

Walks with large steps in the quadrant

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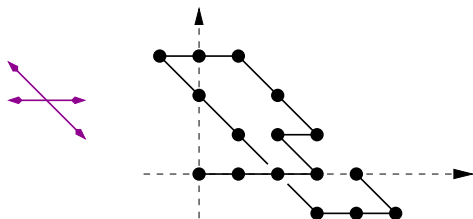
Counting quadrant walks

Let \mathcal{S} be a finite subset of \mathbb{Z}^2 (set of **steps**).

We look at walks starting at $(0, 0)$ and formed of steps of \mathcal{S} .

- What is the number $q(n)$ of n -step walks contained in \mathbb{N}^2 ?
- For $(i, j) \in \mathbb{N}^2$, what is the number $q(i, j; n)$ of such walks that end at (i, j) ?

Example. $\mathcal{S} = \{10, \bar{1}0, 1\bar{1}, \bar{1}\bar{1}\}$



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$$Q(x, y; t) = \sum_{n \geq 0} \sum_{(i, j) \in \mathbb{N}^2} q(i, j; n) x^i y^j t^n$$

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- Functional equation:

$$\left(1 - t\left(x + \frac{1}{x} + \frac{y}{x} + \frac{x}{y}\right)\right) Q(x, y; t) = 1 - t \frac{1+y}{x} Q(0, y; t) - t \frac{x}{y} Q(x, 0; t)$$

What is the **nature** of this series?

A hierarchy of formal power series

- Rational series

$$A(t) = \frac{P(t)}{Q(t)}$$

- Algebraic series

$$\text{Pol}(t, A(t)) = 0$$

- Differentially finite series (D-finite)

$$\sum_{i=0}^d P_i(t) A^{(i)}(t) = 0$$

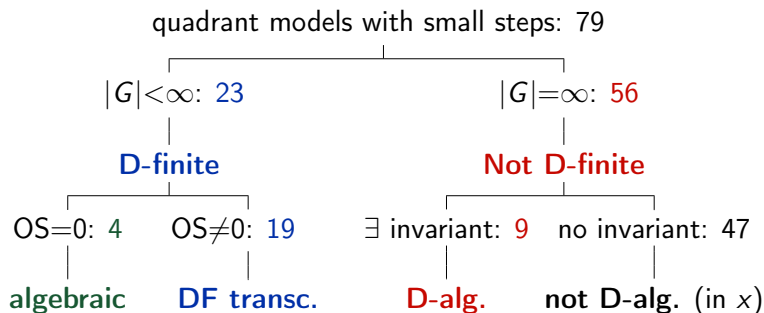
- D-algebraic series

$$\text{Pol}(t, A(t), A'(t), \dots, A^{(d)}(t)) = 0$$

Multi-variate series: one DE per variable



Classification of quadrant walks with small steps



- G the group of the model
- OS the orbit sum

[mbm-Mishna 10] [Bostan-Kauers 10] [Mishna-Rechnitzer 07]

[Melczer-Mishna 13] [Kurkova-Raschel 12] [Bostan-Raschel-Salvy 14]

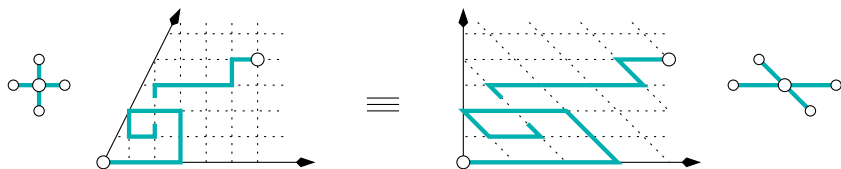
[Bernardi-mbm-Raschel 17(a)] [Dreyfus-Hardouin-Roques-Singer 17(a), 18]

Quadrant walks with arbitrary steps

- A mathematical challenge: the small step condition seems crucial in all approaches (apart from computer algebra)
- Large steps occur in “real life”: simple walk models,

