Rational and algebraic series in enumerative combinatorics

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Enumeration and generating functions

Let $\mathcal{A}$ be a set of discrete objects equipped with a size:

$$\begin{array}{rcl} size : \mathcal{A} & \to & \mathbb{N} \\ a & \mapsto & |a| \end{array}$$

Assume that for all n,

$$\mathcal{A}_n := \{a \in \mathcal{A} : |a| = n\} \text{ is finite.}$$

Let $a(n) = |\mathcal{A}_n|$.

The generating function of the objects of $\mathcal{A}$, counted by their size, is

$$\begin{array}{rcl} A(t) & := & \displaystyle\sum_{n \geq 0} a(n) t^n \\ & = & \displaystyle\sum_{a \in \mathcal{A}} t^{|a|}. \end{array}$$

Applications: probability, algebra, computer science (analysis of algorithms), statistical physics... and curiosity

Rational and algebraic formal power series

- The formal power series $A(t)$ is algebraic (over $\mathbb{Q}(t)$) if it satisfies a (non-trivial) polynomial equation:

$$P(t, A(t)) = 0.$$

- The formal power series $A(t)$ is rational if it can be written

$$A(t) = \frac{P(t)}{Q(t)}$$

where $P(t)$ and $Q(t)$ are polynomials in t.

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where $P(t)$ and $Q(t)$ are polynomials in t.

• A class of objects having a rational [algebraic] generating function will be said to be rational [algebraic].

Some charms of rational and algebraic series

- Closure properties ($+, \times, /$, derivatives, composition...)

- "Easy" to handle (partial fraction decomposition, Puiseux expansions, elimination, resultants, Gröbner bases...)

- Algebraicity can be guessed from the first coefficients (GFUN)

- The coefficients can be computed in a linear number of operations.

- (Almost) automatic asymptotics of the coefficients: in general,

$$a(n) \sim \frac{\kappa}{\Gamma(d+1)} \mu^n n^d,$$

where κ and ρ are algebraic over $\mathbb{Q}$ and $d \in \mathbb{Q} \setminus \{-1, -2, \ldots\}$.

What does it mean for a class of objects to have

a rational [algebraic] generating function?

Combinatorial constructions and operations on series: A dictionary

Construction	Numbers	Generating function
Union $\mathcal{A} = \mathcal{B} \uplus \mathcal{C}$	$a(n) = b(n) + c(n)$	$A(t) = B(t) + C(t)$
Product $\mathcal{A} = \mathcal{B} \times \mathcal{C}$ $\lvert(\beta, \gamma)\rvert = \lvert\beta\rvert + \lvert\gamma\rvert$	$a(n) = b(0)c(n) + \cdots + b(n-1)c(1) + b(n)c(0)$	$A(t) = B(t) \cdot C(t)$

"Symbolic Combinatorics" [Flajolet-Sedgewick]

A rational example: integer compositions

• Let $n \in \mathbb{N}$. A composition of size n with k parts is a k-tuple $c = (n_1, \ldots, n_k)$ with $n_i \in \{1, 2, \ldots\}$ for all i and $n_1 + \cdots + n_k = n$.

(1) (2) (1, 1) (3) (2, 1) (1, 2) (1, 1, 1)

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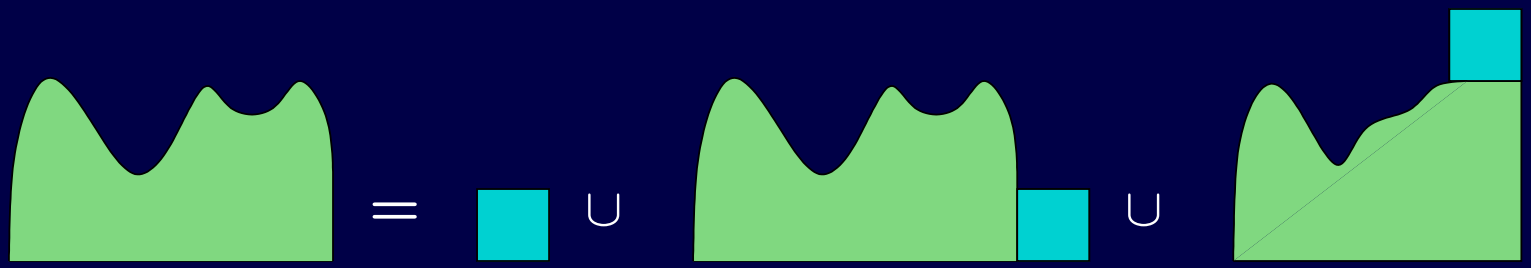
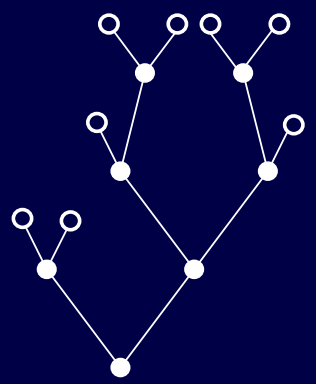
• Let $C(t)$ be the generating function of compositions, counted by their size. Then

