t]] \), and \( A_i \in K[[t]] \) for all \( i \).

For any series \( X \equiv X(t) \) such that

- the series \( F(X) \equiv F(X; t) \) is well-defined
- \( \frac{\partial P}{\partial x_0}(F(X), A_1, \ldots, A_k, t, X) = 0 \),

one has

\[ \frac{\partial P}{\partial x}(F(X), A_1, \ldots, A_k, t, X) = 0. \]
Polynomial equations with one catalytic variable

• Assume

\[ P(F(x), A_1, \ldots, A_k, t, x) = 0 \]

where \( P(x_0, x_1, \ldots, x_k, t, x) \) is a polynomial with coefficients in \( \mathbb{K} \),
\( F(x) \equiv F(x; t) \in \mathbb{K}[x][[t]] \), and \( A_i \in \mathbb{K}[[t]] \) for all \( i \).

• For any series \( X \equiv X(t) \) such that
  
  - the series \( F(X) \equiv F(X; t) \) is well-defined
  
  \( \frac{\partial P}{\partial x_0} (F(X), A_1, \ldots, A_k, t, X) = 0 \),

one has

\[ \frac{\partial P}{\partial x} (F(X), A_1, \ldots, A_k, t, X) = 0. \]

Proof: differentiate (1) with respect to \( x \)

\[ F'(x) \frac{\partial P}{\partial x_0} (F(x), A_1, \ldots, A_k, t, x) + \frac{\partial P}{\partial x} (F(x), A_1, \ldots, A_k, t, x) = 0. \]
Polynomial equations with one catalytic variable

- Assume there exist $k$ series $X_1, \ldots, X_k$ such that

$$\frac{\partial P}{\partial x_0}(F(X_i), A_1, \ldots, A_k, t, X_i) = 0.$$ 

In this case, for each $X_i$,

$$\frac{\partial P}{\partial x}(F(X_i), A_1, \ldots, A_k, t, X_i) = 0$$

and

$$P(F(X_i), A_1, \ldots, A_k, t, X_i) = 0.$$

- This system of $3k$ polynomial equations in $3k$ unknowns $A_1, \ldots, A_k$, $X_1, \ldots, X_k$, $F(X_1), \ldots, F(X_k)$ may imply (together with the fact that the $X_i$ are distinct) the algebraicity of the $A_i$.

[mbm-Jehanne 06]
A computer algebra approach

Example. When \( S = \{10, 01, 11\} \), the equation reads

\[
(1 - t(\bar{x} + \bar{y} + xy))xyA(x, y; t) = xy - tyA(0, y; t) - txA(x, 0; t).
\]

Naïve route: guess and check!

- Guess a polynomial equation \( \text{Pol} \) satisfied by \( A(x, y; t) \)
  (degrees \([18, 18, 17, 12]\) in \( x, y, t, A \))
- Let \( F(x, y; t) \) be the solution of \( \text{Pol} \) that coincides with \( A(x, y; t) \) up to high order (in \( t \))
- Prove that \( F(x, y; t) \) is a formal power series in \( t \) with polynomial coefficients in \( x \) and \( y \) \( \Rightarrow \) \( F(x, 0; t) \) and \( F(0, y; t) \) are well-defined
- By taking resultants, prove that \( F(x, y; t) \) satisfies the above functional equation.
Example. When $S = \{\bar{1}0, 0\bar{1}, 11\}$, the equation reads

$$(1 - t(\bar{x} + \bar{y} + xy))xyA(x, y; t) = xy - tyA(0, y; t) - txA(x, 0; t).$$

Naïve route: guess and check!

