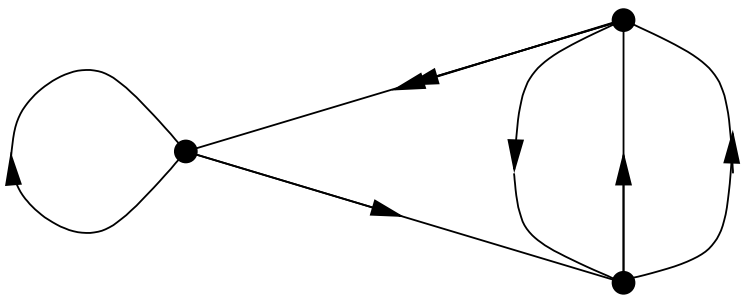
Eulerian orientations of planar maps

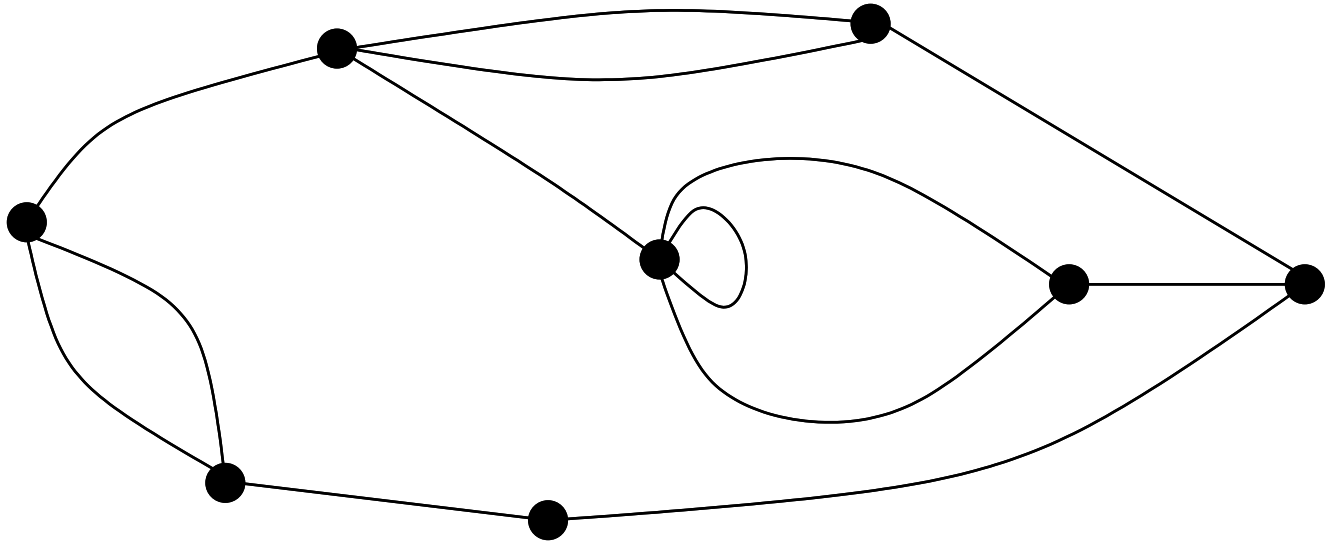
Mireille Bousquet-Mélou, CNRS, Bordeaux, France