$$C(t) = t + tC(t) + tC(t).$$



$$C(t) = \frac{t}{1 - 2t}.$$

An algebraic example: binary trees



• Let $B(t)$ be the generating function of binary trees, counted by the number of (inner) nodes. Then

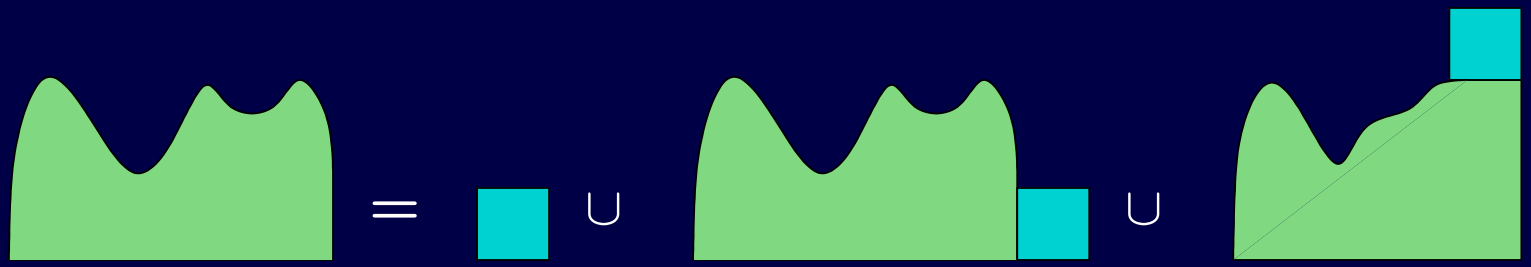
$$B(t) = 1 + tB(t)^2$$

$$B(t) = \frac{1-\sqrt{1-4t}}{2t} = \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n} t^n$$

Catalan number.

Rational and algebraic series: a combinatorial intuition

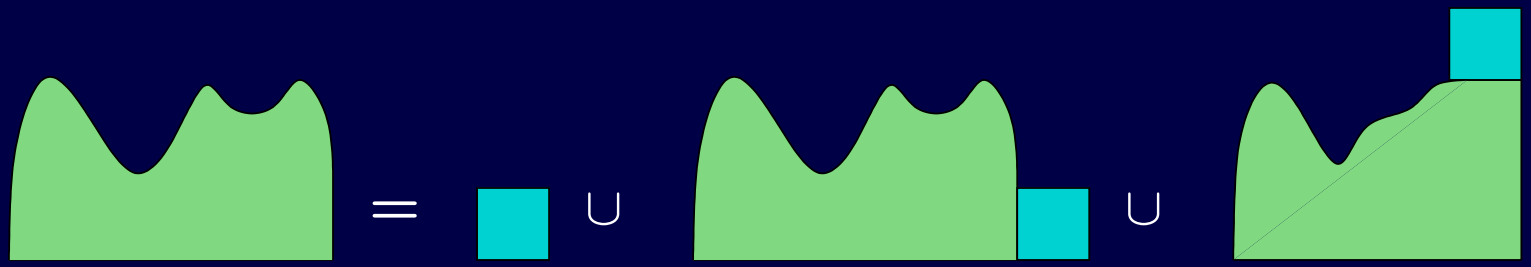
- "Linear" structure of rational objects



- "Tree-like" structure of algebraic objects

Rational and algebraic series: a combinatorial intuition

- "Linear" structure of rational objects



- "Tree-like" structure of algebraic objects

1. **Formalize this intuition:** tree-like structures, $\mathbb{N}$-algebraic series
2. **Are all algebraic objects tree-like?**
3. How does one prove algebraicity?