- Guess a polynomial equation $\text{Pol}$ satisfied by $A(x, y; t)$
  (degrees $[18, 18, 17, 12]$ in $x, y, t, A$)
- Let $F(x, y; t)$ be the solution of $\text{Pol}$ that coincides with $A(x, y; t)$ up to high order (in $t$)
- Prove that $F(x, y; t)$ is a formal power series in $t$ with polynomial coefficients in $x$ and $y$ $\Rightarrow$ $F(x, 0; t)$ and $F(0, y; t)$ are well-defined
- By taking resultants, prove that $F(x, y; t)$ satisfies the above functional equation.
A computer algebra approach

Example. When $S = \{\bar{1}0, 0\bar{1}, 11\}$, the equation reads

$$(1 - t(\bar{x} + \bar{y} + xy))xyA(x, y; t) = xy - tyA(0, y; t) - txA(x, 0; t).$$

Naïve route: guess and check!

- Guess a polynomial equation Pol satisfied by $A(x, y; t)$
  (degrees $[18, 18, 17, 12]$ in $x, y, t, A$)
- Let $F(x, y; t)$ be the solution of Pol that coincides with $A(x, y; t)$ up to high order (in $t$)
- Prove that $F(x, y; t)$ is a formal power series in $t$ with polynomial coefficients in $x$ and $y \Rightarrow F(x, 0; t)$ and $F(0, y; t)$ are well-defined
- By taking resultants, prove that $F(x, y; t)$ satisfies the above functional equation.
A computer algebra approach

Example. When $\mathcal{S} = \{\overline{10}, \overline{01}, 11\}$, the equation reads

$$(1 - t(\overline{x} + \overline{y} + xy))xyA(x, y; t) = xy - tyA(0, y; t) - txA(x, 0; t).$$

Naïve route: guess and check!

- Guess a polynomial equation $\text{Pol}$ satisfied by $A(x, y; t)$
  (degrees $[18, 18, 17, 12]$ in $x, y, t, A$)
- Let $F(x, y; t)$ be the solution of $\text{Pol}$ that coincides with $A(x, y; t)$ up to high order (in $t$)
- Prove that $F(x, y; t)$ is a formal power series in $t$ with polynomial coefficients in $x$ and $y \Rightarrow F(x, 0; t)$ and $F(0, y; t)$ are well-defined
- By taking resultants, prove that $F(x, y; t)$ satisfies the above functional equation.
Example. When $S = \{\bar{1}0, 0\bar{1}, 11\}$, the equation reads

$$(1 - t(\bar{x} + \bar{y} + xy))xyA(x, y; t) = xy - tyA(0, y; t) - txA(x, 0; t).$$

Naïve route: guess and check!

- Guess a polynomial equation $Pol$ satisfied by $A(x, y; t)$
  (degrees $[18, 18, 17, 12]$ in $x, y, t, A$)
- Let $F(x, y; t)$ be the solution of $Pol$ that coincides with $A(x, y; t)$ up to high order (in $t$)
- Prove that $F(x, y; t)$ is a formal power series in $t$ with polynomial coefficients in $x$ and $y \Rightarrow F(x, 0; t)$ and $F(0, y; t)$ are well-defined
- By taking resultants, prove that $F(x, y; t)$ satisfies the above functional equation.
- Then

$$(1 - t(\bar{x} + \bar{y} + xy))xyA(x, y; t) = xy - tyF(y, 0; t) - txF(x, 0; t).$$
A computer algebra approach

Example. When $S = \{\bar{1}0, 0\bar{1}, 11\}$, the equation reads

$$(1 - t(\bar{x} + \bar{y} + xy))xyA(x, y; t) = xy - tyA(0, y; t) - txA(x, 0; t).$$

A less naïve route

- Guess a polynomial equation $Pol$ satisfied by $A(x, 0; t) = A(0, x; t)$ (degrees $[6, 10, 6]$ in $x, t, A$)
- Let $F(x, 0; t)$ be the solution of $Pol$ that coincides with $A(x, 0; t)$ up to high order (in $t$)
- Prove that $F(x, 0; t)$ is a formal power series in $t$ with polynomial coefficients in $x$
- By taking resultants, prove that $F$ satisfies the equation obtained by canceling the kernel:

$$txF(x, 0; t) + tY_0F(Y_0, 0; t) = xY_0.$$ 

- Then

$$ (1 - t(\bar{x} + \bar{y} + xy))xyA(x, y; t) = xy - tyF(y, 0; t) - txF(x, 0; t).$$
A computer algebra approach