Andrew Elvey Price, Bordeaux

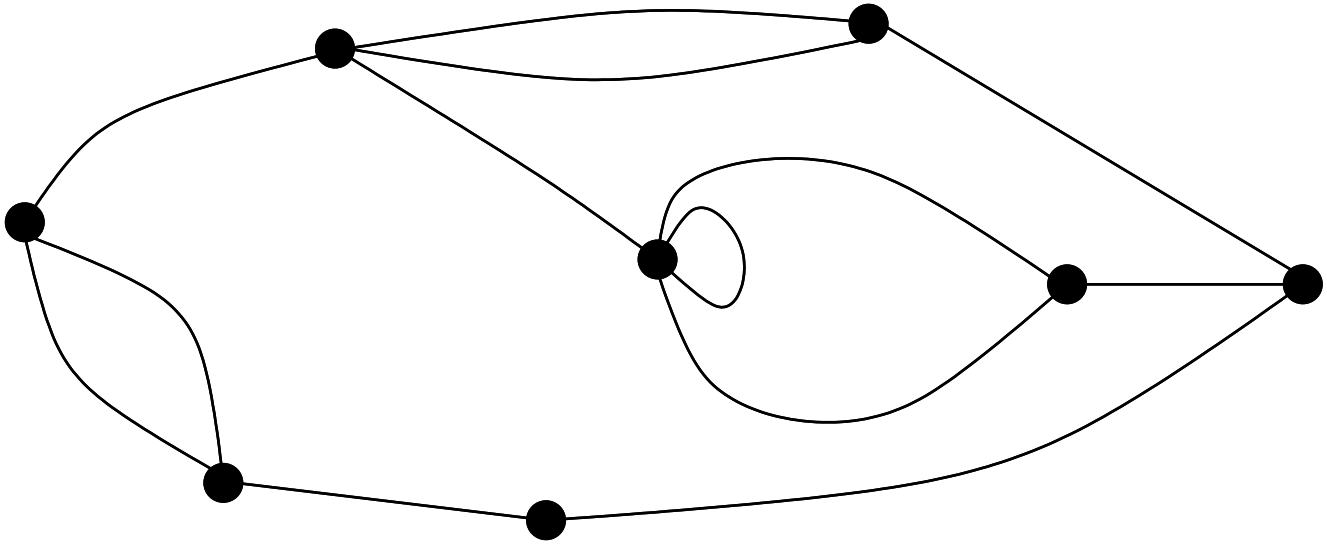


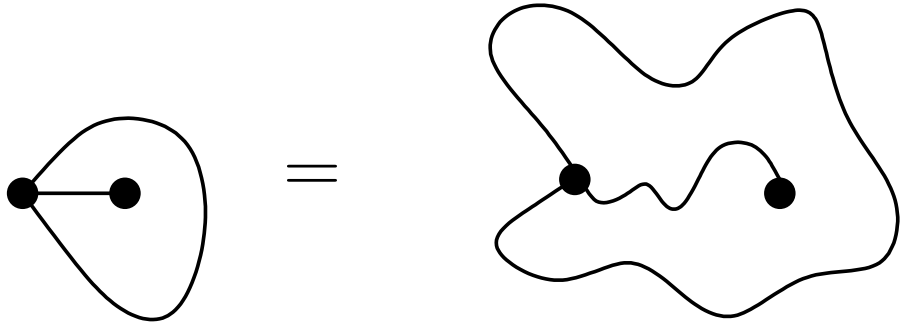
ArXiv: 1803.08265

Planar maps

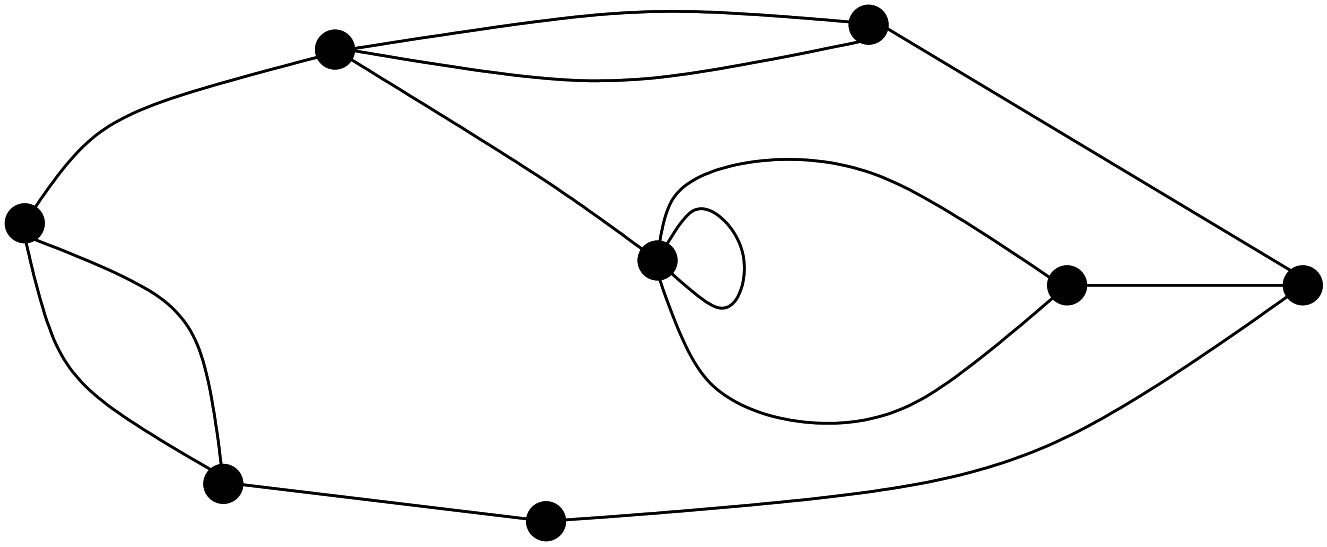


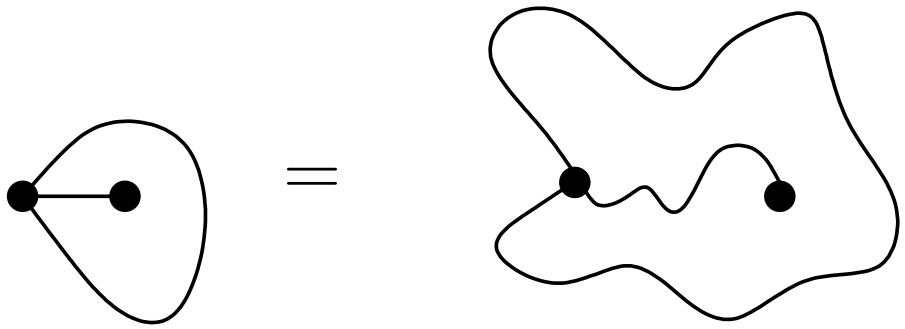
Planar maps





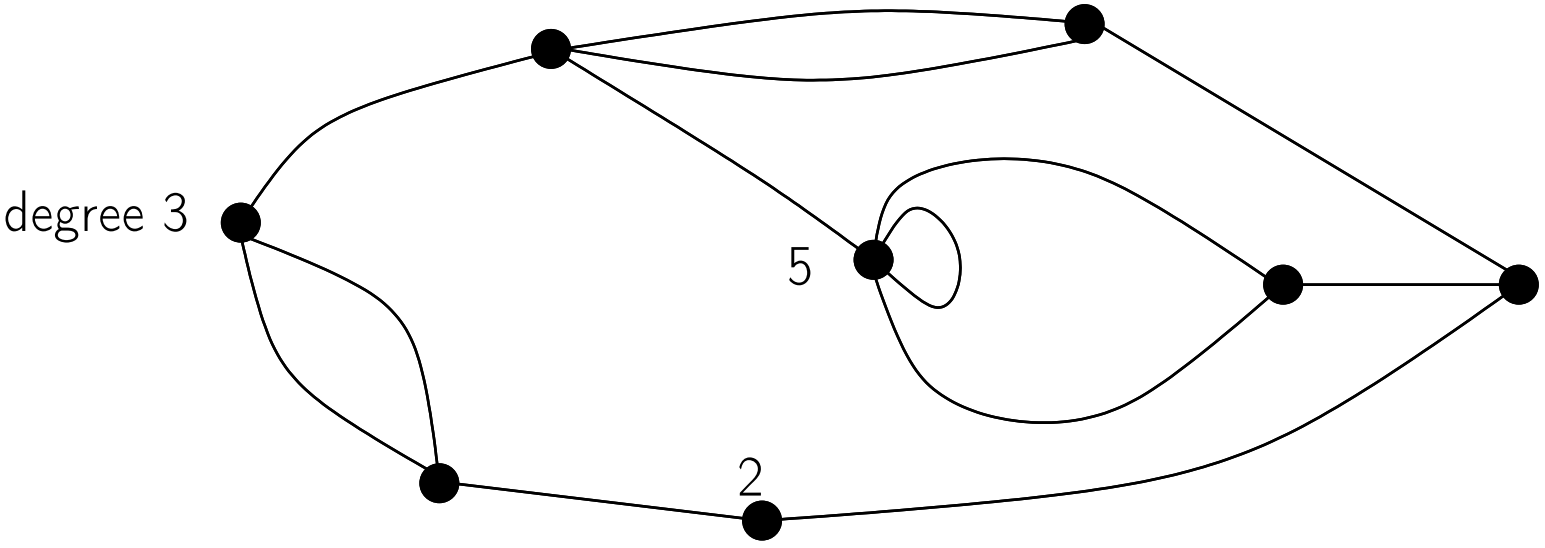
Planar maps



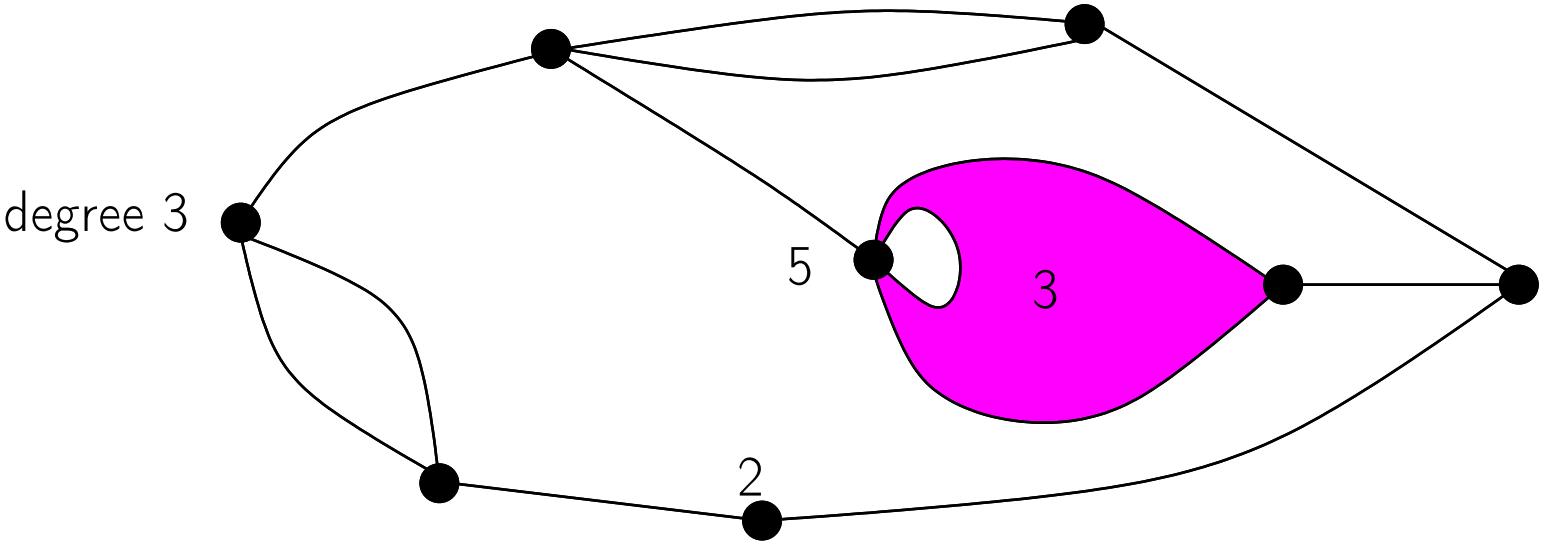


There are finitely many maps with n edges

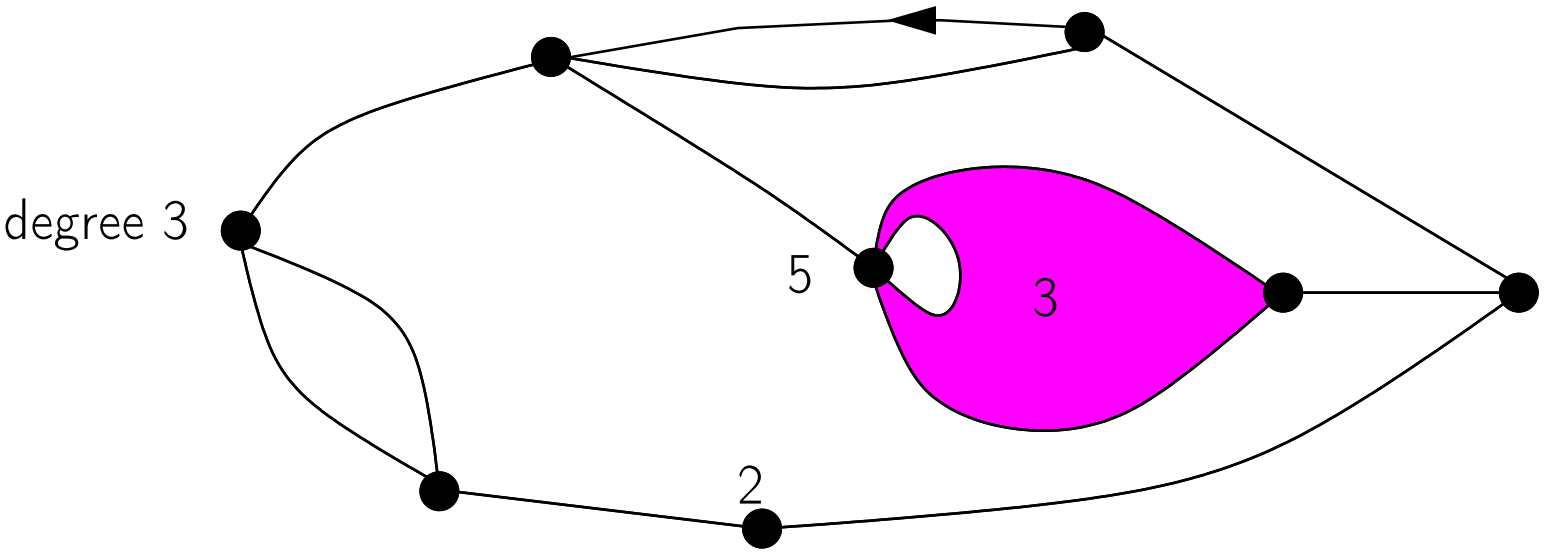
Planar maps



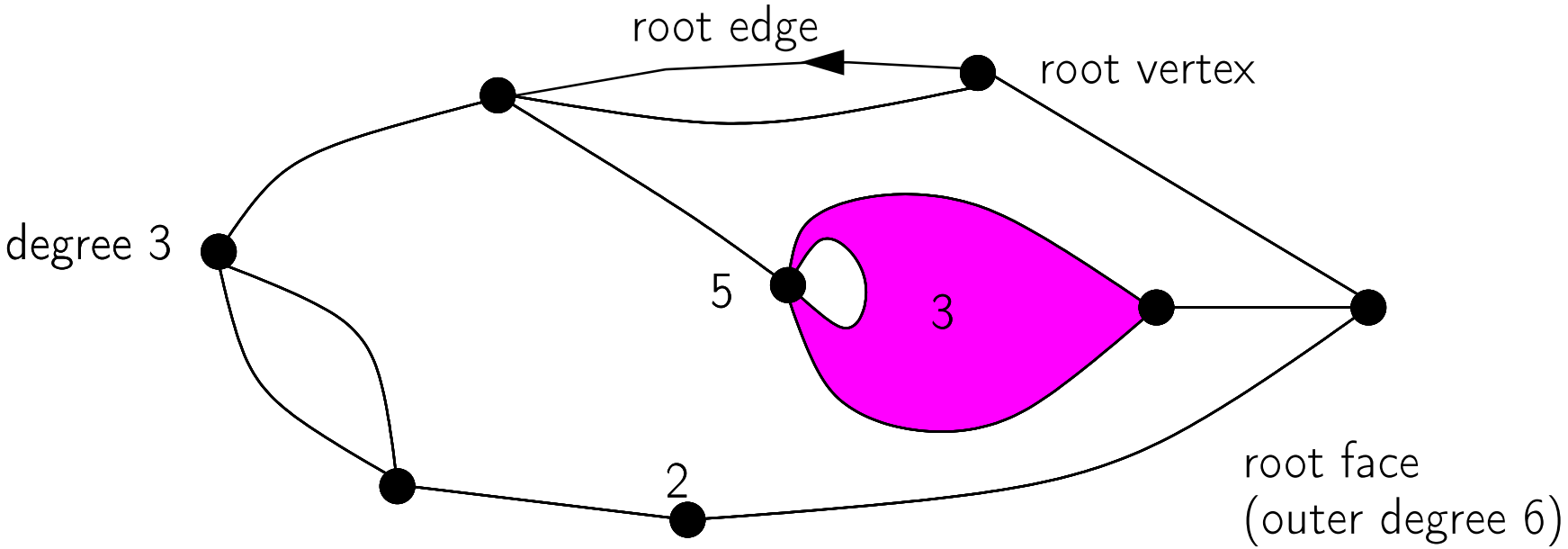
Planar maps



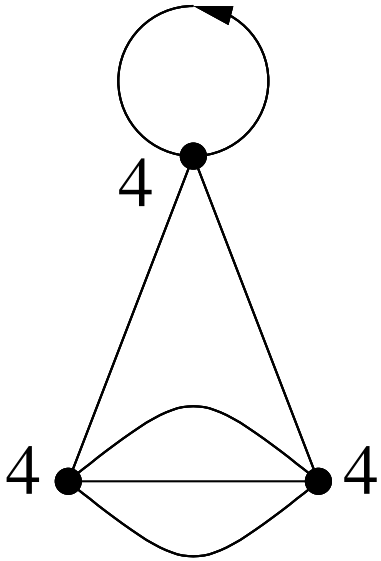
Planar maps: rooted version



Planar maps: rooted version

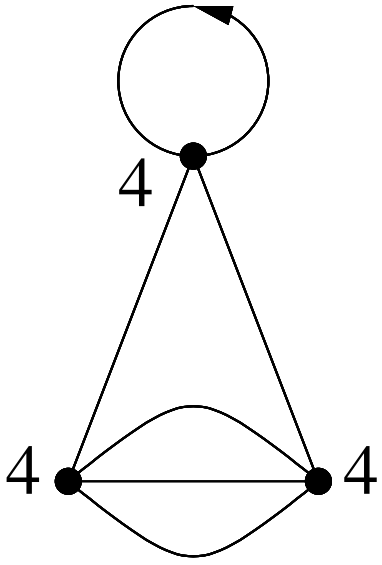


Degree constraints: 4-valent maps and quadrangulations

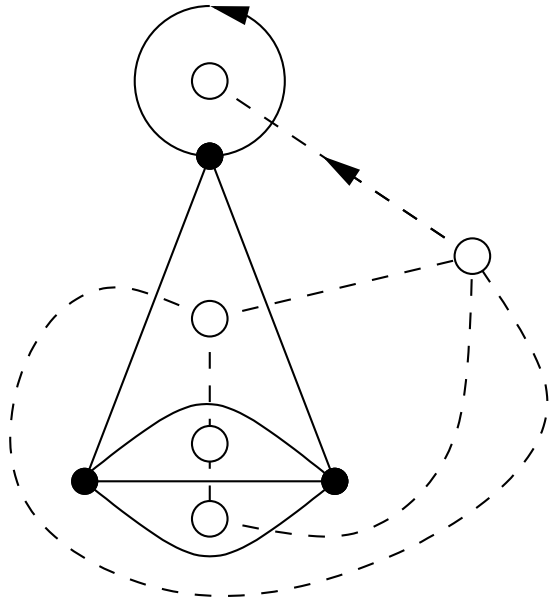


A 4–valent map

Degree constraints: 4-valent maps and quadrangulations

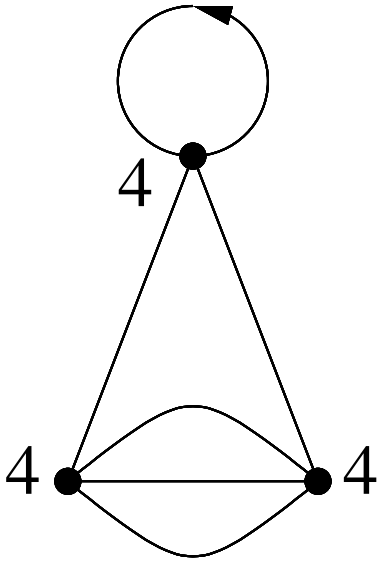


A 4–valent map

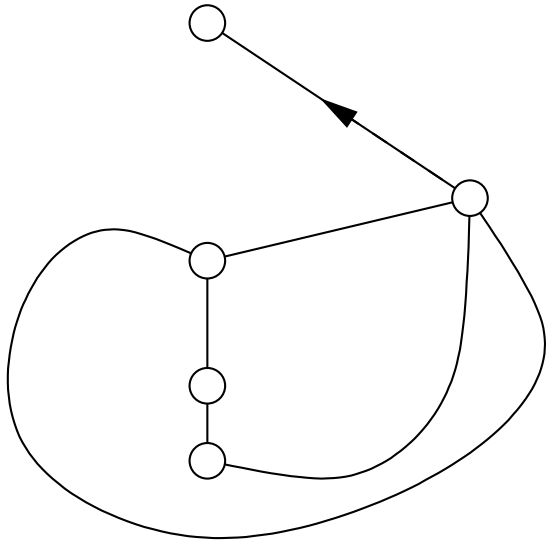


Its dual

Degree constraints: 4-valent maps and quadrangulations

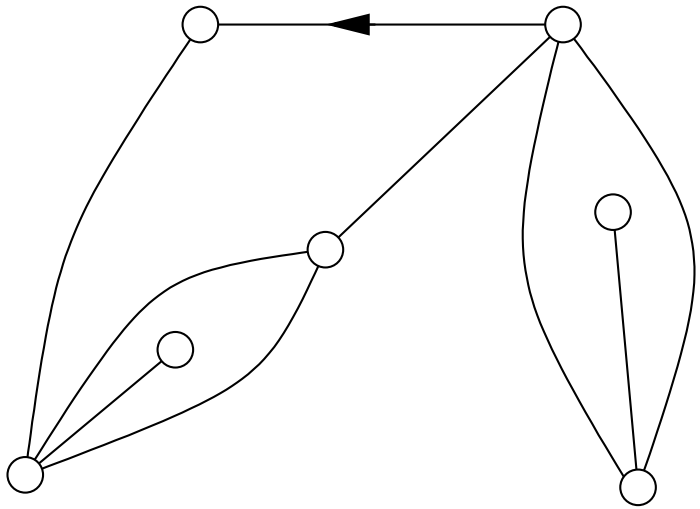


A 4–valent map

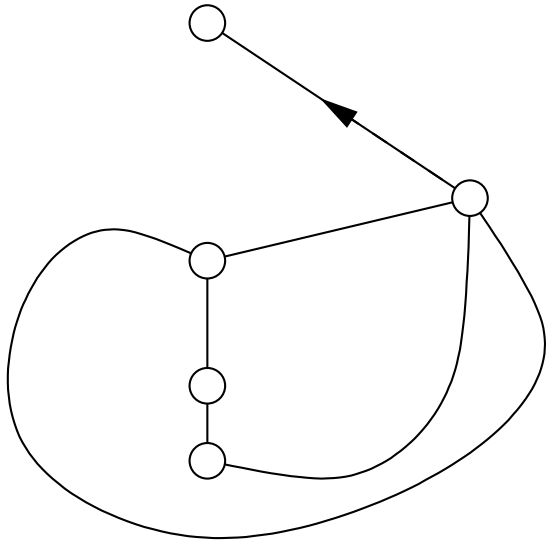


A quadrangulation

Degree constraints: 4-valent maps and quadrangulations

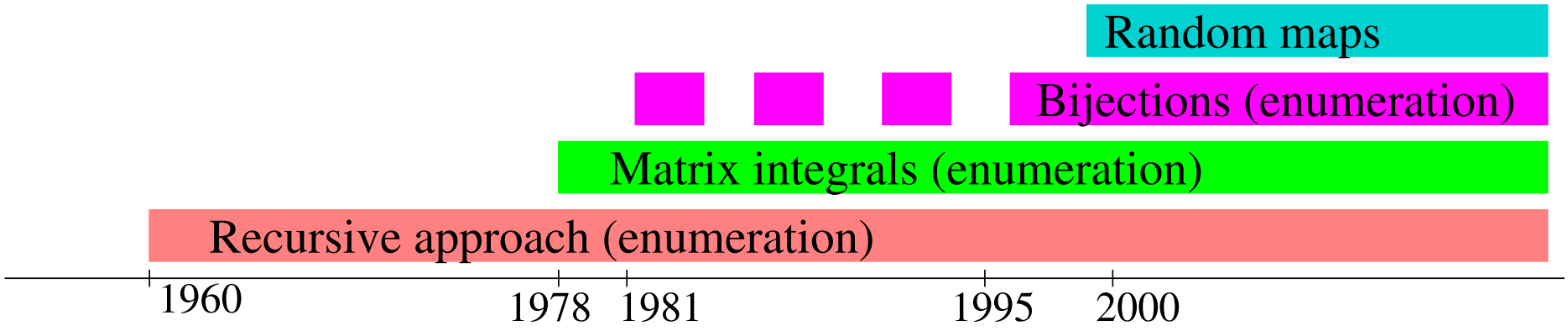


A near–quadrangulation
with outer degree 6



A quadrangulation

A chronology of planar maps



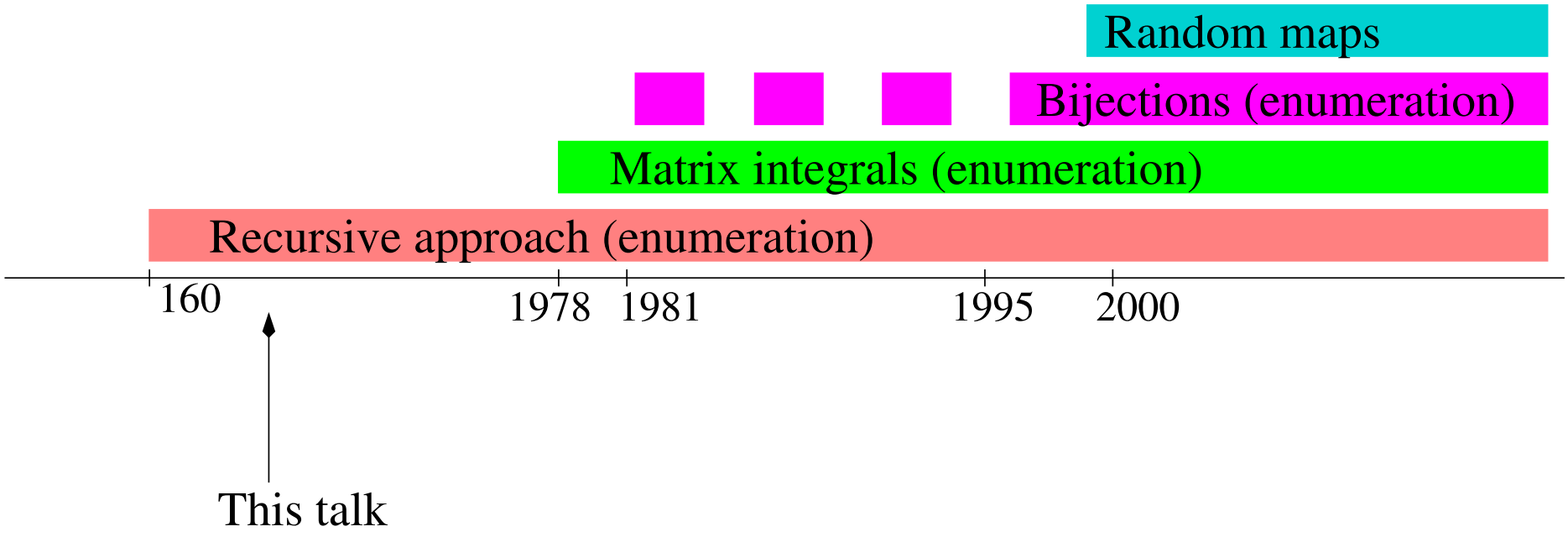
• **Recursive approach:** Tutte, Brown, Bender, Canfield, Richmond, Goulden, Jackson, Wormald, Walsh, Lehman, Gao, Wanless...