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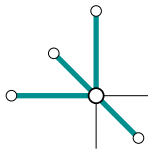
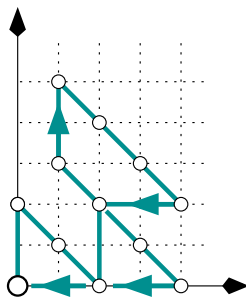
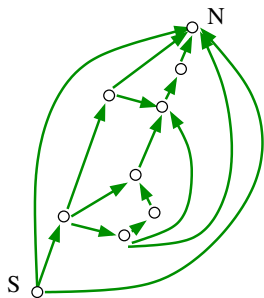
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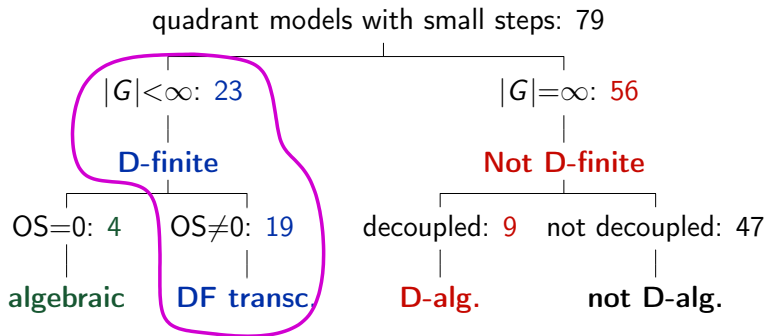
Quadrant walks with arbitrary steps

- A mathematical challenge: the small step condition seems crucial in all approaches (apart from computer algebra)
- Large steps occur in “real life”: simple walk models, queuing theory, bipolar orientations ([Kenyon et al. 15(a)])



A general approach for quadrant walks...

which solves *some* cases.



“The simple branch”
[mbm-Mishna 10]

A four step approach

1. Write a functional equation for the tri-variate series $Q(x, y; t)$.
It involves bi-variate series $Q(x, 0; t)$, $Q(0, y; t)$, ... (called **sections**)
2. Compute the “orbit” of (x, y)
3. Combine the main equation and the orbit to find a functional equation **free from sections**
4. Extract from it $Q(x, y; t)$

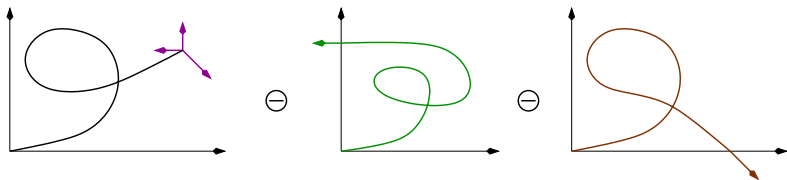
Step 1: Write a step-by-step functional equation



Example: $\mathcal{S} = \{01, \bar{1}0, 1\bar{1}\}$ (bipolar triangulations)

$$Q(x, y; t) \equiv Q(x, y) = 1 + t(y + \bar{x} + x\bar{y})Q(x, y) - t\bar{x}Q(0, y) - tx\bar{y}Q(x, 0)$$

with $\bar{x} = 1/x$ and $\bar{y} = 1/y$.



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or

$$(1 - t(y + \bar{x} + x\bar{y}))Q(x, y) = 1 - t\bar{x}Q(0, y) - tx\bar{y}Q(x, 0)$$

- The polynomial $1 - t(y + \bar{x} + x\bar{y})$ is the **kernel** of this equation

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- The polynomial $1 - t(y + \bar{x} + x\bar{y})$ is the **kernel** of this equation
- The series $Q(0, y)$ and $Q(x, 0)$ are the **sections**.

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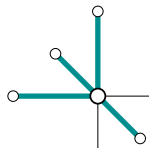
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Example: Bipolar quadrangulations

$$(1 - t(x\bar{y} + \bar{x}^2 + \bar{x}y + y^2))Q(x, y) = 1 - tx\bar{y}Q(x, 0) - t\bar{x}^2(Q_0(y) + xQ_1(y)) - t\bar{x}yQ_0(y),$$

where $Q_i(y)$ counts quadrant walks ending at abscissa i .



