Self-avoiding walks

on the honeycomb lattice

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What is $c(n)$, the number of $n$-step SAW?

Not so easy! $c(n)$ is known up to $n = 105$ [Jensen 06]

Problem: a highly non-markovian model
Simple bounds on $c(n)$

We have

$$F_{n+1} \leq c(n) \leq 3 \cdot 2^{n-1}$$

with $F_0 = F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ (Fibonacci numbers).

- Upper bound: at most two possible extensions at each step
- Lower bound: SAW that never move in the South direction (partially directed).
The connective constant

Clearly,

\[ c(m + n) \leq c(m)c(n) \]

⇒ \( \lim c(n)^{1/n} \) exists and

\[ \mu := \lim c(n)^{1/n} = \inf c(n)^{1/n} \]
The connective constant

\[ \mu := \lim c(n)^{1/n} = \inf c(n)^{1/n} \]

Given the bounds

\[ F_{n+1} \leq c(n) \leq 3 \cdot 2^{n-1}, \]

there holds

\[ \frac{1 + \sqrt{5}}{2} \simeq 1.62 \leq \mu \leq 2 \]

**Theorem** [Duminil-Copin & Smirnov 10]

\[ \mu = \sqrt{2 + \sqrt{2}} \simeq 1.85 \]

Conjectured by Nienhuis in 1982
Sub-families of SAW

Self-avoiding polygons and self-avoiding bridges

- Same growth constant:

  \[ p(n)^{1/n} \to \mu \quad \text{and} \quad b(n)^{1/n} \to \mu \]

[Hammersley 61]
Generating functions

For self-avoiding walks, polygons and bridges:

\[ C(t) = \sum c(n)t^n, \quad P(t) = \sum p(n)t^n, \quad B(t) = \sum b(n)t^n \]

**Theorem** [Duminil-Copin & Smirnov 10]

The radius of convergence of these series is

\[ t_c = \frac{1}{\sqrt{2 + \sqrt{2}}} \]

Moreover:

- \( C(t) \) diverges at \( t_c \),
- \( P(t) \) converges at \( t_c \).

**Conjectures:**

\[ C(t) \sim \frac{1}{(1 - t/t_c)^{43/32}}, \quad P(t)_{\text{sing}} \sim (1 - t/t_c)^{3/2}, \quad B(t) \sim \frac{1}{(1 - t/t_c)^{9/16}} \]
Walks starting and ending at mid-edges
A key identity

Consider the following domain $D_{h,\ell}$. 

\[
\begin{array}{c}
\text{Diagram}
\end{array}
\]
Consider the following domain $D_{h,\ell}$.

Let $L_{h,\ell}(t)$ (resp. $R_{h,\ell}(t)$, $M_{h,\ell}(t)$) be the generating function of walks that start from $a$ and end on the left (resp. right, middle) border of the domain $D_{h,\ell}$. These series are polynomials in $t$.

Then for all $h$ and $\ell$, at $t = t_c = 1/\sqrt{2 + \sqrt{2}},$

$$\frac{\sqrt{2 - \sqrt{2}}}{2} L_{h,\ell}(t_c) + R_{h,\ell}(t_c) + \frac{1}{\sqrt{2}} M_{h,\ell}(t_c) = 1$$
An upper bound on $\mu$

$$\frac{\sqrt{2 - \sqrt{2}}}{2} L_{h,\ell}(t_c) + R_{h,\ell}(t_c) + \frac{1}{\sqrt{2}} M_{h,\ell}(t_c) = 1$$

Consider walks in $D_{h,\ell}$ that go from $a$ to $a^+$ ($\simeq$ self-avoiding polygons). Their generating function $\tilde{P}_{h,\ell}(t)$ satisfies

$$\tilde{P}_{h,\ell}(t_c) \leq L_{h,\ell}(t_c) \leq \frac{2}{\sqrt{2 - \sqrt{2}}}$$

But

$$\tilde{P}_{h,\ell}(t_c) \to \tilde{P}(t_c) \geq t_c^2 P(t_c)$$

as $h, \ell \to \infty$. This implies that $P(t_c)$ is finite. Hence $\mu \leq 1/t_c = \sqrt{2 + \sqrt{2}}$. 
A lower bound on $\mu$

$$\frac{\sqrt{2} - \sqrt{2}}{2} L_{h,\ell}(t_c) + R_{h,\ell}(t_c) + \frac{1}{\sqrt{2}} M_{h,\ell}(t_c) = 1$$

As $h \to \infty$, $L_{h,\ell}(t_c)$ and $R_{h,\ell}(t_c)$ increase to limit values $L_{\ell}(t_c)$ and $R_{\ell}(t_c)$. Hence $M_{h,\ell}(t_c)$ decreases to a limit value $M_{\ell}(t_c)$.

- If $M_{\ell}(t_c) > 0$ for some $\ell$, the series

$$\sum_{h} M_{h,\ell}(t_c)$$

diverges. As it counts certain SAW, this implies that $\mu \geq 1/t_c$.
- If $M_{\ell}(t_c) = 0$ for all $\ell$, another argument shows that $\sum_{\ell} R_{\ell}(t_c)$ (the generating function of bridges) diverges.