• **Matrix integrals:** Brézin, Itzykson, Parisi, Zuber, Bessis, Ginsparg, Zinn-Justin, Boulatov, Kazakov, Mehta, Bouttier, Di Francesco, Guitter, Eynard...

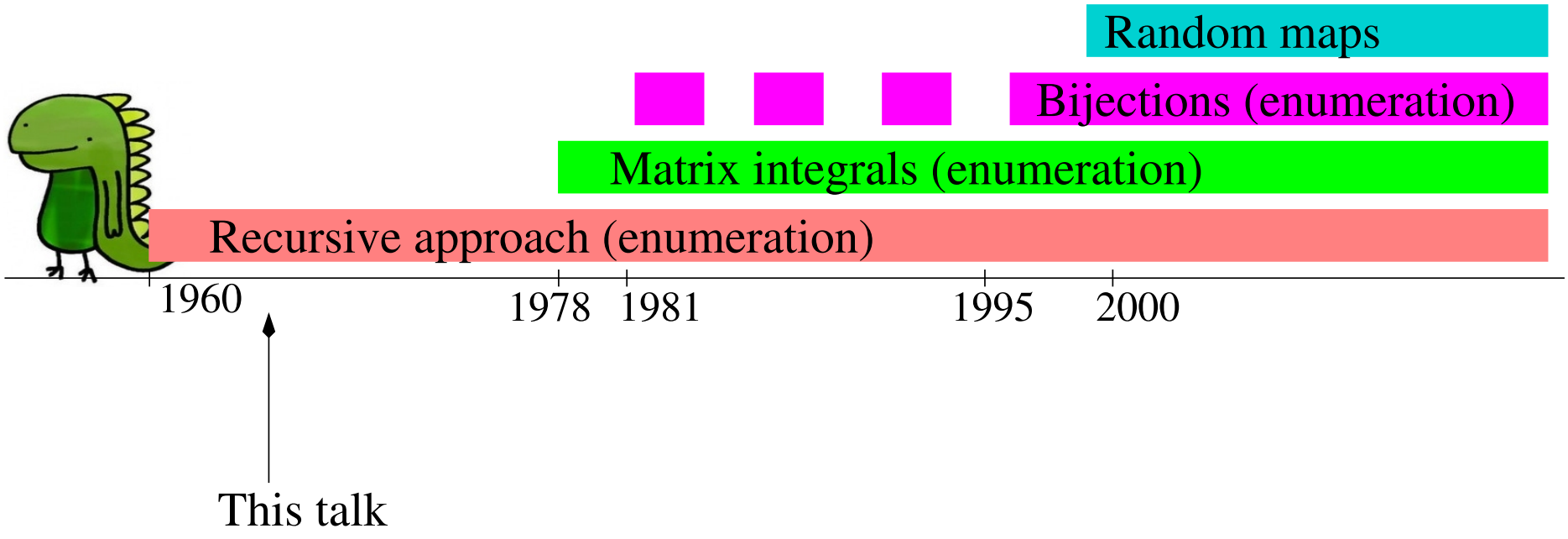
• **Bijections:** Cori & Vauquelin, Schaeffer, Bouttier, Di Francesco & Guitter (BDG), Bernardi, Fusy, Poulalhon, mbm, Chapuy...

• **Geometric properties of random maps:** Chassaing & Schaeffer, BDG, Marckert & Mokkadem, Jean-François Le Gall, Miermont, Curien, Albenque, Bettinelli, Ménard, Angel, Sheffield, Miller, Gwynne...

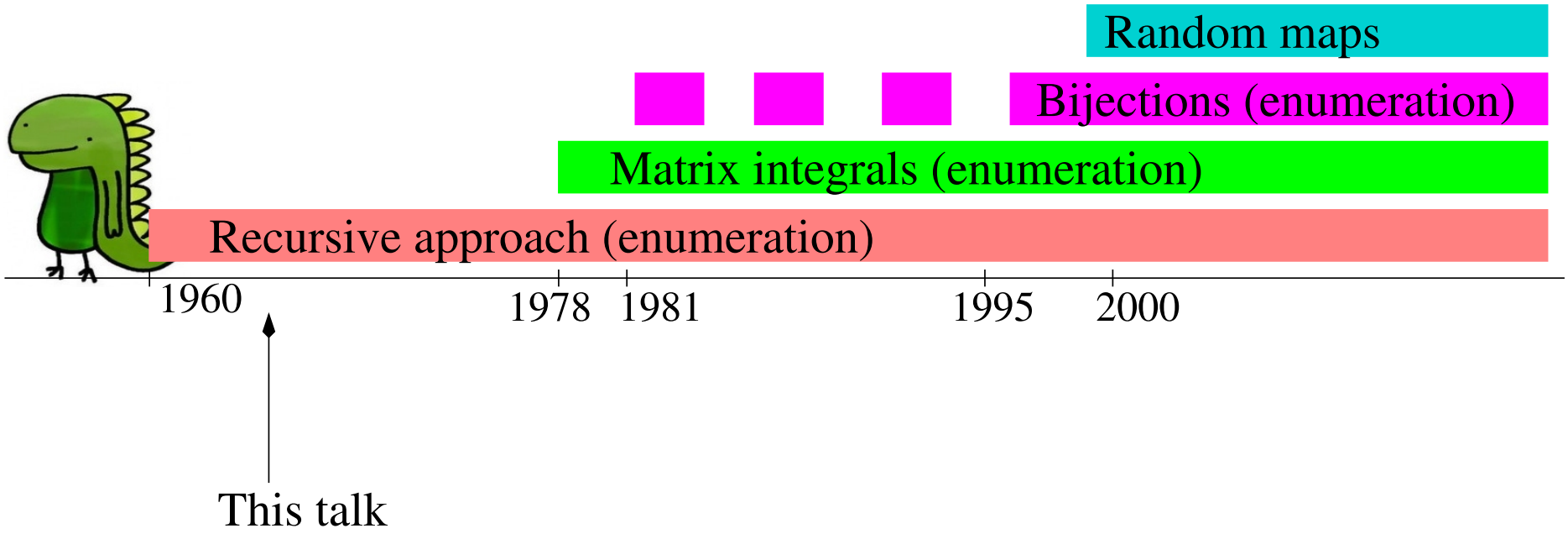
A chronology of planar maps



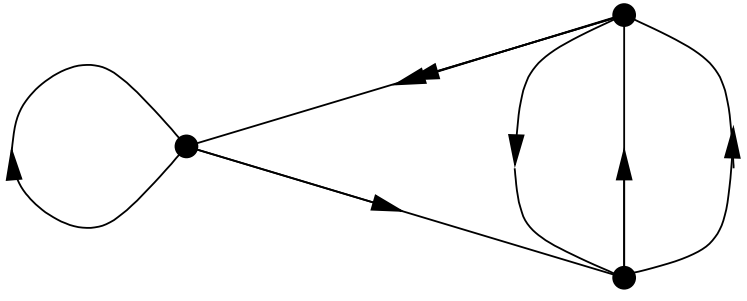
A chronology of planar maps



A chronology of planar maps

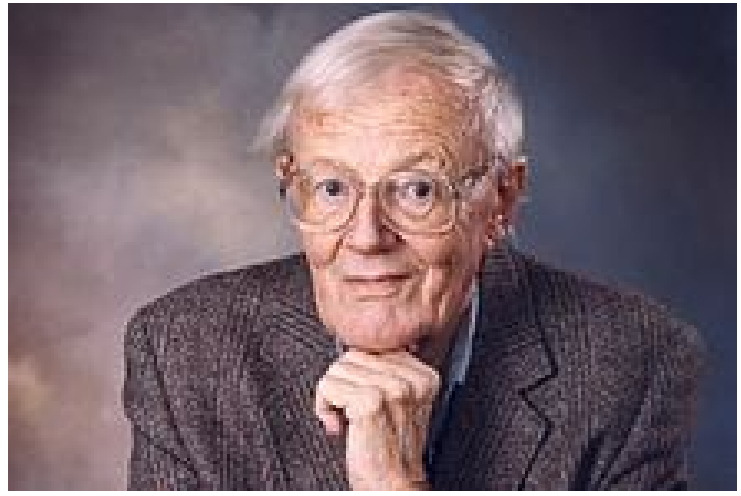


... but for maps equipped with an Eulerian orientation

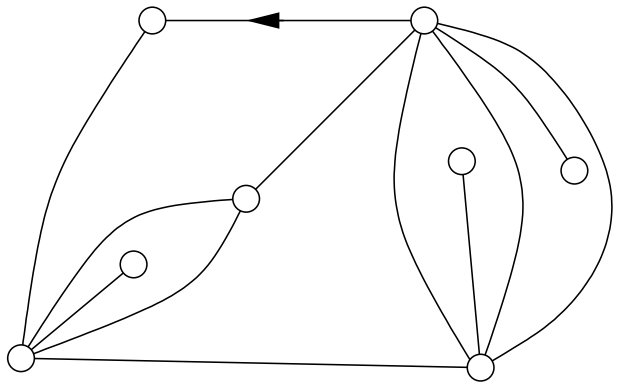


I. Counting planar maps: the early days

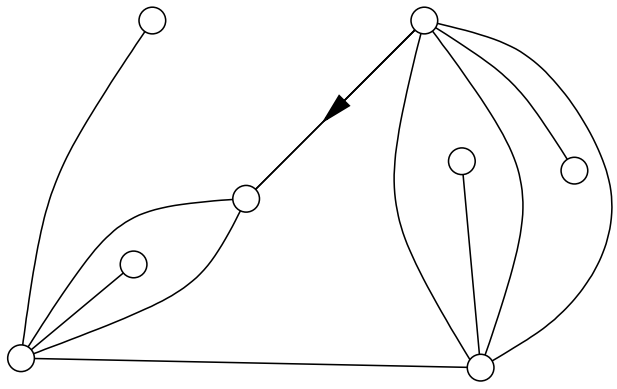
William Tutte and the recursive method



Counting quadrangulations by edge deletion

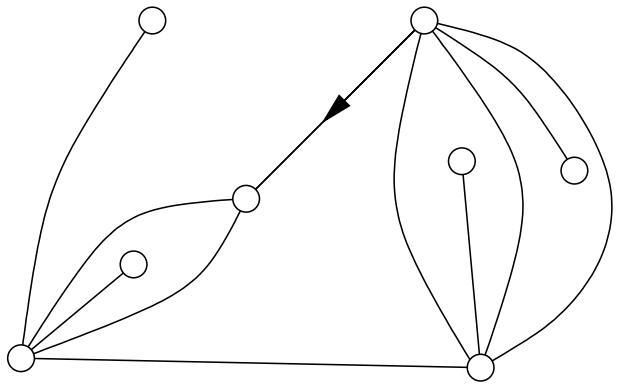


Counting quadrangulations by edge deletion



This increases the outer degree! (by 2)

Counting quadrangulations by edge deletion



This increases the outer degree! (by 2)

OK... then count near-quadrangulations, and record the outer degree:
$q(n, d)$ near-quadrangulations with n inner faces and outer degree d

Counting quadrangulations by edge deletion

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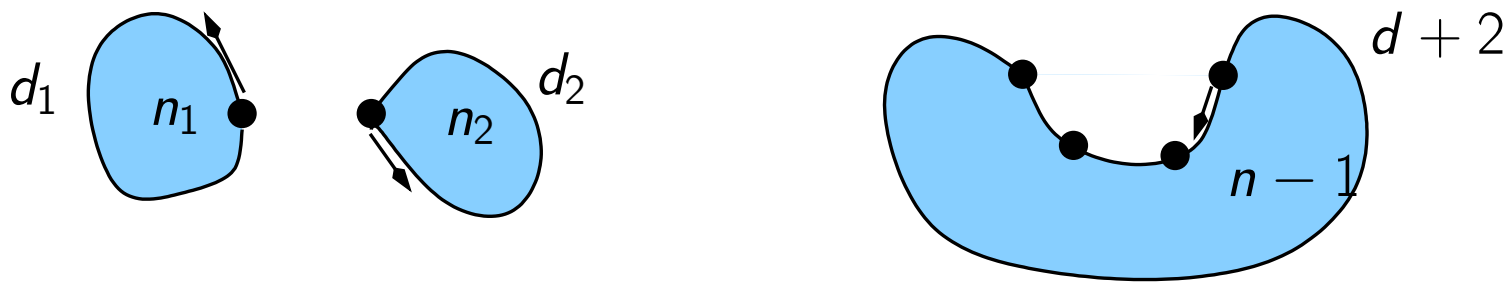
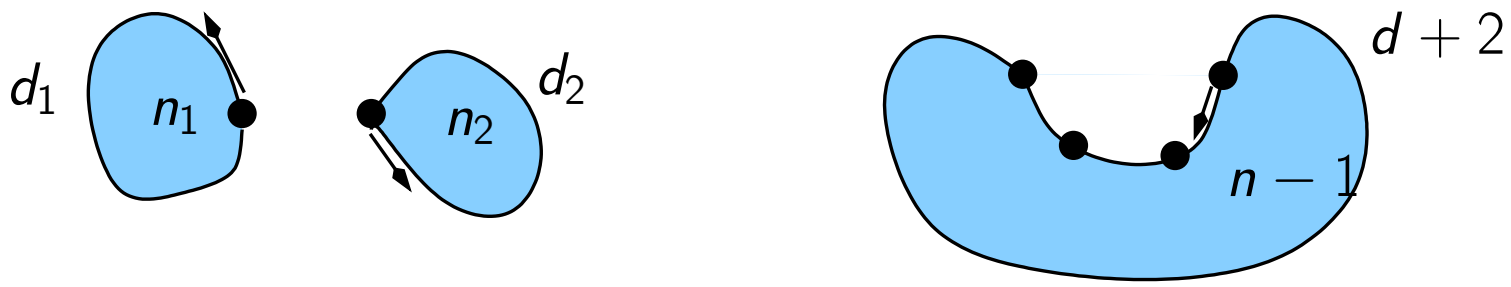
Counting quadrangulations by edge deletion