\Rightarrow More sections, kernel of higher degree

Step 2: the group of the model (and its orbit)



- The step polynomial:

$$S(x, y) = \bar{x} + y + x\bar{y}$$

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$$\Phi : (x, y) \mapsto (\bar{x}y, y) \quad \text{and} \quad \Psi : (x, y) \mapsto (x, x\bar{y}).$$

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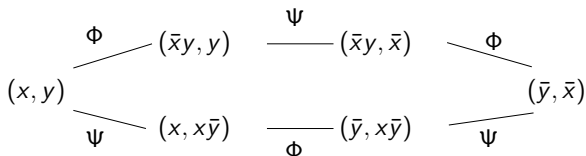
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- Φ and Ψ are involutions
- They generate a (dihedral) group



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 - they have one coordinate in common
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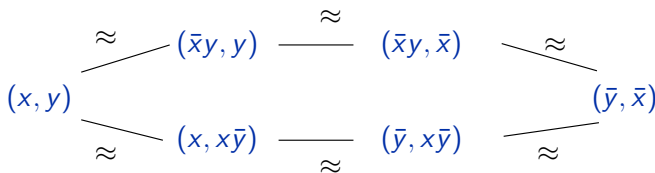
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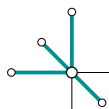
- The pairs $(\bar{x}y, y)$ and $(x, x\bar{y})$ are “adjacent” to (x, y) :
 - they have one coordinate in common
 - they give the same value to the step polynomial S
- Let \sim be the transitive closure of the adjacency relation. The orbit of (x, y) is its equivalence class.



Step 2: the orbit of the model

Example: Bipolar quadrangulations

$$S(x, y) = \bar{x}^2 + \bar{x}y + y^2 + x\bar{y}$$



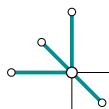
The equation (in x') $S(x, y) = S(x', y)$, has 3 solutions, namely x and

$$x_{1,2} = \frac{xy^2 + y \pm \sqrt{y(x^2y^3 + 4x^3 + 2xy^2 + y)}}{2x^2}.$$

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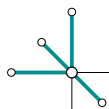
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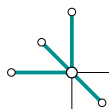
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- there are also two pairs (x, y') that are adjacent to (x, y) , which happen to be (x, \bar{x}_1) and (x, \bar{x}_2)

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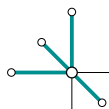
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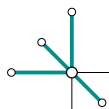
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- which pairs (x_1, y') are adjacent to (x_1, y) ?
- and so on.

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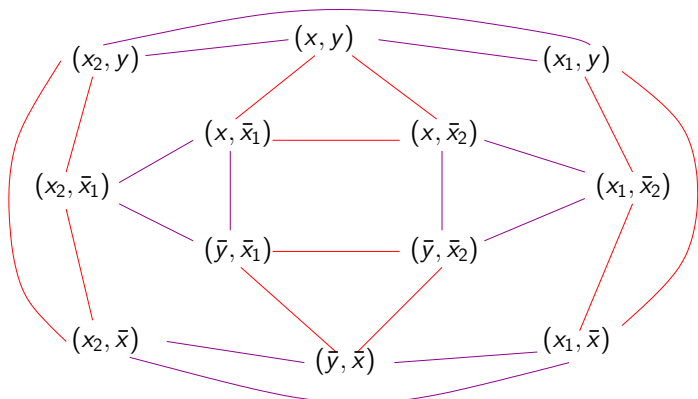
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• Orbit



Step 3: Find a functional equation free from sections



- The equation reads (with $K(x, y) = 1 - tS(x, y)$):

$$K(x, y)xyQ(x, y) = xy - tx^2Q(x, 0) - tyQ(0, y)$$

- The orbit of (x, y) is

$$(x, y) \approx (\bar{x}y, y) \approx (\bar{x}y, \bar{x}) \approx (\bar{y}, \bar{x}) \approx (\bar{y}, x\bar{y}) \approx (x, x\bar{y})$$