Example. When \( S = \{\bar{1}0, 0\bar{1}, 11\} \), the equation reads

\[
(1 - t(\bar{x} + \bar{y} + xy))xyA(x, y; t) = xy - tyA(0, y; t) - txA(x, 0; t).
\]

A less naïve route

- Guess a polynomial equation \( \text{Pol} \) satisfied by \( A(x, 0; t) = A(0, x; t) \)
  (degrees \([6, 10, 6]\) in \( x, t, A \))
- Let \( F(x, 0; t) \) be the solution of \( \text{Pol} \) that coincides with \( A(x, 0; t) \) up to high order (in \( t \))
- Prove that \( F(x, 0; t) \) is a formal power series in \( t \) with polynomial coefficients in \( x \)
- By taking resultants, prove that \( F \) satisfies the equation obtained by canceling the kernel:

\[
txF(x, 0; t) + tY_0F(Y_0, 0; t) = xY_0.
\]

- Then

\[
(1 - t(\bar{x} + \bar{y} + xy))xyA(x, y; t) = xy - tyF(y, 0; t) - txF(x, 0; t).
\]
Example. When $\mathcal{S} = \{\bar{1}0, 0\bar{1}, 11\}$, the equation reads
\[
(1 - t(\bar{x} + \bar{y} + xy))xyA(x, y; t) = xy - tyA(0, y; t) - txA(x, 0; t).
\]

A less naïve route

- Guess a polynomial equation $\text{Pol}$ satisfied by $A(x, 0; t) = A(0, x; t)$ (degrees $[6, 10, 6]$ in $x, t, A$)
- Let $F(x, 0; t)$ be the solution of $\text{Pol}$ that coincides with $A(x, 0; t)$ up to high order (in $t$)
- Prove that $F(x, 0; t)$ is a formal power series in $t$ with polynomial coefficients in $x$
- By taking resultants, prove that $F$ satisfies the equation obtained by canceling the kernel:
\[
 txF(x, 0; t) + tY_0F(Y_0, 0; t) = xY_0.
\]

Then
\[
(1 - t(\bar{x} + \bar{y} + xy))xyA(x, y; t) = xy - tyF(y, 0; t) - txF(x, 0; t).
\]
A computer algebra approach

Example. When $S = \{\bar{1}0, 0\bar{1}, 11\}$, the equation reads

$$(1 - t(\bar{x} + \bar{y} + xy))xyA(x, y; t) = xy - tyA(0, y; t) - txA(x, 0; t).$$

A less naïve route

- Guess a polynomial equation $Pol$ satisfied by $A(x, 0; t) = A(0, x; t)$ (degrees $[6, 10, 6]$ in $x, t, A$)
- Let $F(x, 0; t)$ be the solution of $Pol$ that coincides with $A(x, 0; t)$ up to high order (in $t$)
- Prove that $F(x, 0; t)$ is a formal power series in $t$ with polynomial coefficients in $x$
- By taking resultants, prove that $F$ satisfies the equation obtained by canceling the kernel:

$$txF(x, 0; t) + tY_0F(Y_0, 0; t) = xY_0.$$ 

Then

$$(1 - t(\bar{x} + \bar{y} + xy))xyA(x, y; t) = xy - tyF(y, 0; t) - txF(x, 0; t).$$
A computer algebra approach

Example. When \( S = \{\bar{1}0, 0\bar{1}, 11\} \), the equation reads

\[
(1 - t(\bar{x} + \bar{y} + xy))xyA(x, y; t) = xy - tyA(0, y; t) - txA(x, 0; t).
\]