$q(n, d)$ near-quadrangulations with n inner faces and outer degree d

Counting quadrangulations by edge deletion

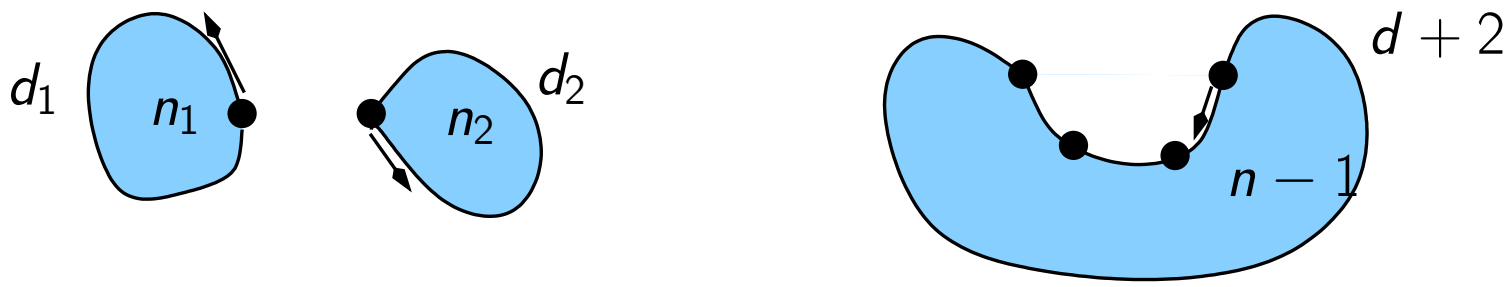


$q(n,d)$ near-quadrangulations with n inner faces and outer degree d

Recurrence relation: $q(0,0)=1$ and

$$q(n,d)=\sum_{n_1+n_2=n}\ \sum_{d_1+d_2=d-2} q(n_1,d_1)q(n_2,d_2)+q(n-1,d+2)$$

Counting quadrangulations by edge deletion



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Generating function: $Q(t,x)=\sum_{n,d\geq 0} q(n,d)t^n x^d$

$$Q(t,x)=1+x^2Q(t,x)^2+\frac{t}{x^2}\left(Q(t,x)-1-x^2Q_2(t)\right),$$

with

$$Q_2(t)=[x^2]Q(t,x)=\sum_{n\geq 0} q(n,2)t^n.$$

Counting quadrangulations by edge deletion

Functional equation:

$$Q(t,x) = 1 + x^2 Q(t,x)^2 + \frac{t}{x^2}\left(Q(t,x) - 1 - x^2 Q_2(t)\right),$$

with

$$Q_2(t) = \sum_{n \geq 0} q(n,2) t^n.$$

Counting quadrangulations by edge deletion

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The original solution: guess and check

$$Q_2(t) = \frac{(1-12t)^{3/2} - 1 + 18t}{54t^2} \qquad \text{and} \qquad q(n,2) = 2 \cdot 3^n \frac{(2n)!}{n!(n+2)!}.$$

Counting quadrangulations by edge deletion

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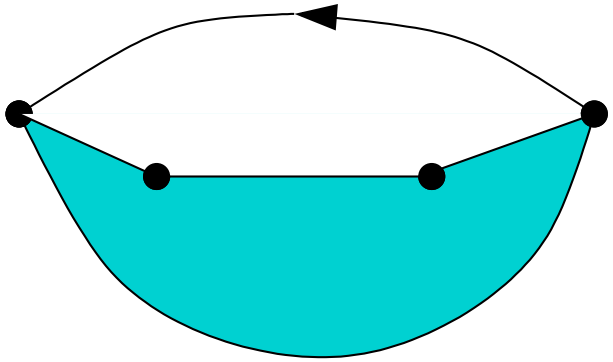
This is also the number of quadrangulations with n faces.



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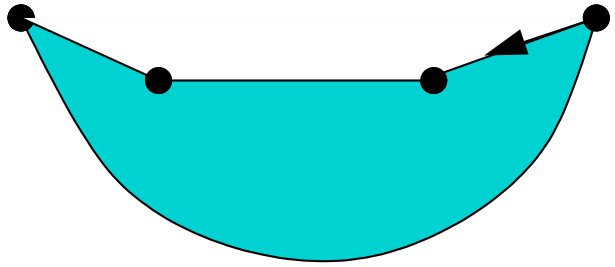


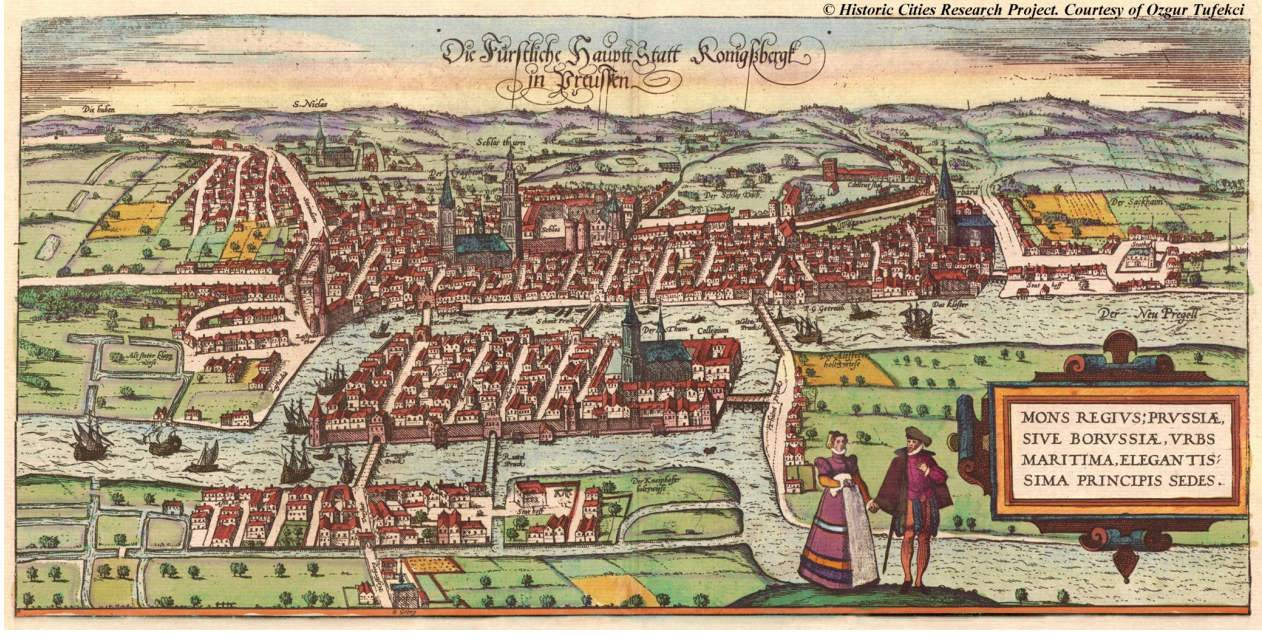
The recursive method

- One may have to
  - introduce more general objects (near-quadrangulations)
  - introduce new enumeration parameters (the outer degree)
- Solving can be non-trivial!

$$Q(t,x) = 1 + x^2 Q(t,x)^2 + \frac{t}{x^2}\left(Q(t,x) - 1 - x^2 Q_2(t)\right)$$

II. Maps with an additional structure

Eulerian orientations and the 6-vertex model

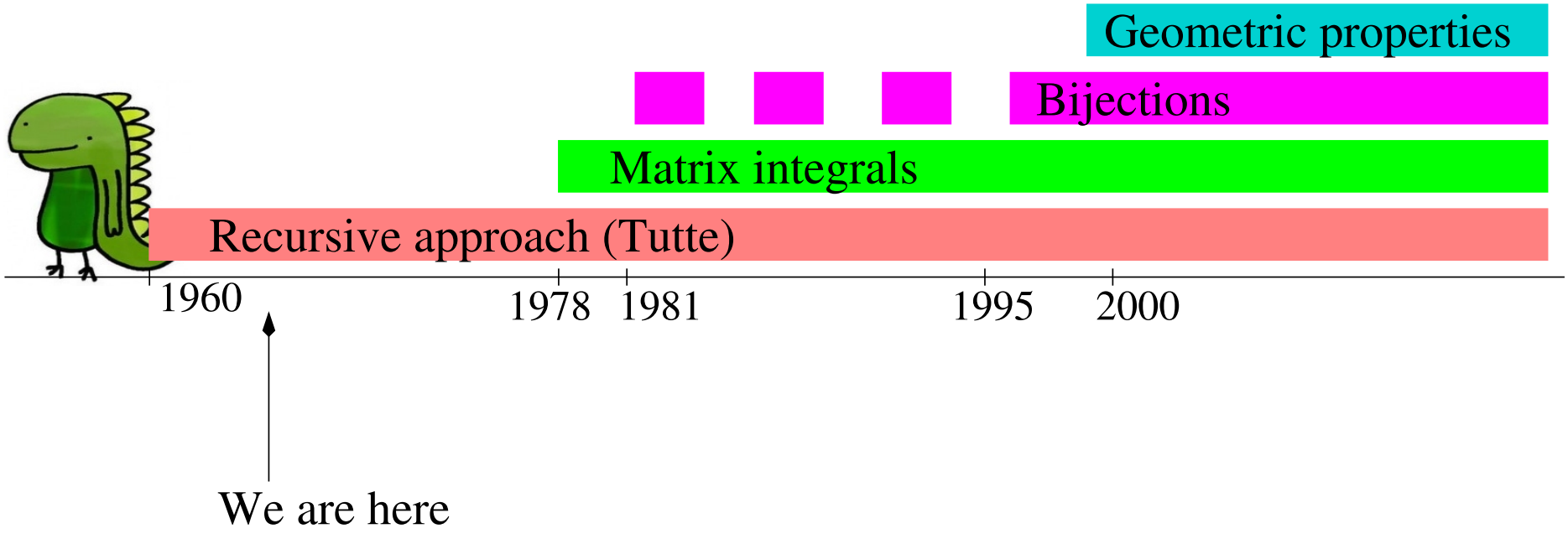


Maps equipped with an additional structure

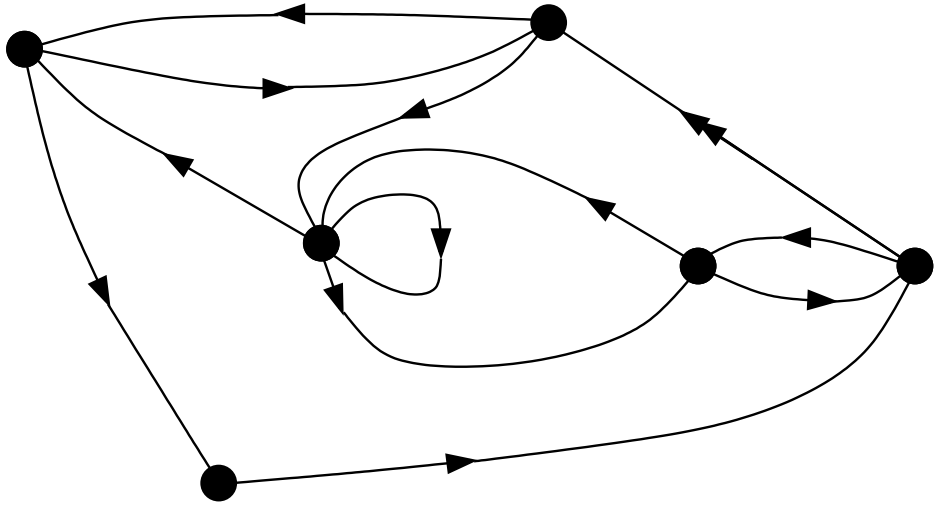
In combinatorics, probability and statistical physics

- **How many maps equipped with...**
  - a spanning tree [Mullin 67, Bernardi]
  - a spanning forest? [Bouttier et al., Sportiello et al., mbm & Courtiel]
  - a self-avoiding walk? [Duplantier & Kostov; Gwynne & Miller]
  - a proper q-colouring? [Tutte 74-83, Bouttier et al.]
  - a bipolar orientation? [Kenyon, Miller, Sheffield, Wilson, Fusy, mbm...]
- **What is the expected partition function of...**
  - the Ising model? [Boulatov, Kazakov, mbm, Schaeffer, Chen, Turunen, Bouttier et al., Albenque, Ménard...]
  - the hard-particle model? [mbm, Schaeffer, Jehanne, Bouttier et al.]
  - the Potts model? [Eynard-Bonnet, Baxter, mbm-Bernardi, Guionnet et al., Borot et al., ...]
  - ...

Maps equipped with an additional structure

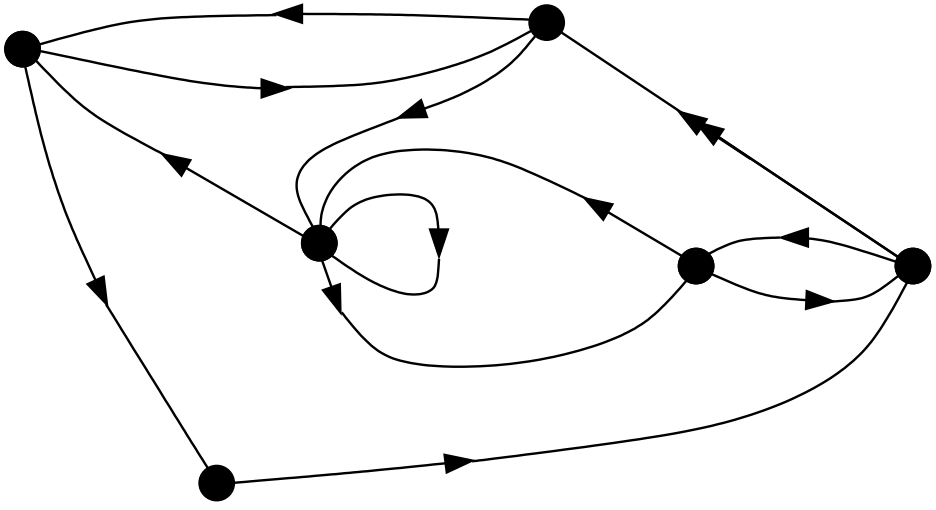


Eulerian orientations

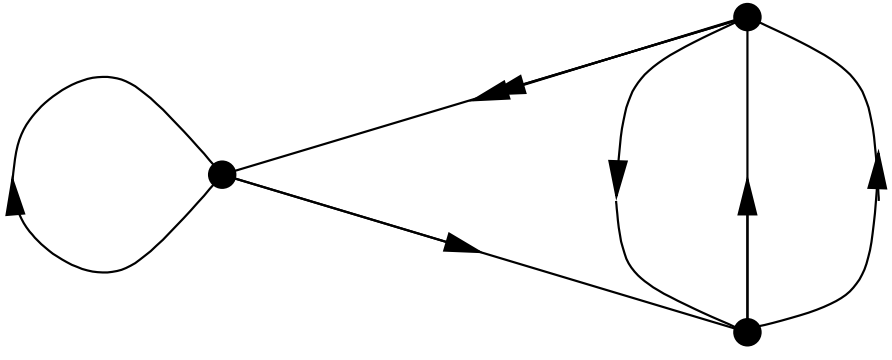


Each vertex has equally many incoming as outgoing edges.