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- The value of $S(x, y)$ (and $K(x, y)$) is the same over the orbit. Hence

$$\begin{aligned} K(x, y) xyQ(x, y) &= xy - tx^2Q(x, 0) - tyQ(0, y) \\ K(x, y) \bar{x}y^2Q(\bar{x}y, y) &= \bar{x}y^2 - t\bar{x}^2y^2Q(\bar{x}y, 0) - tyQ(0, y) \\ K(x, y) \bar{x}^2yQ(\bar{x}y, \bar{x}) &= \bar{x}^2y - t\bar{x}^2y^2Q(\bar{x}y, 0) - t\bar{x}Q(0, \bar{x}) \\ \dots &= \dots \\ K(x, y) x^2\bar{y}Q(x, x\bar{y}) &= x^2\bar{y} - tx^2Q(x, 0) - tx\bar{y}Q(0, x\bar{y}). \end{aligned}$$

Step 3: Find a functional equation free from sections



⇒ Form the alternating sum of the equation over all elements of the orbit:

$$\begin{aligned} K(x, y) & \left(xyQ(x, y) - \bar{x}y^2Q(\bar{x}y, y) + \bar{x}^2yQ(\bar{x}y, \bar{x}) \right. \\ & \left. - \bar{x}\bar{y}Q(\bar{y}, \bar{x}) + x\bar{y}^2Q(\bar{y}, x\bar{y}) - x^2\bar{y}Q(x, x\bar{y}) \right) = \\ & \qquad \qquad \qquad xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y} \\ & \qquad \qquad \qquad \text{(the orbit sum)}. \end{aligned}$$

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Step 4: Extract $Q(x, y)$



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- Both sides are power series in t , with coefficients in $\mathbb{Q}[x, \bar{x}, y, \bar{y}]$.

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- Both sides are power series in t , with coefficients in $\mathbb{Q}[x, \bar{x}, y, \bar{y}]$.
- Extract the part with positive powers of x and y :

$$xyQ(x, y) = [x^{>0}y^{>0}] \frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})}$$

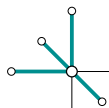
is a D-finite series.

[Lipshitz 88]

Step 3: Find a functional equation free from sections

Example: Bipolar quadrangulations

$$S(x, y) = \bar{x}^2 + \bar{x}y + y^2 + x\bar{y}$$



Functional equation:

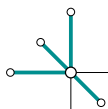
$$(1 - t(x\bar{y} + \bar{x}^2 + \bar{x}y + y^2))Q(x, y) = 1 - tx\bar{y}Q(x, 0) - t\bar{x}(\bar{x} + y)Q_0(y) - t\bar{x}Q_1(y),$$

where $Q_i(y)$ counts quadrant walks ending at abscissa i .

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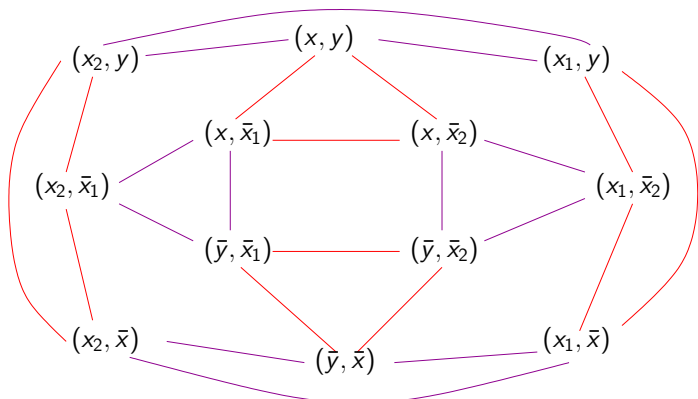
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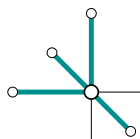
• Orbit



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Two section-free linear combinations (+ linear combinations):