A less naïve route

- Guess a polynomial equation Pol satisfied by \( A(x, 0; t) = A(0, x; t) \)
  (degrees \([6, 10, 6]\) in \( x, t, A \))
- Let \( F(x, 0; t) \) be the solution of Pol that coincides with \( A(x, 0; t) \) up to high order (in \( t \))
- Prove that \( F(x, 0; t) \) is a formal power series in \( t \) with polynomial coefficients in \( x \)
- By taking resultants, prove that \( F \) satisfies the equation obtained by canceling the kernel:

\[
txF(x, 0; t) + tY_0F(Y_0, 0; t) = xY_0.
\]

Then

\[
(1 - t(\bar{x} + \bar{y} + xy))xyA(x, y; t) = xy - tyF(y, 0; t) - txF(x, 0; t).
\]
A computer algebra approach: climax

Algebraicity of Gessel’s model [Bostan-Kauers 10]

- When $S = \{10, \bar{10}, 11, \bar{11}\}$, the series $A(x, y; t)$ is algebraic (degree 72).
- In particular, the series $A(0, 0; t)$, which counts loops, has degree 8, and the following expansion:

$$A(0, 0; t) = \sum_{n \geq 0} 16^n \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} t^{2n}, \quad (1)$$

with $(i)_n = i(i + 1) \cdots (i + n - 1)$.


The algebraicity of $A(x, y; t)$ has now been re-proved using a complex analysis approach [Bostan, Kurkova & Raschel 13(a)]
A complex analysis approach

Markov chains with small steps in the quadrant: stationary distribution(s) [Malyshev 71+]

Le petit livre jaune
[Fayolle, Iasnogorodski & Malyshev 99]

⇒ Reduction to a boundary value problem of the Riemann-Carleman type
A complex analysis approach

An expression of $Q$ for any non-singular model $S$

$$\tilde{K}(x, 0; t)A(x, 0; t) - \tilde{K}(0, 0; t)A(0, 0; t) = xY_0(x; t) +$$

$$\frac{1}{2i\pi} \int_{x_1(t)}^{x_2(t)} u \left[ Y_0(u; t) - Y_1(u; t) \right] \left[ \frac{\partial_u w(u; t)}{w(u; t) - w(x; t)} - \frac{\partial_u w(u; t)}{w(u; t) - w(0; t)} \right] du$$

where $Y_0$, $Y_1$, $x_1$ and $x_2$ are explicit algebraic series and $w$ is explicit/very well understood.

[Raschel 12] + Fayolle, Kurkova

+ other formulas that complete the characterization of $A(x, y; t)$
A complex analysis approach

An expression of $Q$ for any non-singular model $S$

$$
\tilde{K}(x, 0; t)A(x, 0; t) - \tilde{K}(0, 0; t)A(0, 0; t) = xY_0(x; t) +
\frac{1}{2i\pi} \int_{x_1(t)}^{x_2(t)} u [Y_0(u; t) - Y_1(u; t)] \left[ \frac{\partial_u w(u; t)}{w(u; t) - w(x; t)} - \frac{\partial_u w(u; t)}{w(u; t) - w(0; t)} \right] du
$$

where $Y_0$, $Y_1$, $x_1$ and $x_2$ are explicit algebraic series and $w$ is explicit/very well understood.

In particular, $w$ is D-finite (in fact, algebraic!) iff the group is finite.

[Raschel 12] + Fayolle, Kurkova

+ other formulas that complete the characterization of $A(x, y; t)$
A complex analysis approach: climax(es)

Theorems

- If $S$ has an infinite group and is not singular, then $A(x, y; t)$ is not D-finite in $x$ (≡ no differential equation with respect to $x$)

[Bostan, Kurkova & Raschel 13(a)]

- A new proof of the algebraicity of Gessel’s model

[Bostan, Kurkova & Raschel 13(a)]