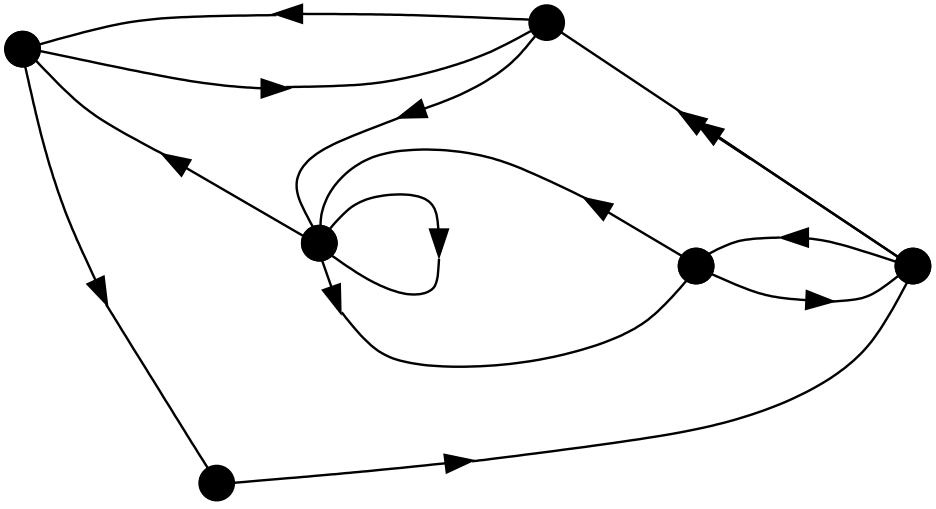
Eulerian orientations



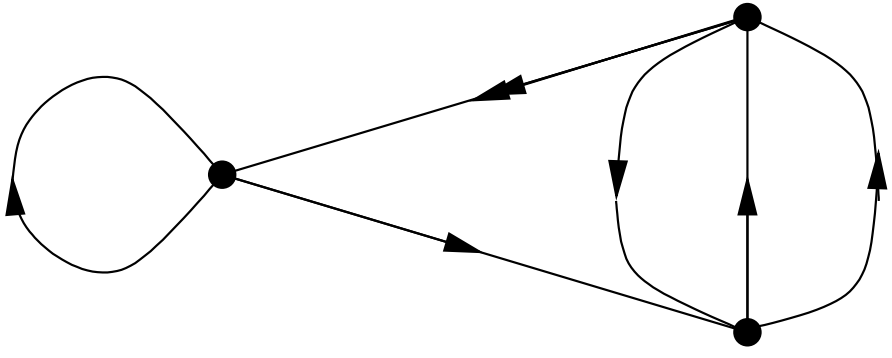
The 4-valent case: the ice model



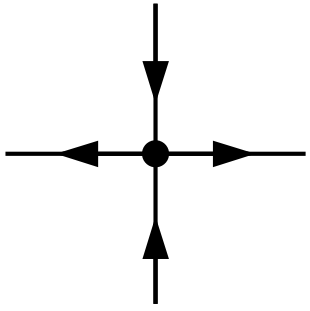
Eulerian orientations



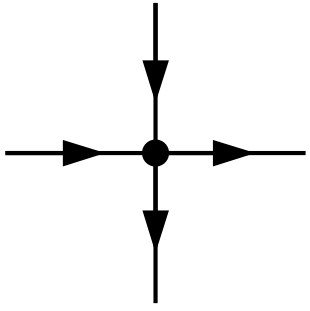
The 4-valent case: the ice model



The 6-vertex model

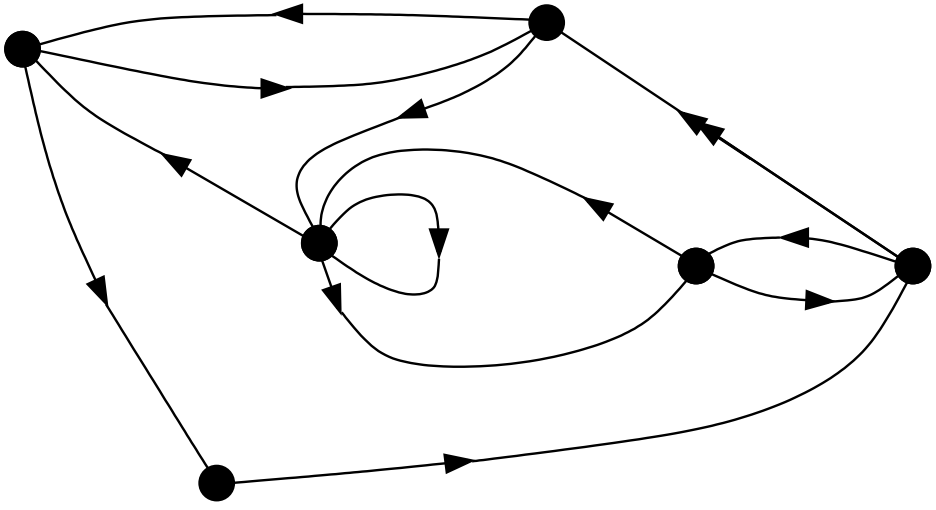


alternating (weight $t\omega$)

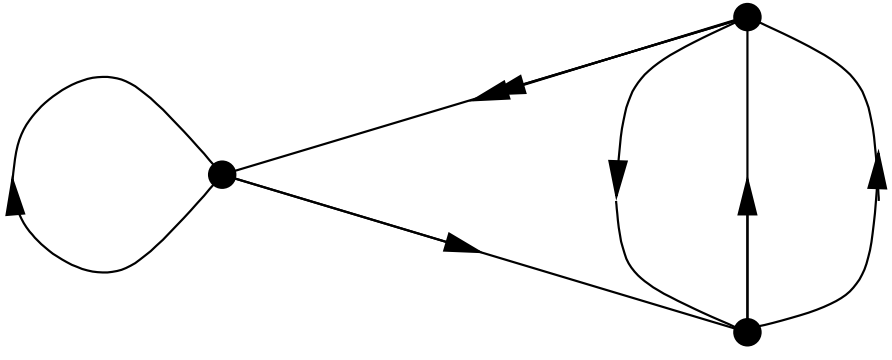


non-alternating (weight t)

Eulerian orientations

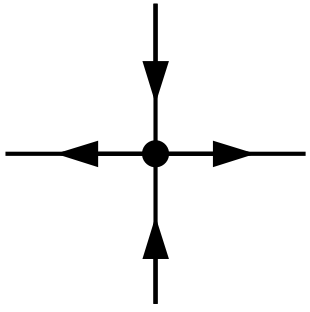


The 4-valent case: the ice model

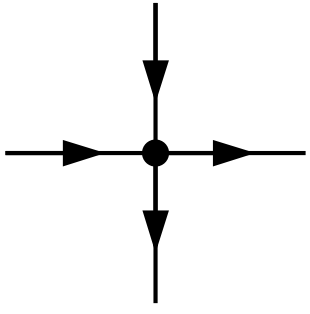


$\omega = 1$

The 6-vertex model

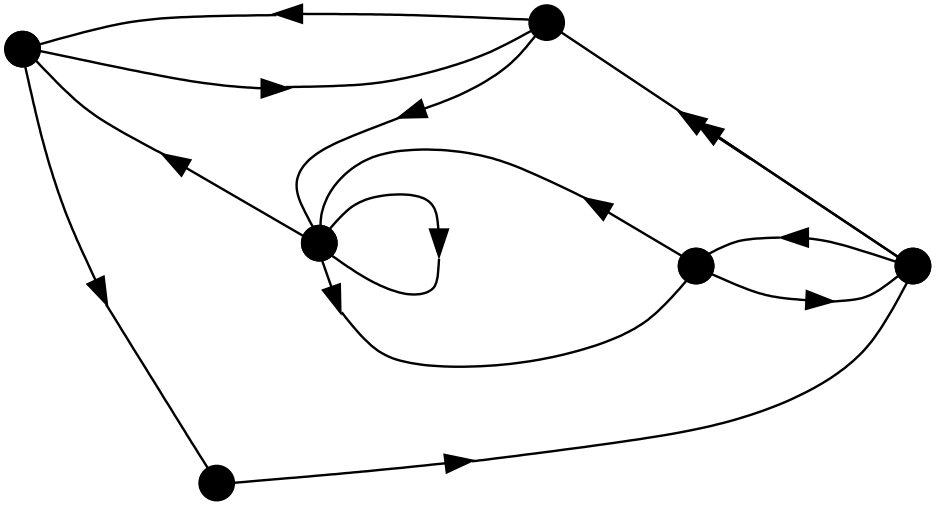


alternating (weight $t\omega$)



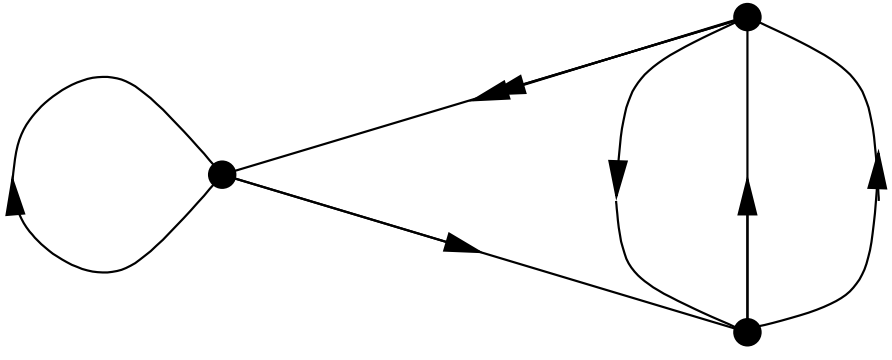
non-alternating (weight t)

Eulerian orientations



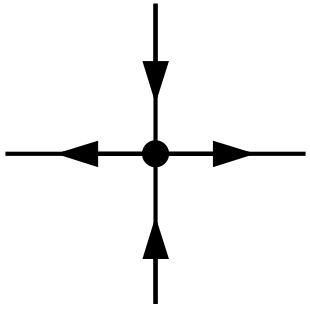
$\omega = 0$

The 4-valent case: the ice model

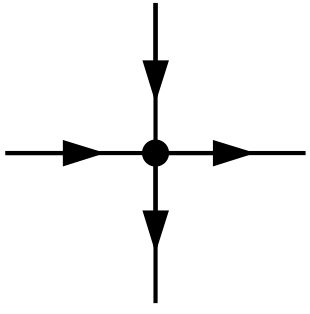


$\omega = 1$

The 6-vertex model



alternating (weight $t\omega$)



non-alternating (weight t)

The number of planar Eulerian orientations

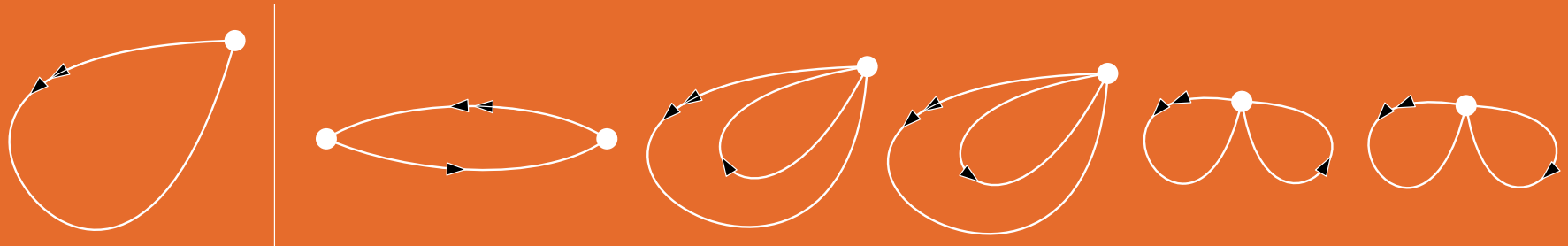
Theorem [mbm & Elvey Price 18]

Let $S \equiv S(t) = t - 2t^2 - 4t^3 - 20t^4 - 132t^5 - \cdots$ be the unique power series with constant term 0 satisfying

$$t = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n}^2 S^{n+1}.$$

The generating function of planar Eulerian orientations counted by edges is

$$\begin{aligned} G(t) &= \frac{1}{4t^2}(t - 2t^2 - S(t)) \\ &= t + 5t^2 + 33t^3 + \cdots \end{aligned}$$



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Asymptotics: $G(t) = \sum_n g_n t^n$ where

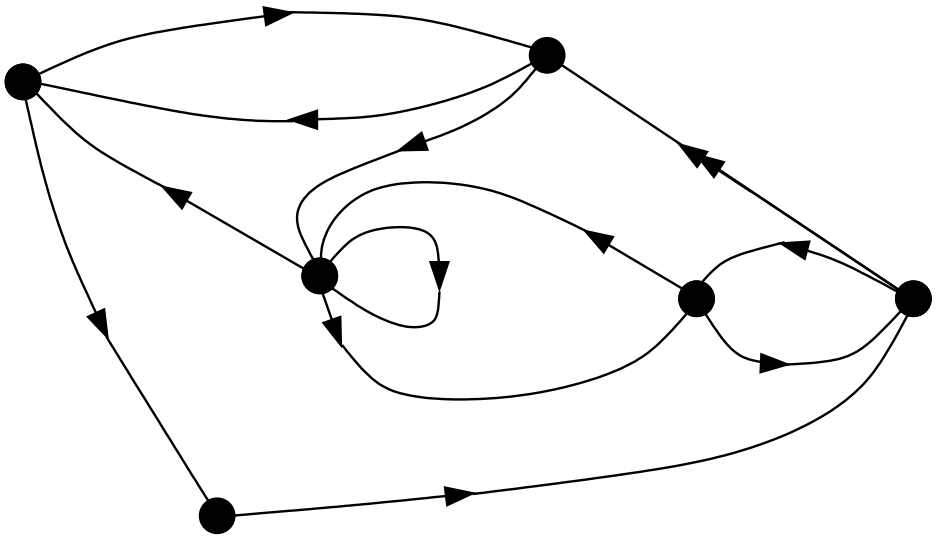
$$g_n \sim \frac{1}{16} \frac{(4\pi)^{n+2}}{n^2 (\log n)^2}$$

III. Two bijections

1. The dual of an Eulerian orientation: a labelled map

2. A bijection between 4-valent Eulerian orientations with no alternating vertex (n vertices) and general Eulerian orientations with n edges

1. From Eulerian orientations to labelled maps



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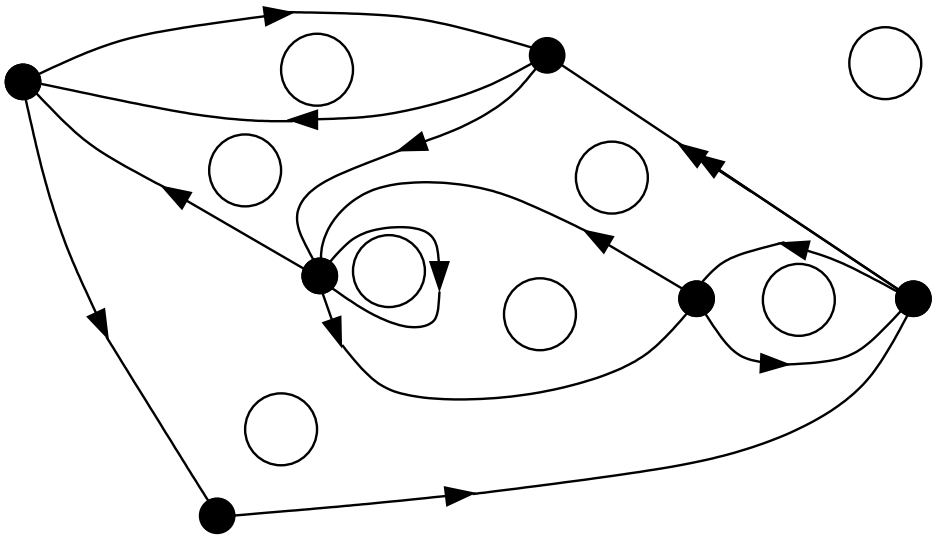