Similar questions for rational objects and linear structures

Part 1: Tree-like structures and $\mathbb{N}$-algebraic series

[Formal languages – Regular and context-free languages]

Tree-like constructions

- Building blocks: a finite collection of initial objects (atoms)
- Take (disjoint) unions: $(A, B) \to A + B$
- Take cartesian products: $(A, B) \to A \cdot B$
- The construction may involve k different classes of objects $\mathcal{A}_1, \ldots, \mathcal{A}_k$

Example:
$$\begin{aligned} A_1 &= 1 + 2t + 4tA_1 + (12t + t^2)A_2 + t^2 A_1 A_2^2 \\ A_2 &= \qquad\quad 14tA_1 + 8tA_2 \qquad\quad + 3tA_1^3 \end{aligned}$$

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Definition: A series A is $\mathbb{N}$-algebraic if there exists a k-tuple of formal power series $\mathcal{A} = (A_1, \ldots, A_k)$ such that $A = A_1$ and for $1 \leq j \leq k$

$$\mathcal{A} = \begin{pmatrix} A_1 \\ \vdots \\ A_k \end{pmatrix} = \Phi(\mathcal{A}) = \begin{pmatrix} P_1(t, A_1, \ldots, A_k) \\ \vdots \\ P_k(t, A_1, \ldots, A_k) \end{pmatrix}$$

where

- each P_j is a polynomial with coefficients in $\mathbb{N}$
- the map $\mathcal{B} \mapsto \Phi^m(\mathcal{B})$ is a contraction for some m.

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Property: The series A_i are uniquely defined by this system, have coefficients in $\mathbb{N}$ and are algebraic.

Linear constructions

- Building blocks: a finite collection of initial objects (atoms)
- Take (disjoint) unions: $(A, B) \to A + B$
- Take cartesian products with atoms only: $A \to t^d A$
- The construction may involve k different types of objects

Example:
$$\begin{array}{lcccccc} A_1 & = & 1 + 2t & + & 4tA_1 & + & (12t + t^2)A_2 \\ A_2 & = & & & 14tA_1 & + & 8tA_2 \end{array}$$

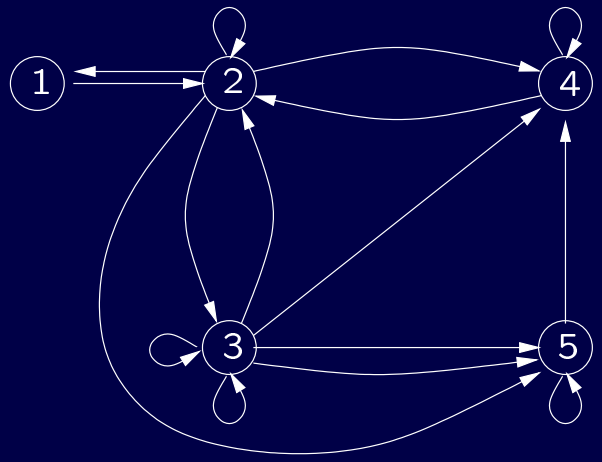
Definition: A series A is $\mathbb{N}$-rational if there exists a k-tuple of formal power series $\mathcal{A} = (A_1, \ldots, A_k)$ such that $A = A_1$ and for $1 \leq j \leq k$

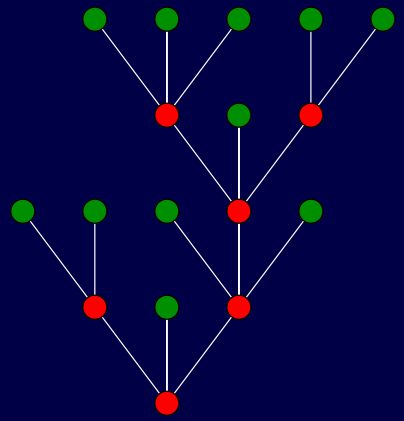
$$\mathcal{A} = \begin{pmatrix} A_1 \\ \vdots \\ A_k \end{pmatrix} = \Phi(\mathcal{A}) = \begin{pmatrix} P_1(t, A_1, \ldots, A_k) \\ \vdots \\ P_k(t, A_1, \ldots, A_k) \end{pmatrix}$$

where
- each $P_j(t, x_1, \ldots, x_k)$ is a polynomial with coefficients in $\mathbb{N}$ of total degree at most 1 (in the x_i),
- the map $\mathcal{B} \mapsto \Phi^m(\mathcal{B})$ is contractant for some m.