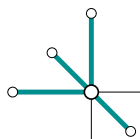
$$\begin{aligned} & xyQ(x, y) - \bar{x}_1xQ(x, \bar{x}_1) - \frac{x_1^2y(x-x_2)Q(x_1, y)}{x(x_1-x_2)} + \frac{x_1^2(x-x_2)Q(x_1, \bar{x})}{(x_1-x_2)x^2} \\ & + \frac{x_2^2y(x-x_1)Q(x_2, y)}{x(x_1-x_2)} - \frac{(x_1y-1)x_2^2(x-x_2)Q(x_2, \bar{x})}{x^2(x_1-x_2)(x_2y-1)} \\ & + \frac{(xy-1)x_2^2Q(x_2, \bar{x}_1)}{x_1x(x_2y-1)} + \frac{(x-x_2)Q(\bar{y}, \bar{x})}{y(x_2y-1)x^2} - \frac{(x-x_2)Q(\bar{y}, \bar{x}_1)}{yx_1x(x_2y-1)} \\ & = \frac{(y-\bar{x}_1)(xy-1)(\bar{y}-\bar{x}^2y-2\bar{x}^3)}{K(x, y)}, \end{aligned}$$

and the same equation with x_1 and x_2 exchanged.

Step 4: Extract $Q(x, y)$

Example: Bipolar quadrangulations

$$S(x, y) = \bar{x}^2 + \bar{x}y + y^2 + x\bar{y}$$



- A section-free equation:

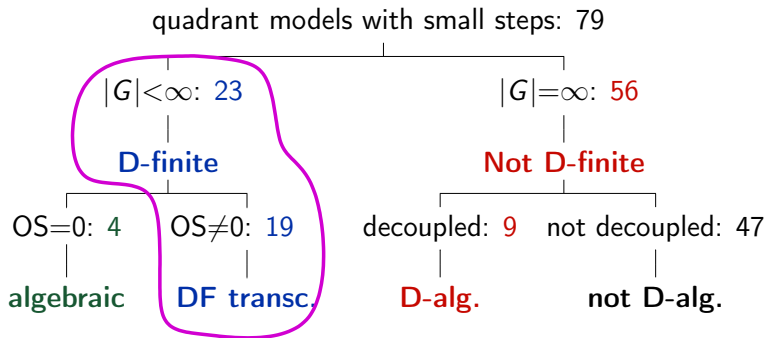
$$\begin{aligned} xyQ(x, y) - \bar{x}_1xQ(x, \bar{x}_1) - \frac{x_1^2y(x - x_2)Q(x_1, y)}{x(x_1 - x_2)} + \dots \\ = \frac{(y - \bar{x}_1)(xy - 1)(\bar{y} - \bar{x}^2y - 2\bar{x}^3)}{K(x, y)} \end{aligned}$$

- Then

$$xyQ(x, y) = [x^{>0}y^{>0}] \frac{(y - \bar{x}_1)(xy - 1)(\bar{y} - \bar{x}^2y - 2\bar{x}^3)}{K(x, y)},$$

provided the RHS is expanded first in t , then in \bar{y} , and finally in x .

What can go wrong?



1. Functional equation: OK
2. The “orbit” may be infinite
3. There may be several section-free equations (or none?) [NEW]
4. The extraction may be tricky [NEW], or impossible

Some cases that work

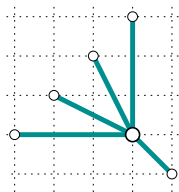
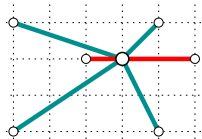
- Hadamard walks:

$$S(x, y) = U(x) + V(x)T(y)$$

(Small steps: 16 out of the 19 “simple” models)

- Bipolar maps [mbm, Fusy, Raschel 18]

$$\mathcal{S}_p = \{(-p, 0), (-p + 1, 1), \dots, (0, p), (1, -1)\}$$

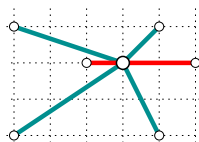


In all those cases, the orbit is finite and the series $Q(x, y; t)$ D-finite, expressed as the non-negative part of an algebraic series.

Hadamard walks

Assume

$$S(x, y) = U(x) + V(x)T(y)$$



Proposition [Bostan, mbm, Melczer 18]

The series $Q(x, y)$ is D-finite, and reads

$$Q(x, y) = [x \geq y \geq] \frac{\prod_{i=1}^m (1 - \bar{x}x_i(y)) \prod_{j=1}^{m'} (1 - \bar{y}y_j)}{1 - tS(x, y)},$$

where

- the $x_i(y)$ are the roots of $S(x, y) = S(x', y)$ (solved for x'), whose expansion in \bar{x} involves no positive power of x ,
- the y_j are the roots of $S(x, y) = S(x, y')$, or $T(y) = T(y')$ (solved for y') whose expansion in \bar{y} involve no positive powers of y .