The orientation defines a height function on the faces

[Elvey Price & Guttmann 18]

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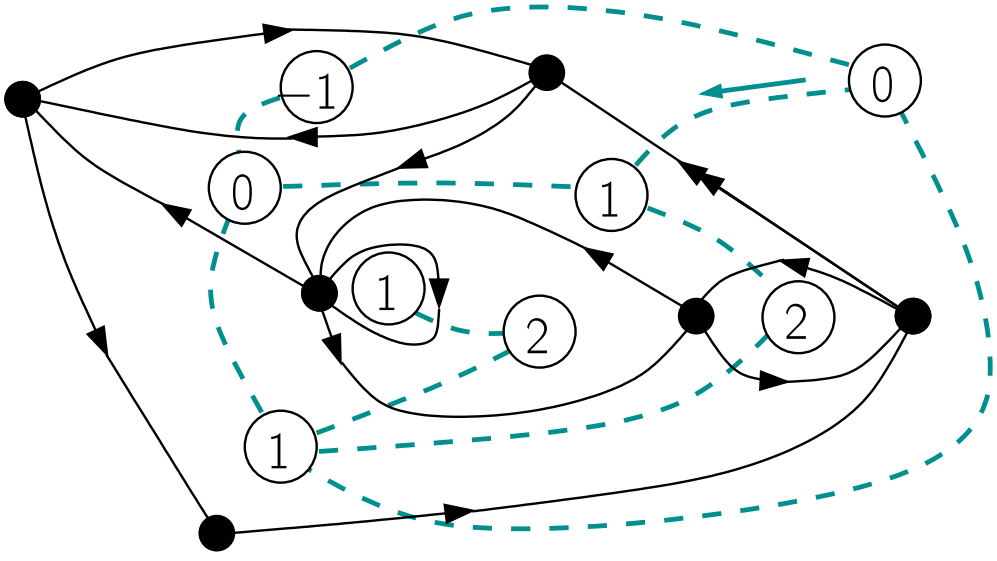

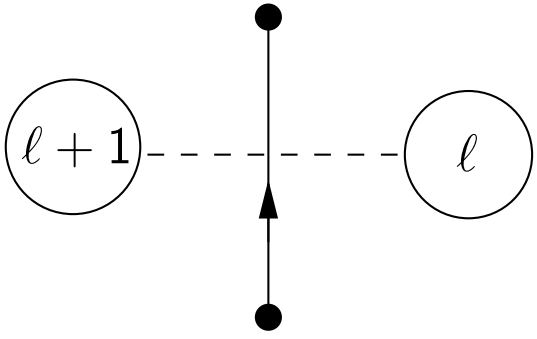

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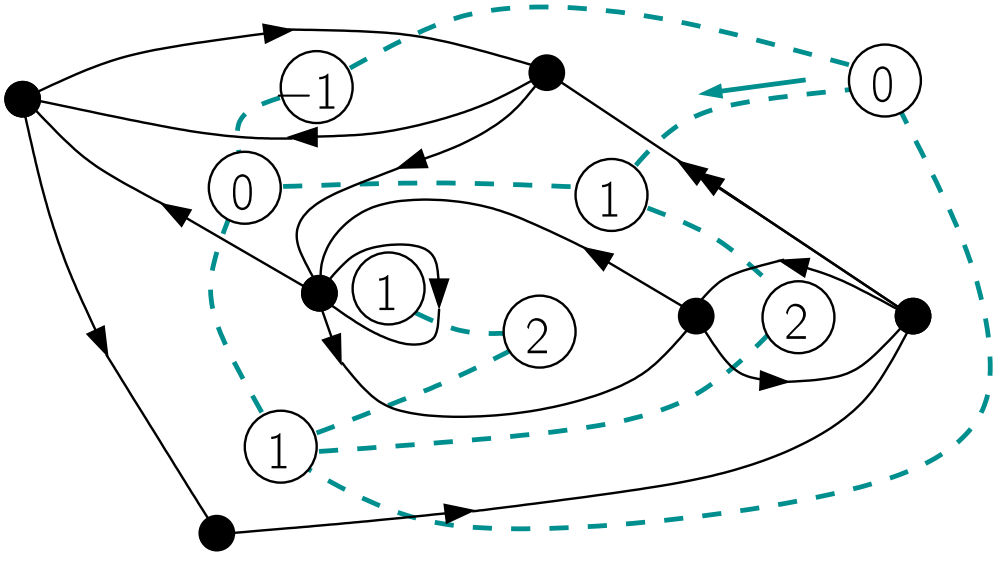
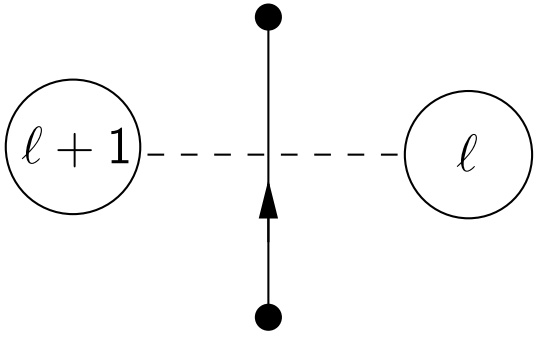
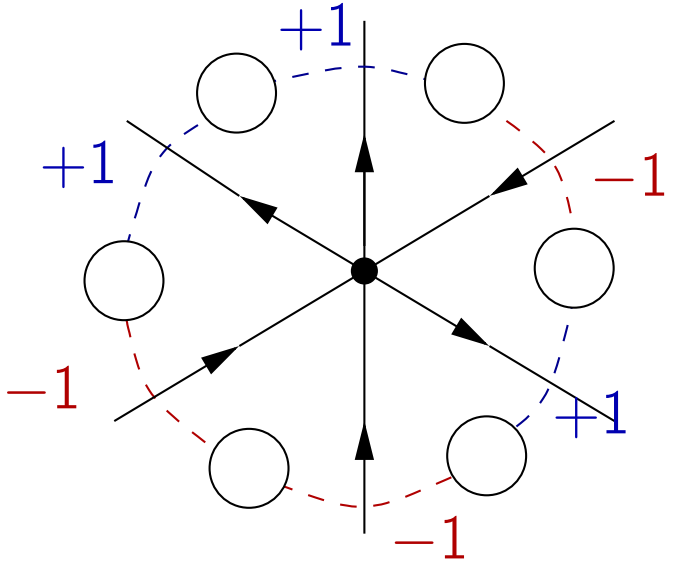


The orientation defines a height function on the faces





[Elvey Price & Guttmann 18]

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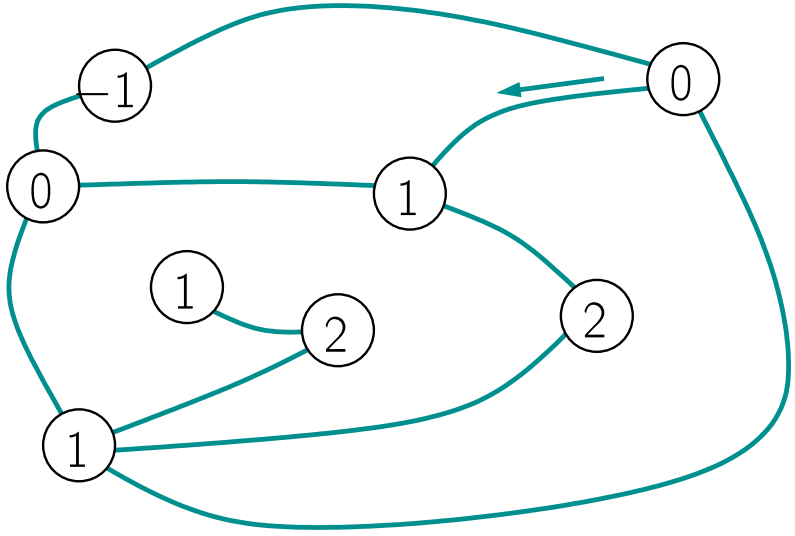

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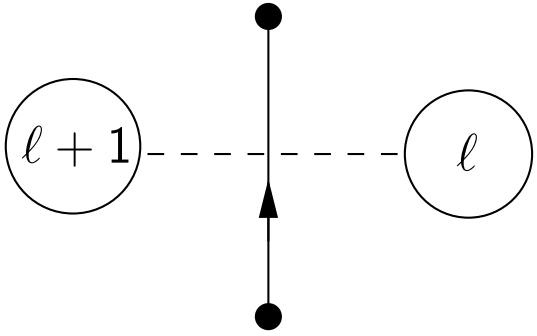

Labelled maps are rooted planar maps with labelled vertices such that:

- the root edge is labelled from 0 to 1,
- adjacent labels differ by ± 1.

[Elvey Price & Guttmann 18]

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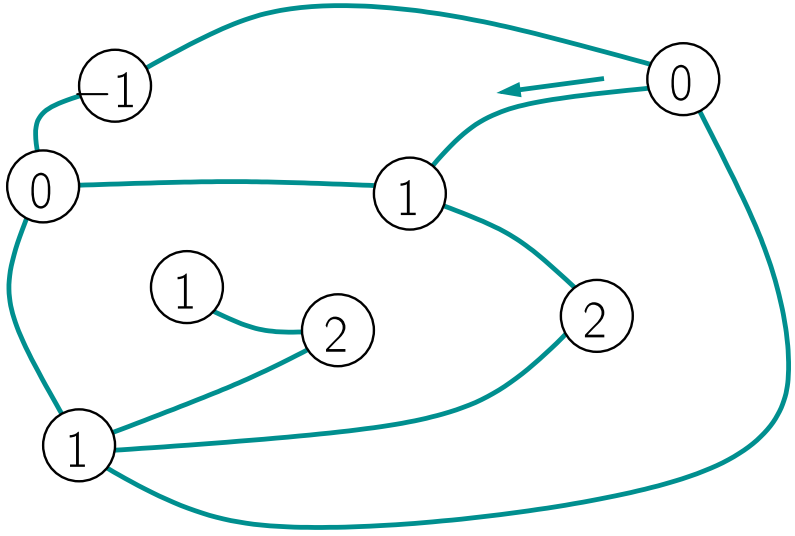
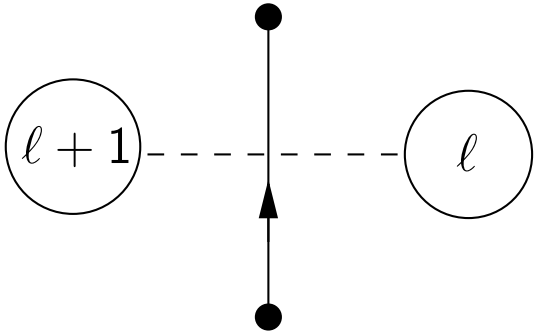
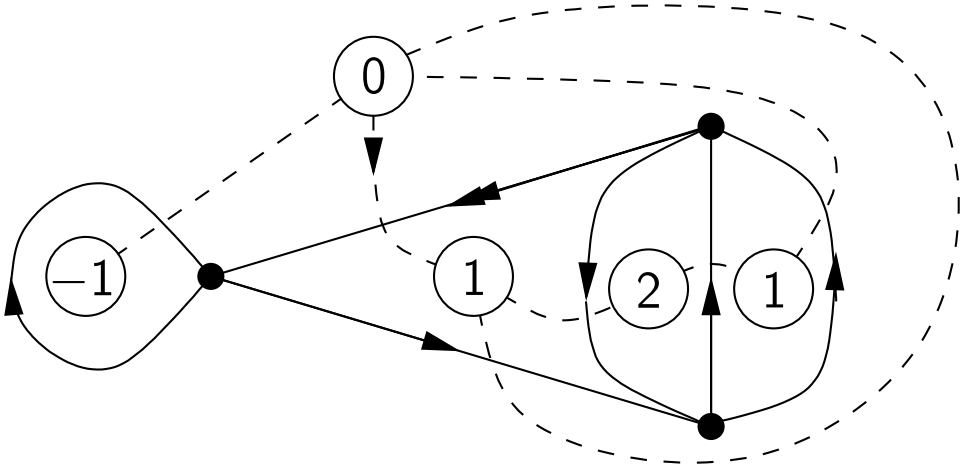




The 4-valent case: labelled quadrangulations



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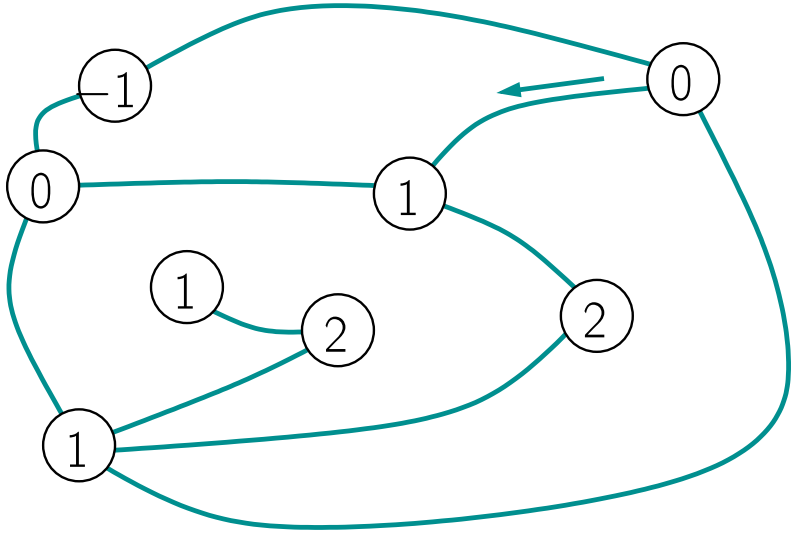

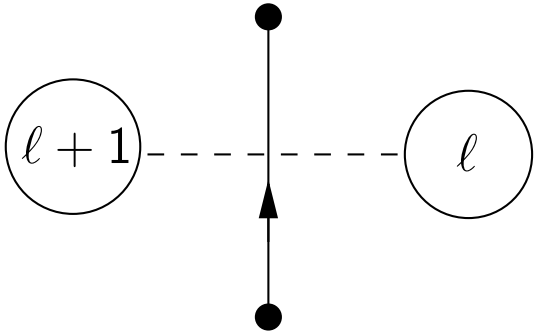

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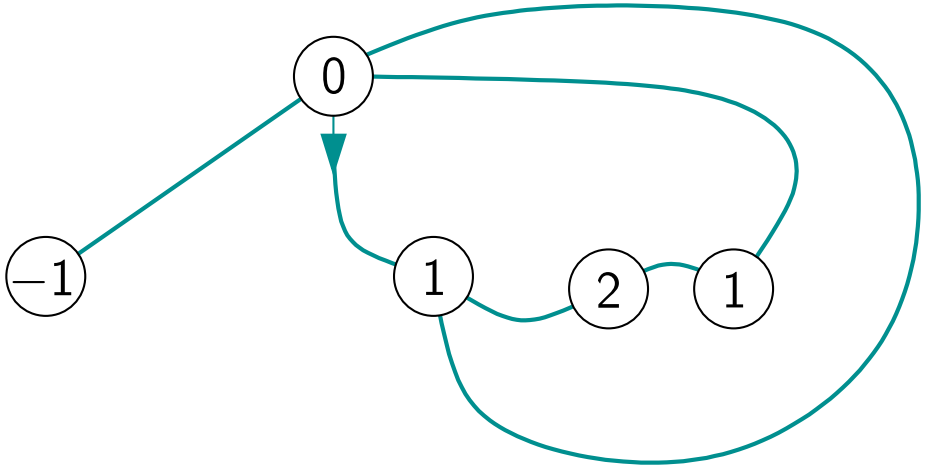