Property: The series A_i are uniquely defined by this system, have coefficients in $\mathbb{N}$ and are rational.

Combinatorial models for $\mathbb{N}$-rational and $\mathbb{N}$-algebraic series





• Let $A(t)$ be an $\mathbb{N}$-rational series. There exists a directed graph G, and two vertices v_1 and v_2 in G such that $A(t) - A(0)$ counts walks going from v_1 to v_2 in the graph (by their length).

• Let $A(t)$ be an $\mathbb{N}$-algebraic series. There exists a family of coloured trees that has generating function $A(t) - A(0)$.

Part 2: Are all algebraic objects tree-like?

Are all algebraic objects tree-like?

Let $\mathcal{A}$ be a class of objects with an algebraic generating function $A(t)$.

1. $\mathbb{N}$-algebraicity

Is the series $A(t)$ $\mathbb{N}$-algebraic?

2. Tree-like structure

Is a tree-like structure visible directly on the objects?

How easy is it to detect that $\mathcal{A}$ has an algebraic generating function?

Similar questions for rational GFs and linear structures.

Are all algebraic objects tree-like?

Let $\mathcal{A}$ be a class of objects with a rational [algebraic] generating function $A(t)$.

1. $\mathbb{N}$-rationality/$\mathbb{N}$-algebraicity

Is the series $A(t)$ $\mathbb{N}$-rational [$\mathbb{N}$-algebraic]?

Rational GF	Algebraic GF
$\approx$ YES	???
I have never met a counting problem with a rational, but not $\mathbb{N}$-rational solution	**I do not know how to detect $\mathbb{N}$-algebraicity**

A criterion for $\mathbb{N}$-rationality [Soittola]

A rational series $A(t) = \sum_k a(k)t^k$ with coefficients in $\mathbb{N}$ is $\mathbb{N}$-rational iff

- either $\sum_k a(k)t^k$ has a unique pole of minimal modulus (a *dominant* pole),
- or each of the series $\sum_k a(2k)t^k$ and $\sum_k a(2k+1)t^k$ have a unique dominant pole,
- or each of the series $\sum_k a(3k)t^k$, $\sum_k a(3k+1)t^k$ and $\sum_k a(3k+2)t^k$ have a unique dominant pole,
- etc.

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- or each of the series $\sum_k a(3k)t^k$, $\sum_k a(3k+1)t^k$ and $\sum_k a(3k+2)t^k$ have a unique dominant pole,
- etc.

Application: The series

$$A(t) = \frac{1 - 2t + 225t^2}{(1 - 25t)(625t^2 + 14t + 1)}$$

has non-negative integer coefficients, but is NOT $\mathbb{N}$-rational.

A criterion for $\mathbb{N}$-algebraicity?

Question: Given an algebraic series $A(t)$, how can we decide whether it is $\mathbb{N}$-algebraic?
At least, can we state a necessary condition for an algebraic series to be $\mathbb{N}$-algebraic?

Suggestion: look at the singularities of $A(t)$

What are the possible singular behaviours of an $\mathbb{N}$-algebraic series?

Singular behaviour of algebraic series

Theorem

Let $A(t) = \sum_n a(n)t^n$ be an algebraic series with coefficients in $\mathbb{N}$.
Let ρ be its radius of convergence. Then $A(t)$ is singular at ρ, with a singular expansion of the form

$$A_{\text{sing}}(t) = C(1 - t/\rho)^\gamma + o((1 - t/\rho)^\gamma)$$

where $\gamma \in \mathbb{Q} \setminus \mathbb{N}$.

[Newton-Puiseux]

Singular behaviour of $\mathbb{N}$-algebraic series

(Future) theorem

Let $A(t) = \sum_n a(n) t^n$ be an $\mathbb{N}$-algebraic series.