In both cases, $\mu \geq 1/t_c = \sqrt{2} + \sqrt{2}$. 
A lower bound on \( \mu \) (continued)

\[
\alpha \ L_{h,\ell}(t_c) + R_{h,\ell}(t_c) + \frac{1}{\sqrt{2}} \ M_{h,\ell}(t_c) = 1
\]

- If \( M_\ell(t_c) = 0 \) for all \( \ell \), then

\[
\alpha \ L_\ell(t_c) + R_\ell(t_c) = 1
\]

\[
\Rightarrow \alpha \ (L_{\ell+1}(t_c) - L_\ell(t_c)) = R_\ell(t_c) - R_{\ell+1}(t_c)
\]

\[
\Rightarrow 2\alpha t_c R_\ell R_{\ell+1} \geq R_\ell(t_c) - R_{\ell+1}(t_c)
\]

\[
\Rightarrow 2\alpha t_c \geq \frac{1}{R_{\ell+1}(t_c)} - \frac{1}{R_\ell(t_c)}
\]

\[
\Rightarrow 2\ell\alpha t_c + \frac{1}{R_1} \geq \frac{1}{R_\ell}
\]

\[
\Rightarrow R_\ell \geq \frac{1}{2\ell\alpha t_c + \beta'}
\]

so that \( \sum_\ell R_\ell(t_c) \) (the generating function of bridges) diverges.
A key identity

\[ \frac{\sqrt{2} - \sqrt{2}}{2} L_{h,\ell}(t_c) + R_{h,\ell}(t_c) + \frac{1}{\sqrt{2}} M_{h,\ell}(t_c) = 1 \]

Where does it come from?

From a local identity that is re-summed over all vertices of the domain.
A local identity

Let $D$ be a finite, simply connected domain, and $a$ a point on the boundary of $D$. For $p$ a point in the domain, let

$$F(p) \equiv F(t, \alpha; p) = \sum_{\omega: a \rightarrow p} t|\omega| e^{i\alpha W(\omega)},$$

where $|\omega|$ is the length of $\omega$, and $W(\omega)$ its winding number.

\[ W(\omega) = 0 \quad W(\omega) = -2\pi \quad W(\omega) = -\pi \]
A local identity

Let $D$ be a finite, simply connected domain, and $a$ a point on the boundary of $D$. For $p$ a point in the domain, let

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where $|\omega|$ is the length of $\omega$, and $W(\omega)$ its winding number.

$W(\omega) = 0 \quad W(\omega) = -2\pi \quad W(\omega) = -\pi$

Rem. $W$ is additive:

$$W(\omega) = \frac{\pi}{3} \text{(left turns} - \text{right turns}).$$
A local identity

Let

\[ F(p) \equiv F(t, \alpha; p) = \sum_{\omega: a \rightarrow p \text{ in } D} t|\omega|e^{i\alpha W(\omega)}, \]

If \( p, q \) and \( r \) are the 3 mid-edges around a vertex \( v \) of the honeycomb lattice, taken in counterclockwise order, then, for \( t = t_c \) and \( \alpha = -5/8 \),

\[ F(p) + jF(q) + j^2F(r) = 0, \]

where \( j = e^{2i\pi / 3} \), or, more symmetrically,

\[ (p - v)F(p) + (q - v)F(q) + (r - v)F(r) = 0. \]
A local identity

Proof: Group walks three by three or two by two as follows

- Walks that avoid \( v \) + walks in which \( v \) is the last visited vertex:

- Walks that visit \( v \) before their last vertex:

The contribution of all walks in a group is zero.
A local identity

- Walks that avoid $v$ + walks in which $v$ is the last visited vertex:

\[
1 + j t_c \, e^{-i\alpha \pi/3} + j^2 t_c \, e^{i\alpha \pi/3} = 0
\]
A local identity

- Walks that avoid $v$ + walks in which $v$ is the last visited vertex:

\[ 1 + j t_c e^{-i\alpha \pi/3} + j^2 t_c e^{i\alpha \pi/3} = 0 \]

- Walks that visit $v$ before their last vertex:

\[ e^{4i\alpha \pi/3} + j^2 e^{-4i\alpha \pi/3} = 2/j \cos \left( \frac{\pi}{3} + 4\alpha \frac{\pi}{3} \right) = 0 \]
Proof of the key identity

Sum the local identity

\[(p - v)F(p) + (q - v)F(q) + (r - v)F(r) = 0\]

over all vertices \(v\) of the domain \(D_{h,\ell}\).

- The inner mid-edges do not contribute.
- The winding number of walks ending on the boundary is known.
Proof of the key identity

Sum the local identity

\[(p - v)F(p) + (q - v)F(q) + (r - v)F(r) = 0\]
over all vertices \(v\) of the domain \(D_{h,\ell}\).

- The inner mid-edges do not contribute.
- The domain has an up-down symmetry.
- The winding number of walks ending on the boundary is known.

This gives:

\[
\sqrt{2 - \sqrt{2}} \frac{L_{h,\ell}(t_c) + R_{h,\ell}(t_c) + \frac{1}{\sqrt{2}} M_{h,\ell}(t_c)}{2} = 1,
\]
where \(L_{h,\ell}(t)\) (resp. \(R_{h,\ell}(t), M_{h,\ell}(t)\)) are the generating function of walks that start from \(a\) and end on the left (resp. right, middle) border of the domain \(D_{h,\ell}\).
References

- ArXiv 1007.0575

- Some slides:
  - Smirnov’s lecture at ICM
  - A video of Duminil-Copin at Microsoft