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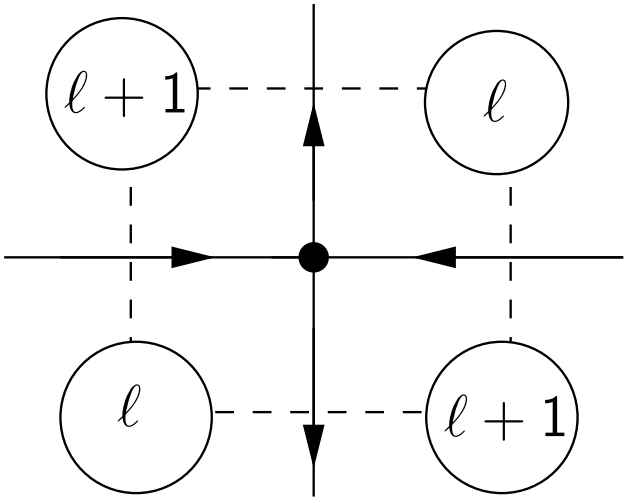
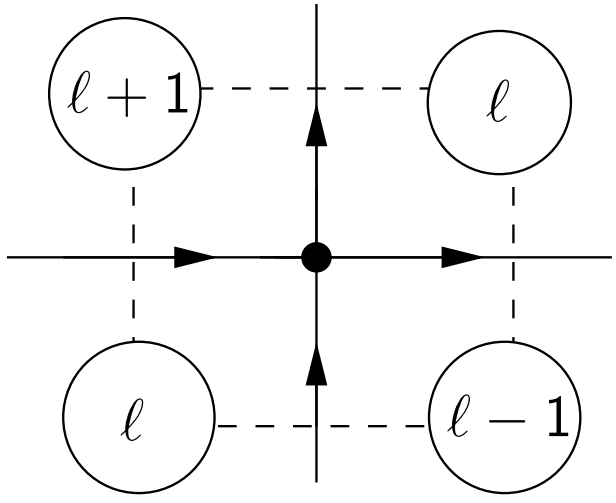
For the 2-vertex model



Alternating (weight $t\omega$)

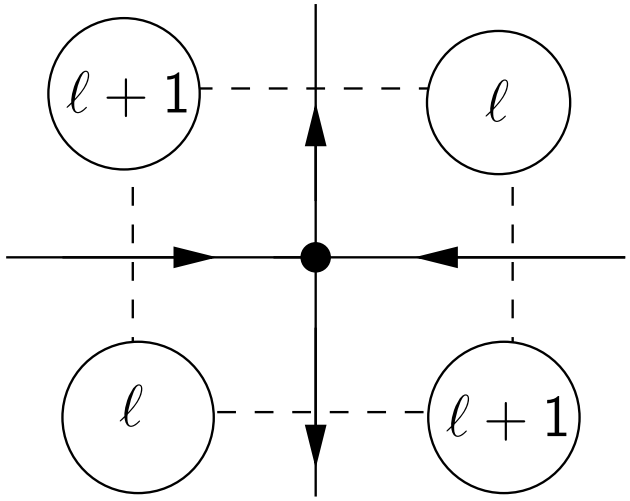
bicoloured face



Non-alternating (weight t)

colourful face (3 labels)

For the 2-vertex model



Alternating (weight $t\omega$)

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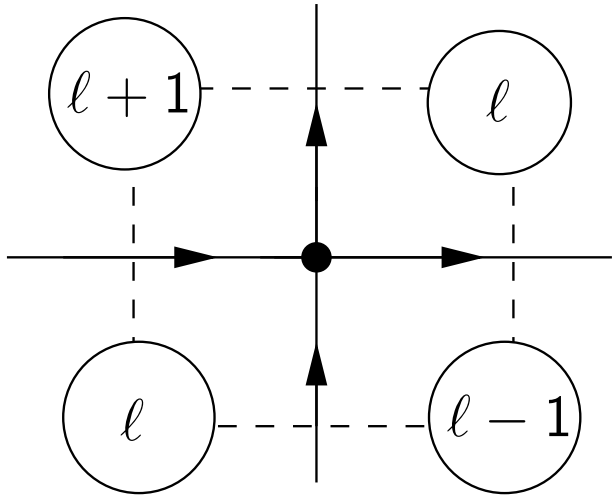

Non-alternating (weight t)

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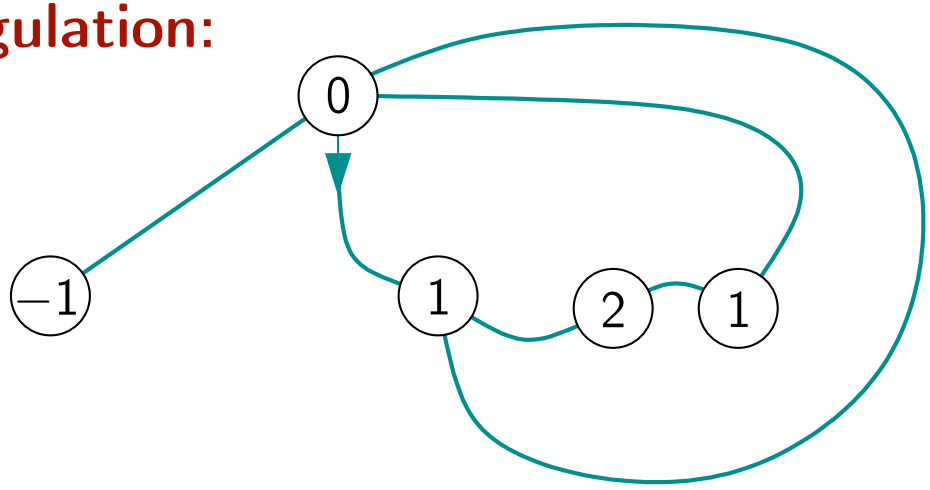
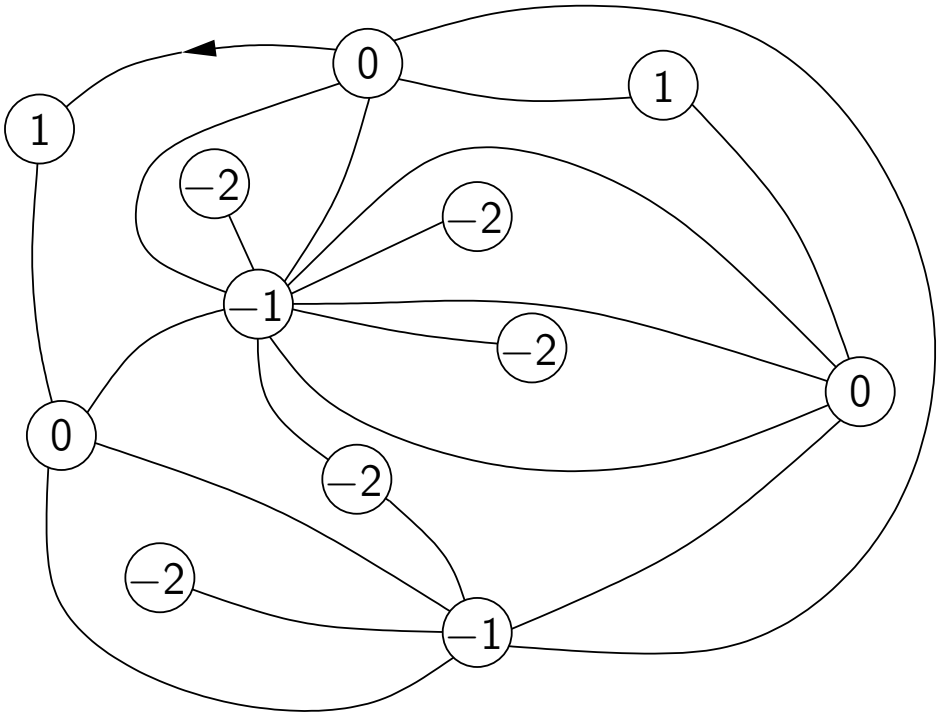
Colourful quadrangulation:



2. From colourful labelled quadrangulations to labelled maps



A bijection à la Cori-Vauquelin-Schaeffer [Miermont 09, Ambjörn & Budd 13]

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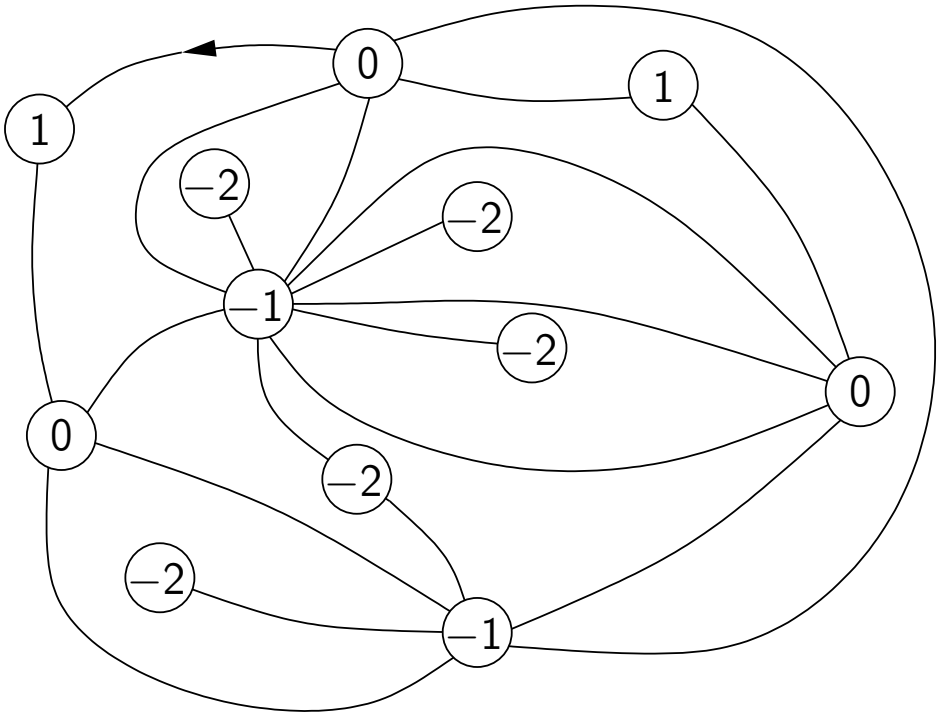

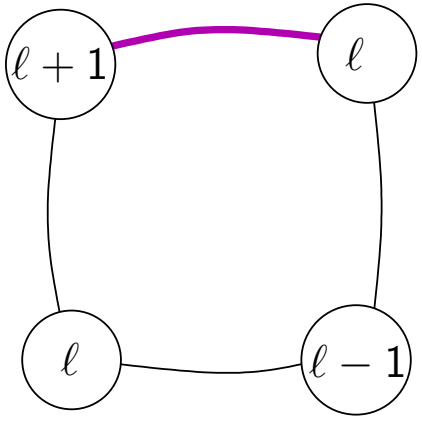

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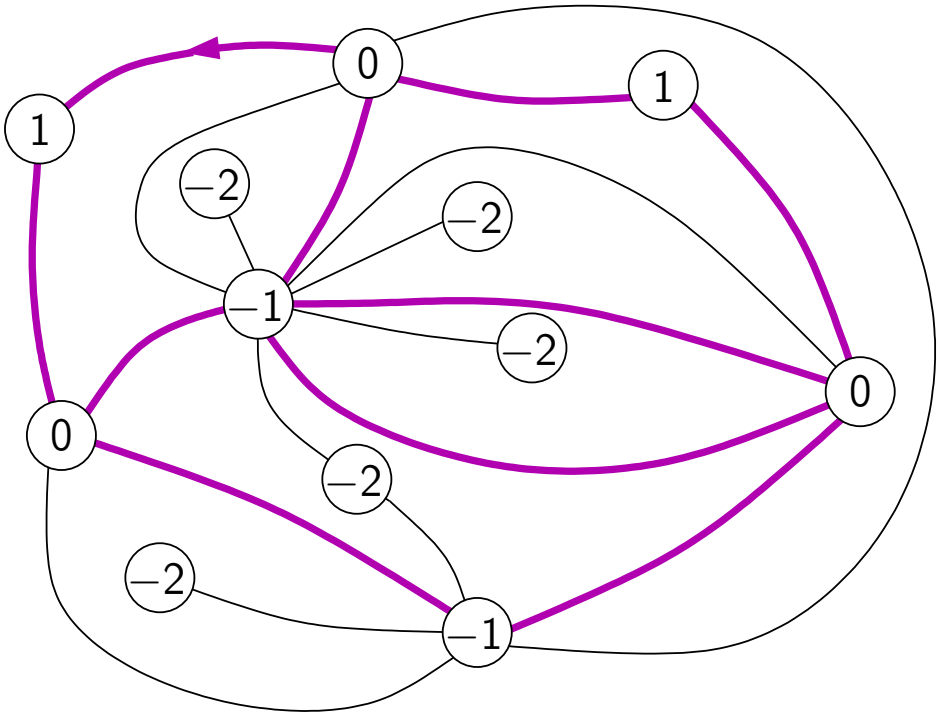

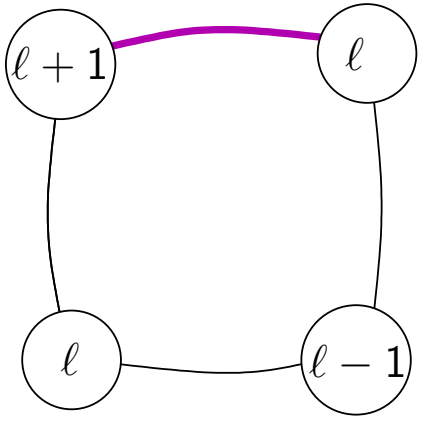

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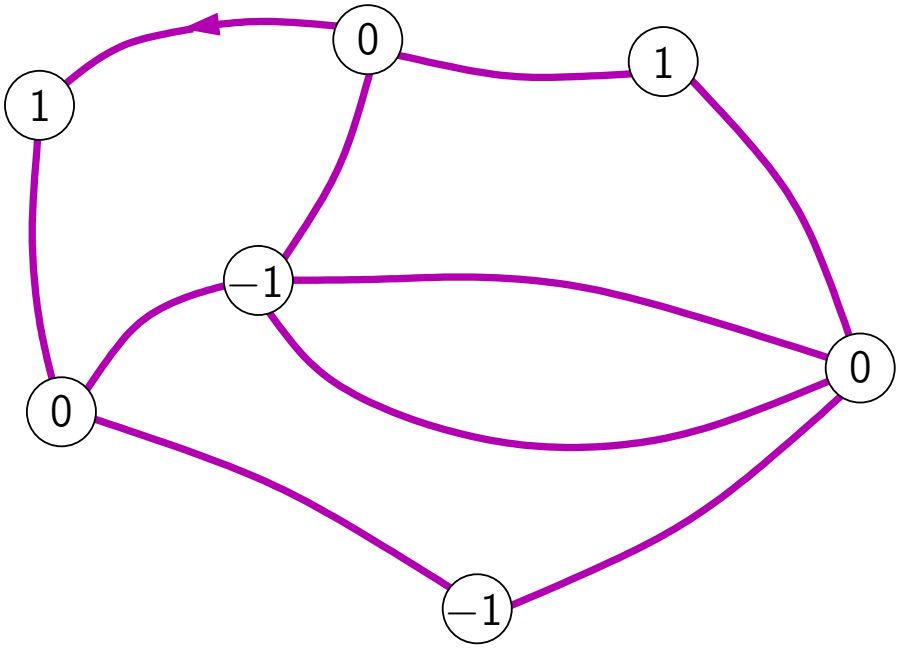