Let ρ be its radius of convergence. Then $A(t)$ is singular at ρ, with a singular expansion of the form

$$A_{\text{sing}}(t) = C(1 - t/\rho)^\gamma + o((1 - t/\rho)^\gamma)$$

where

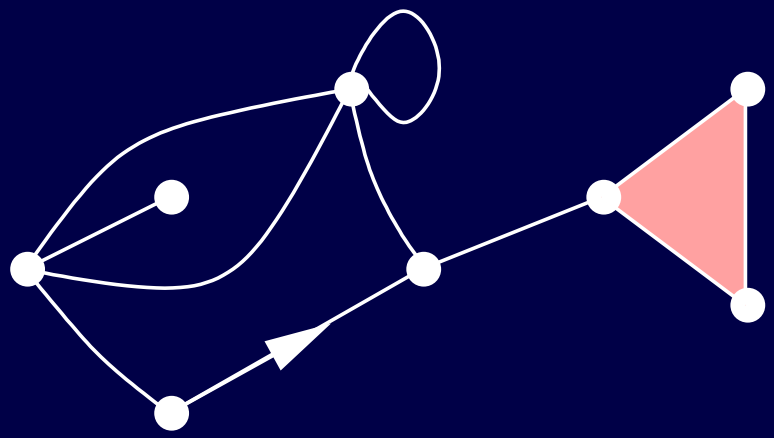
$$\gamma = \frac{k}{2^d},$$

for $k \in \{1\} \cup \{-1, -2, -3, \ldots\}$ and $d \in \{1, 2, 3, \ldots\}$

[mbm-Schaeffer, in preparation]

Rooted planar maps: a candidate for non-$\mathbb{N}$-algebraicity

Proper embedding of a connected planar graph in the plane.
One edge is oriented, with the infinite face on its right.



n edges

$m(n)$

Enumeration of planar maps by edges [Tutte 63]

$$M = A - tA^3, \quad \text{with} \quad A = 1 + 3tA^2.$$

Remark. $A_{\text{sing}}(t) \sim C(1-12t)^{3/2}$ $\Rightarrow$ A good candidate for non-$\mathbb{N}$-algebraicity

Are all algebraic objects tree-like?

2. Tree-like structure

How easy is it to detect that $\mathcal{A}$ has a rational [algebraic] generating function?
Is a linear [tree-like] structure visible directly on the objects?

Rational GF	Algebraic GF
$\approx$ y e s	NO
Detecting rational GFs and a linear structure is (usually) easy	**Many mysterious algebraic objects**

Part 3: A gallery of "hard" algebraic objects

Algebraic objects with no obvious tree-like structure

1. Brute force approach

Describe any recursive construction of the objects. Convert it into any functional equation for the generating function. Prove that the solution of this equation is algebraic.

2. More combinatorics

Describe a recursive construction of the objects based on concatenation that gives directly a system of algebraic equation(s).

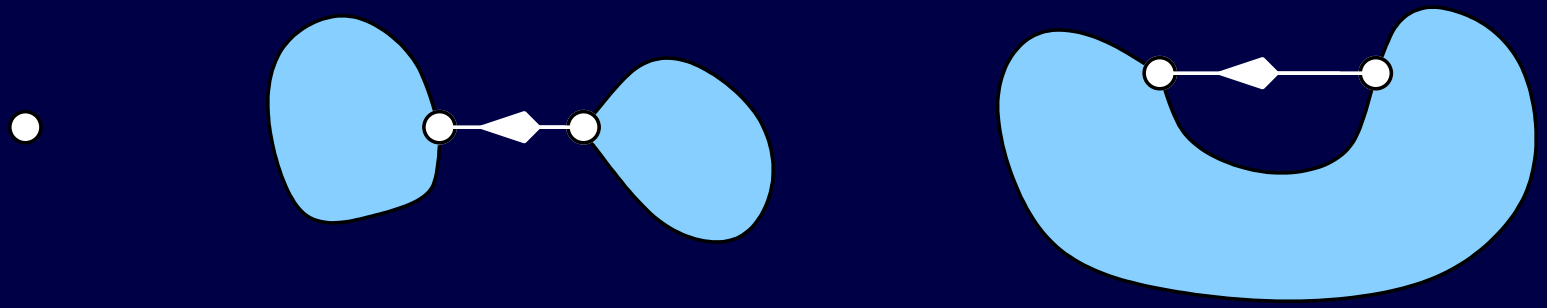
Both approaches may require to describe first a bijection with another class of objects.