Bipolar maps

$$\mathcal{S}_p = \{(-p, 0), (-p + 1, 1), \dots, (0, p), (1, -1)\}$$

Proposition [mbm, Fusy, Raschel 18]

The generating function of quadrant walks with steps in \mathcal{S}_p is

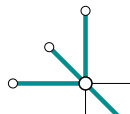
$$Q(x, y) = [x \geq y \geq] \frac{(y - \bar{x}_1)(1 - \bar{x}\bar{y}) S_x(x, y)}{1 - tS(x, y)},$$

where $S(x, y)$ is the step polynomial:

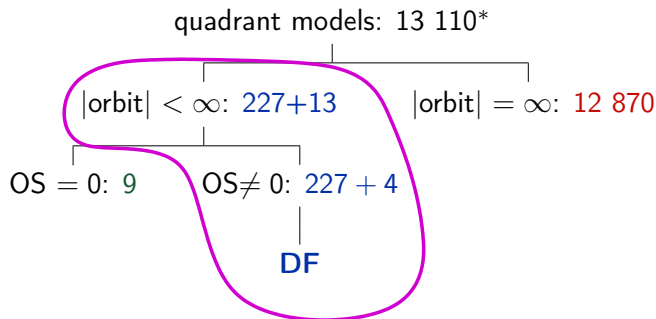
$$S(x, y) = x\bar{y} + \bar{x}^p + \bar{x}^{p-1}y + \dots + \bar{x}y^{p-1} + y^p.$$

and x_1 is the only root of $S(x, y) = S(x', y)$ (solved for x') whose expansion in \bar{y} involves a positive power of y .

It is D-finite.



Quadrant walks with steps in $\{-2, -1, 0, 1\}^2$

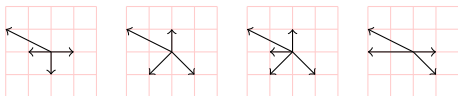


- In all cases, a unique section-free equation
- 227 Hadamard models

(*) Models with at least one occurrence of -2

Some interesting models

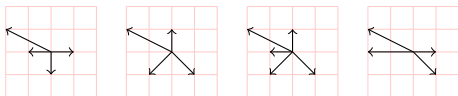
- Non-Hadamard, solvable via our approach (and D-finite):



$\Rightarrow Q(x, y; t)$ is the non-negative part (in x and y) of a rational series

Some interesting models

- Non-Hadamard, solvable via our approach (and D-finite):



$\Rightarrow Q(x, y; t)$ is the non-negative part (in x and y) of a rational series

For the first model,

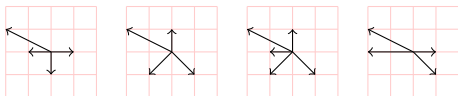
$$Q(x, y) = [x^{\geq 0} y^{\geq 0}] \frac{(x^3 - 2y^2 - x)(y^2 - x)(x^2y^2 - y^2 - 2x)}{x^5y^4(1 - t(y + x\bar{y} + \bar{x}\bar{y} + \bar{x}^2y))}.$$

The coefficients are nice: for $n = 2i + j + 4m$,

$$q(i, j; n) = \frac{(i+1)(j+1)(i+j+2)n!(n+2)!}{m!(3m+2i+j+2)!(2m+i+1)!(2m+i+j+2)!}.$$

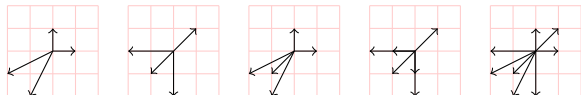
Some interesting models

- Non-Hadamard, solvable via our approach (and D-finite):



⇒ $Q(x, y; t)$ is the non-negative part (in x and y) of a rational series

- Non-Hadamard, orbit sum zero: let's guess! (what $Q(0, 0; t)$ is)



DF

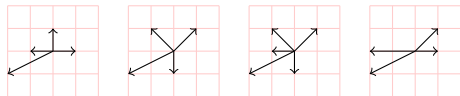
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DF

DF

DF

x/y symmetric



DF

Alg

Alg

DF

not symmetric

Thanks to Axel Bacher!