Planar maps: the brute force approach

Let

$$F(t,u) \equiv F(u) = \sum_M t^{e(M)} u^{d(M)}$$

where $e(M)$ is the number of edges of M and $d(M)$ is the degree of the infinite face. Then



$$F(u) \quad = \quad 1 \quad + \quad tu^2 F(u)^2 \quad + \quad \frac{tu}{u-1}\left(uF(u) - F(1)\right)$$

Polynomial equation with one catalytic variable, u.
[Tutte 68]

Polynomial equations with one catalytic variable [MBM-Jehanne 05]

- Example

$$F(u) = 1 + tuF(u)^3 + \frac{tu\,(F(u) - F(1))\,(2\,F(u) + F(1))}{u-1} + \frac{tu\,(F(u) - F(1) - (u-1)\,F'(1))}{(u-1)^2}.$$

- General framework

Assume a $(k+1)$-tuple $(F(u), F_1, \ldots, F_k)$ of formal power series in t is completely defined by the equation

$$P(F(u), F_1, \ldots, F_k, t, u) = 0 \qquad (1)$$

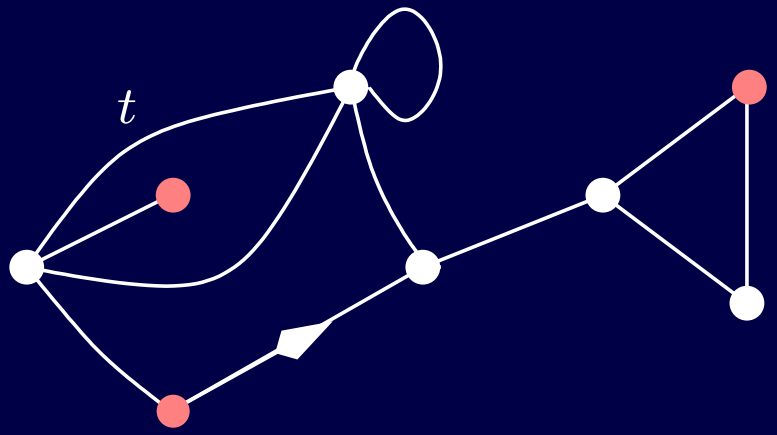
Typically, $F(u) \equiv F(t,u)$ has polynomial coefficients in u, and $F_i \equiv F_i(t)$ is the coefficient of u^{i-1} in $F(t,u)$.

- Results

1. The solution of every proper equation of this type is algebraic.

2. A practical strategy allows to solve specific examples (that is, to derive from (1) an algebraic equation for $F(u)$, or $F_1, \ldots, F_k$).

Hard particles on planar maps

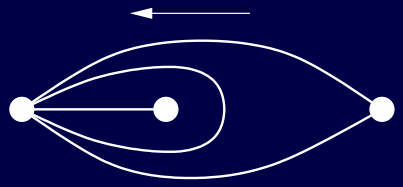


- $F(u) = 1 + G(u) + tu^2 F(u)^2 + \dfrac{tu\,(uF(u) - F_1)}{u-1}$
- $G(u) = tuF(u) + tuF(u)G(u) + \dfrac{tu\,(G(u) - G_1)}{u-1}$

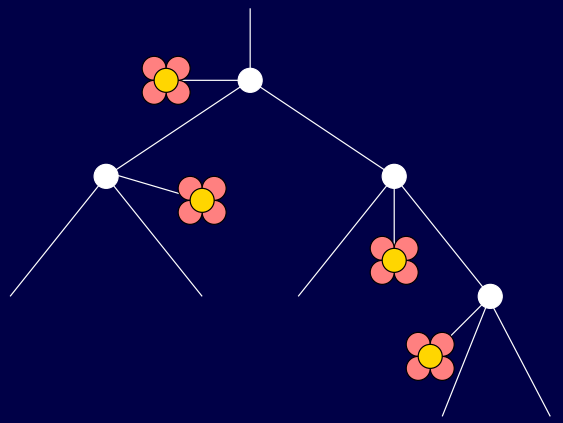
$t^2 F(1) = T^2(1-7T+16T^2+T-15\,T^3+4\,T^4)$ with $T(1-2T)(1-3T+3T^2) = t.$

Rooted planar maps: a combinatorial explanation

• Bijection between planar maps (n edges) and *balanced* blossoming trees (n nodes) [Schaeffer 97]

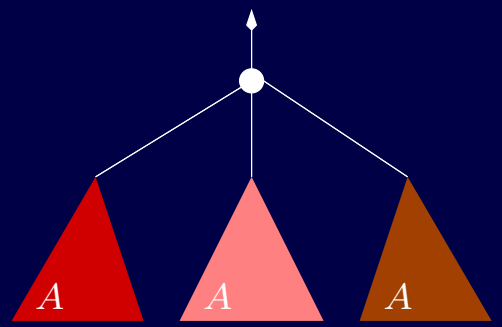


A planar map (GF M)



A blossoming tree $A = 1 + 3tA^2$

• An *unbalanced* blossoming tree is, in essence, a 3-tuple of blossoming subtrees [Bouttier *et al.* 02]:

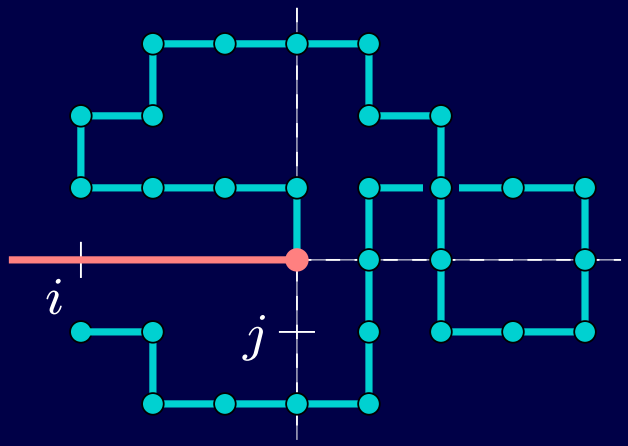


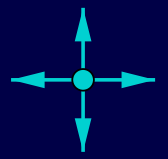
$$\Rightarrow M = A - tA^3.$$

Find combinatorial explanations for...

Walks on the slit plane

$S(u,v,t) := \sum_{i,j,n} a(i,j,n) u^i v^j t^n$





n steps

$a(i,j,n)$

- Why is $S(u,v;t)$ algebraic? Let $\bar{u} = 1/u$ and $\bar{v} = 1/v$. Then

$$S(u,v;t) = \frac{\left(1-2t(1+\bar{u})+\sqrt{1-4t}\right)^{1/2}\left(1+2t(1-\bar{u})+\sqrt{1+4t}\right)^{1/2}}{1-t(u+\bar{u}+v+\bar{v})}.$$

- Why is $S_{i,j}(t) := \sum_n a(i,j;n) t^n$ algebraic for all i and j?

$a(1,0;2n+1) = C_{2n+1}, \quad a(0,1;2n+1) = 4^n C_n, \quad a(-1,1;2n) = C_{2n} \ldots$

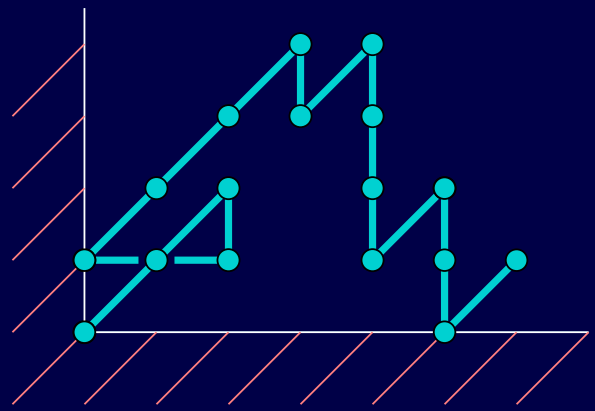
[Barcucci et al. 01]

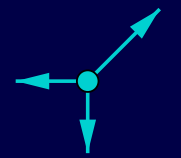
[MBM-Schaeffer 02] [MBM 01]