Final comments

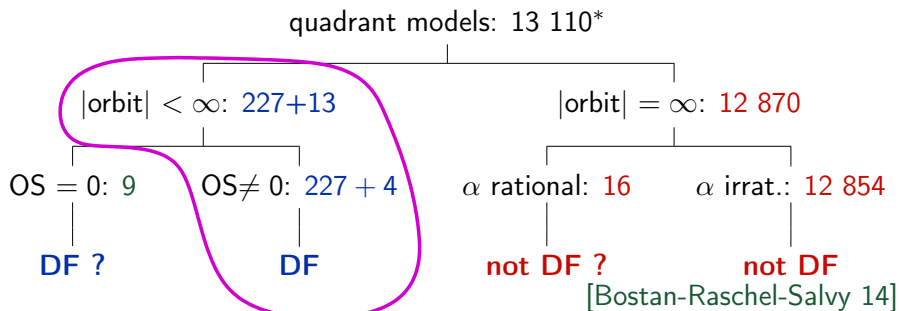
Still a lot to be done...

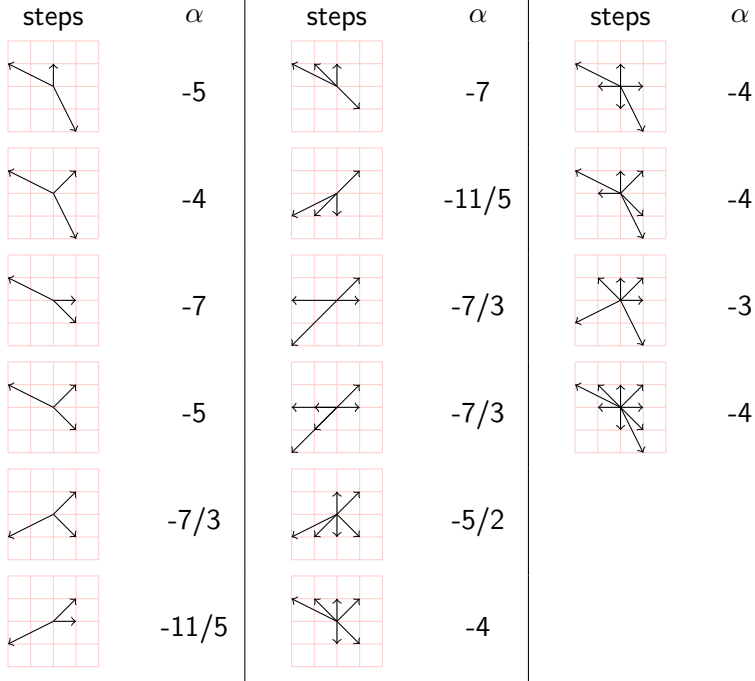
- Is there a unique section free equation when there are no large forward step?

Final comments

Still a lot to be done...

- Is there a unique section free equation when there are no large forward step?
- Closer study for tricky examples (the 9 analogues of Kreweras' and Gessel's algebraic models)
- Nature of models where α is rational but the orbit infinite





Walks on a half-line ($d = 1$)

Let $\mathcal{S} \subset \mathbb{Z}$ with $\min \mathcal{S} = -m$.

Proposition [Bostan, mbm, Melczer 18]

$$Q(x) = [x^{\geq 0}] \frac{\prod_{j=1}^m (1 - \bar{x}x_j)}{1 - tS(x)},$$

where the x_j are the roots of $S(x_j) = S(x)$ whose expansion in \bar{x} involves no positive power of x .

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Classical solution [Gessel 80, mbm-Petkovšek 00, Banderier-Flajolet 02...]

$$Q(x) = \frac{\prod_{j=1}^m (1 - \bar{x}X_j)}{1 - tS(x)}$$

where the $X_j \equiv X_j(t)$ are the roots of $1 - tS(x)$ whose expansion in t involves no negative power of t . The series $Q(x)$ is algebraic.

These solutions are (of course) equivalent.