$$C_n = \frac{1}{n+1}\binom{2n}{n} = n\text{th Catalan}$$

Kreweras' walks in the quarter plane

$$Q(t) := \sum_n a(n) t^n$$





n steps

$a(n)$

- Why is $Q(t)$ algebraic? Let $W = t(2 + W^3)$. Then

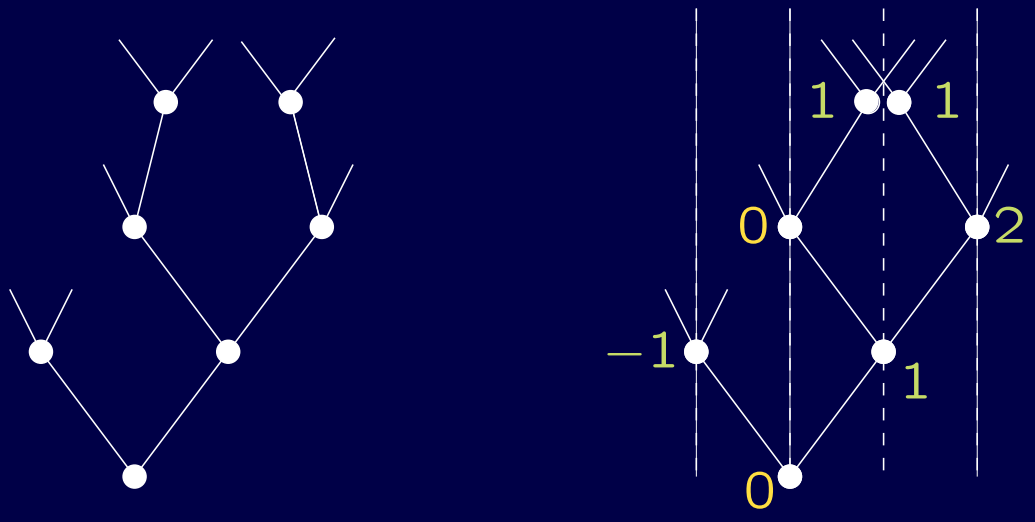
$$Q(t) = 2\,\frac{(1/W - 1)\sqrt{1 - W^2}}{1 - 3t} - \frac{1}{t}.$$

- Why is $a(i, 0; n)$, the number of walks ending at $(i, 0)$, nice and simple?

$$a(i, 0; 3n + 2i) = \frac{4^n (2i + 1)}{(n + i + 1)(2n + 2i + 1)} \binom{2i}{i} \binom{3n + 2i}{n}$$

[Kreweras 65], [Niederhausen 83], [Gessel 86],
[MBM 04], [Bernardi 06]

Embedded binary trees



The generating function of binary trees of right width $\leq j$ is:

$$W_{\leq j} = A\,\frac{(1-Z^{j+2})(1-Z^{j+7})}{(1-Z^{j+4})(1-Z^{j+5})},$$

where A counts binary trees and $Z \equiv Z(t)$ is the unique series in t such that

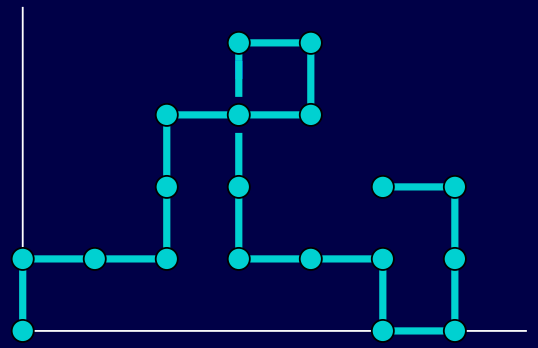
$$Z = t\,\frac{\left(1+Z^2\right)^2}{\left(1-Z+Z^2\right)} \quad \text{and} \quad Z(0) = 0.$$

[mbm 05] But. . . w h y ?

Concluding remarks

- Not all generating functions are algebraic!

Analytic techniques - Asymptotics:
[Flajolet 87]
Algebraic techniques - Reduction modulo p:
[Allouche 97]



- One more step in the hierarchy of generating functions

Type of series	Combinatorial intuition
polynomials	finite sets
rational	linear structures
algebraic	algebraic structures
D-finite	???

D-finite series $\leftrightarrow$ P-recursive sequences:

$$P_0(n)a(n) + P_1(n)a(n-1) + \cdots + P_k(n)a(n-k) = 0.$$

Reverse engineering?

For

$$f_n = \frac{(2n)!}{n!^2}, \quad f_n = \frac{(6n)!(n)!}{(3n)!(2n)!^2}, \quad f_n = \frac{(10n)!(n)!}{(5n)!(4n)!(2n)!}, \quad f_n = \frac{(20n)!(n)!}{(10n)!(7n)!(4n)!},$$

the series

$$F(t) = \sum_{n \geq 0} f_n t^n$$

has coefficients in $\mathbb{N}$ and is algebraic [Rodriguez-Villegas 05].

Does it count something?