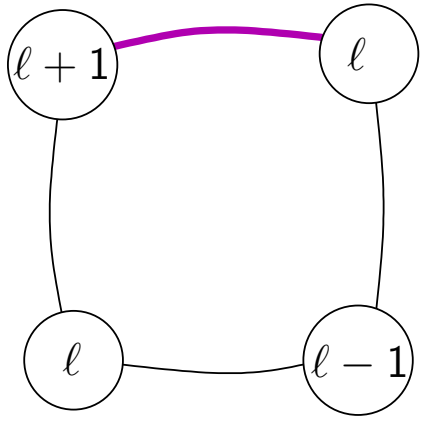

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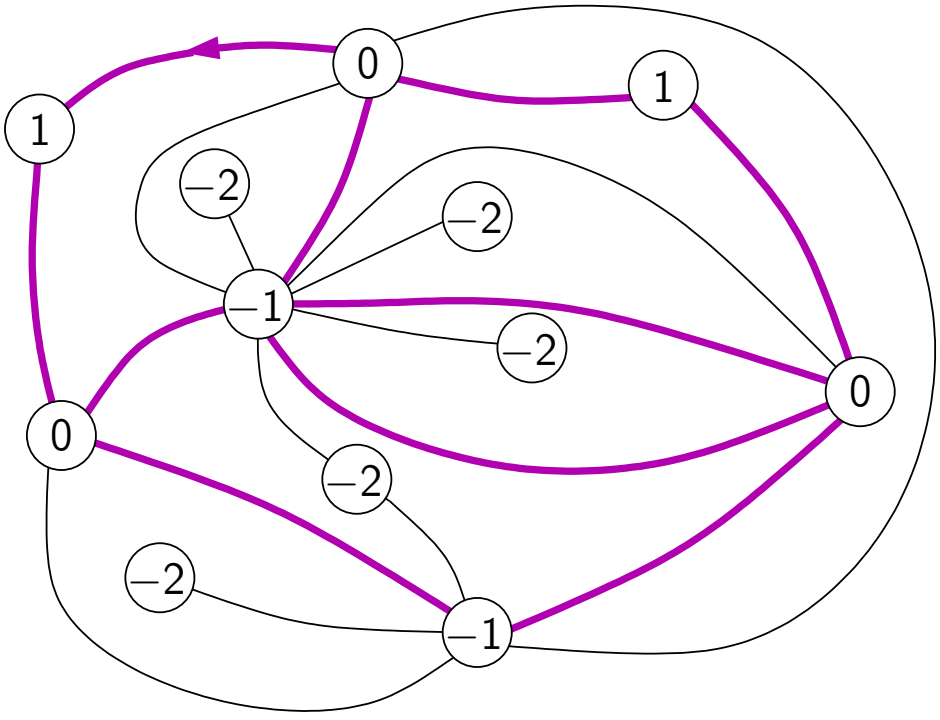
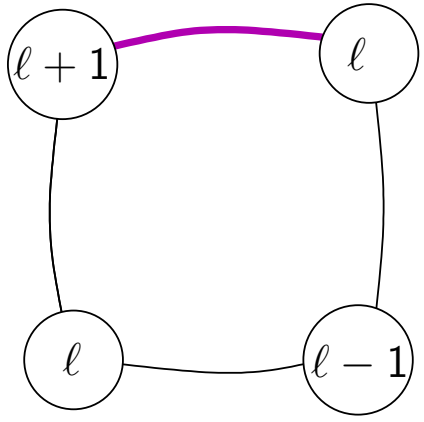


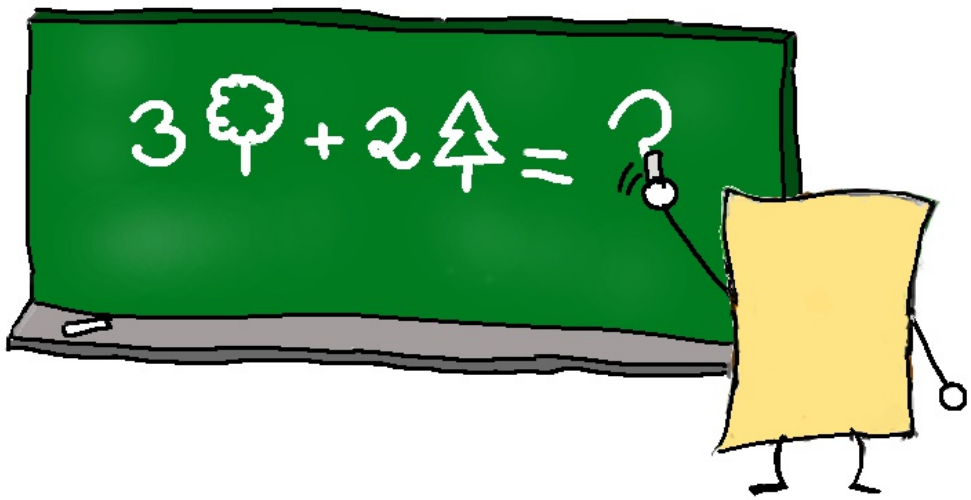


A bijection à la Cori-Vauquelin-Schaeffer [Miermont 09, Ambjörn & Budd 13]

Corollary

The number of labelled maps with n edges is half the number of colourful quadrangulations with n faces.

IV. Functional equations



The recursive method

- One may have to
  - introduce more general objects (3 different classes of labelled maps)
  - introduce new enumeration parameters (2 of them)
- Solving can be non-trivial!

A system for colourful quadrangulations [mbm, Elvey Price]

- There exists a unique 3-tuple of series, denoted $\mathsf{P}(t,y)$, $\mathsf{C}(t,x,y)$ and $\mathsf{D}(t,x,y)$, belonging to $\mathbb{Q}[[y,t]]$, $\mathbb{Q}[x][[y,t]]$ and $\mathbb{Q}[[x,y,t]]$, such that:

$$\mathsf{P}(t,y) = \frac{1}{y}[x^1]\mathsf{C}(t,x,y),$$

$$\mathsf{D}(t,x,y) = \frac{1}{1-\mathsf{C}\left(t,\frac{1}{1-x},y\right)},$$

$$\mathsf{C}(t,x,y) = xy[x^{\geq 0}]\left(\mathsf{P}(t,tx)\mathsf{D}\left(t,\frac{1}{x},y\right)\right),$$

and $\mathsf{P}(t,0) = 1$.

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and $\mathrm{P}(t,0) = 1$.

Notation: for a Laurent series $F(x) = \sum_{i\in\mathbb{Z}} F_i x^i$,

$$[x^1]F(x) := F_1 \qquad \text{and} \qquad [x^{\geq 0}]F(x) := \sum_{i\geq 0} F_i x^i.$$

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and $\mathsf{P}(t,0) = 1$.

- The generating function that counts colourful quadrangulations by faces is

$$\mathsf{Q}^{\mathsf{c}}(t) = [y^1]\mathsf{P}(t,y) - 1.$$

A system for colourful quadrangulations [mbm, Elvey Price]

- There exists a unique 3-tuple of series, denoted $\mathsf{P}(t,y)$, $\mathsf{C}(t,x,y)$ and $\mathsf{D}(t,x,y)$, belonging to $\mathbb{Q}[[y,t]]$, $\mathbb{Q}[x][[y,t]]$ and $\mathbb{Q}[[x,y,t]]$, such that:

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- The generating function that counts colourful quadrangulations by faces is

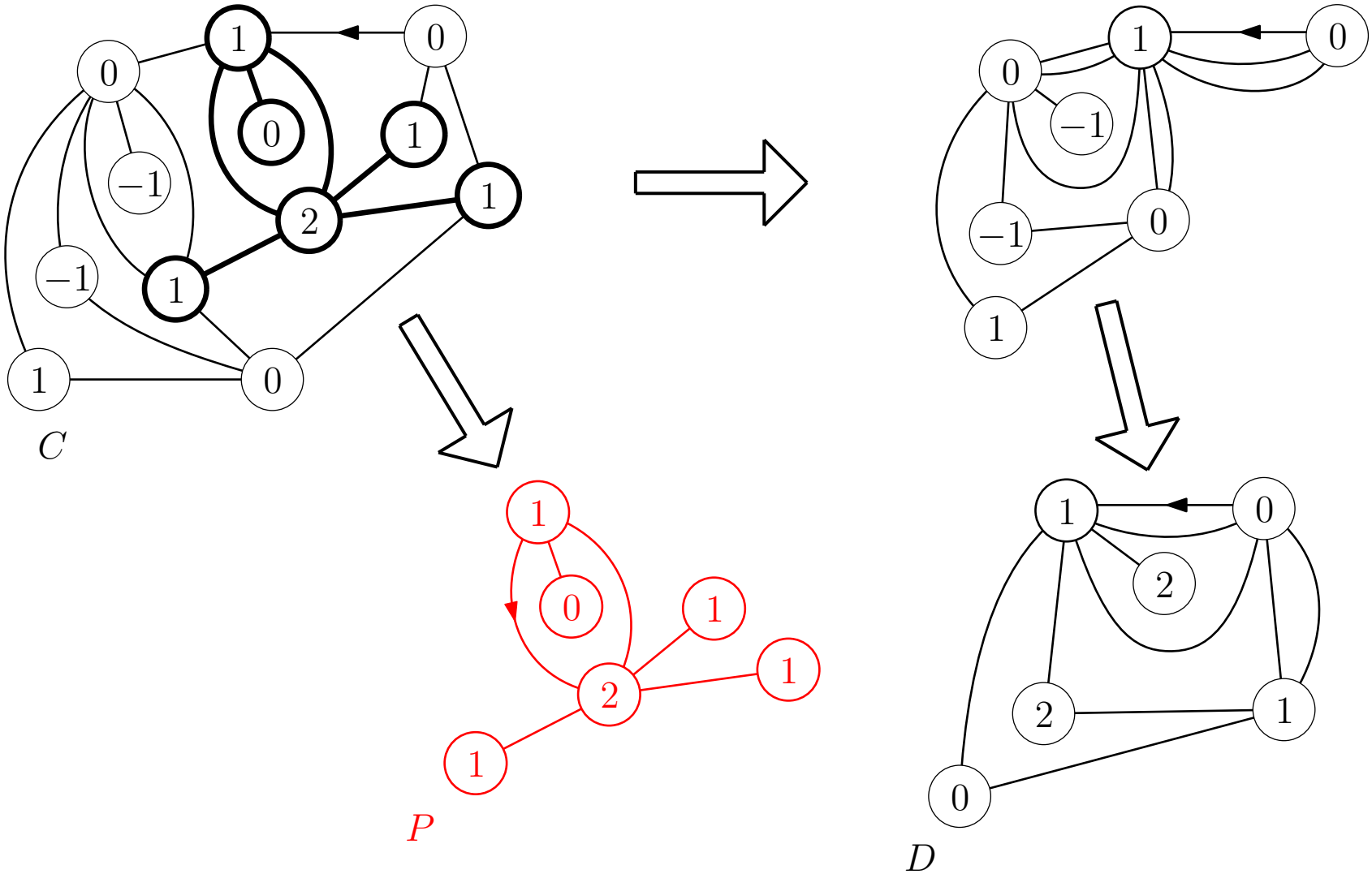
$$\mathsf{Q}^{\mathsf{c}}(t) = [y^1]\mathsf{P}(t,y) - 1.$$

- Recall that $\mathsf{Q}^{\mathsf{c}}(t) = 2\mathsf{G}(t)$, where $\mathsf{G}(t)$ counts Eulerian orientations by edges.

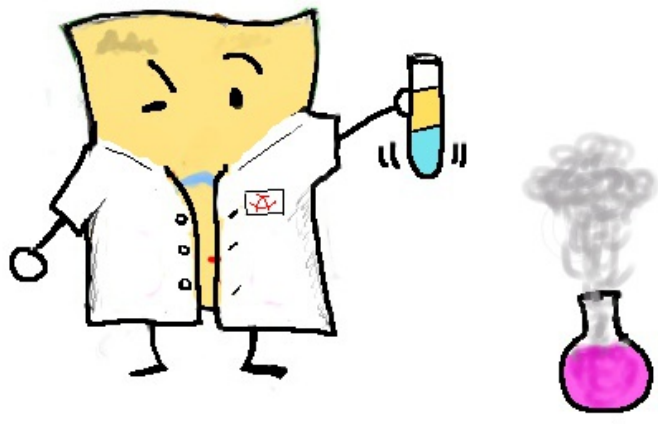
The most interesting equation

$$\mathsf{C}(t,x,y) = xy[x^{\geq 0}]\left(\mathsf{P}(t,tx)\mathsf{D}\left(t,\frac{1}{x},y\right)\right)$$

A contraction operation of a whole submap



V. Solution: guess and check!



Solution for colourful quadrangulations

- Let $\mathsf{S} \equiv \mathsf{S}(t) = t - 2t^2 - 4t^3 - 20t^4 - 132t^5 - \cdots$ be the unique power series with constant term 0 satisfying

$$t = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n}^2 \mathsf{S}^{n+1}.$$

- The generating function of planar Eulerian orientations counted by edges is

$$G(t) = \mathsf{Q}^{\mathsf{c}}(t)/2 = \frac{1}{4t^2}(t - 2t^2 - \mathsf{S})$$

Solution for colourful quadrangulations

• Let $\mathsf{S} \equiv \mathsf{S}(t) = t - 2t^2 - 4t^3 - 20t^4 - 132t^5 - \cdots$ be the unique power series with constant term 0 satisfying

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• There exists a unique 3-tuple of series, denoted $\mathsf{P}(t,y)$, $\mathsf{C}(t,x,y)$ and $\mathsf{D}(t,x,y)$, belonging to $\mathbb{Q}[[y,t]]$, $\mathbb{Q}[x][[y,t]]$ and $\mathbb{Q}[[x,y,t]]$, such that:

$$\mathsf{P}(t,y) = \frac{1}{y}[x^1]\mathsf{C}(t,x,y),$$

$$\mathsf{D}(t,x,y) = \frac{1}{1 - \mathsf{C}\left(t, \frac{1}{1-x}, y\right)},$$

$$\mathsf{C}(t,x,y) = xy[x^{\geq 0}]\left(\mathsf{P}(t,tx)\mathsf{D}\left(t, \frac{1}{x}, y\right)\right),$$

and $\mathsf{P}(t,0) = 1$.

Solution for colourful quadrangulations

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- The whole solution:

$$t\mathsf{P}(t, ty) = \sum_{n \geq 0} \sum_{j=0}^{n} \frac{1}{n+1} \binom{2n}{n} \binom{2n-j}{n} y^j \mathsf{S}^{n+1},$$

$$\mathsf{C}(t, x, ty) = 1 - \exp\left(- \sum_{n \geq 0} \sum_{j=0}^{n} \sum_{i=0}^{n} \frac{1}{n+1} \binom{2n-i}{n} \binom{2n-j}{n} x^{i+1} y^{j+1} \mathsf{S}^{n+1} \right)$$

$$\mathsf{D}(t, x, ty) = \exp\left(\sum_{n \geq 0} \sum_{j=0}^{n} \sum_{i \geq 0} \frac{1}{n+1} \binom{2n-j}{n} \binom{2n+i+1}{n} x^i y^{j+1} \mathsf{S}^{n+1} \right)$$

A similar solution for all labelled quadrangulations

(with a more complicated system of functional equations)

• Let $\mathrm{R}(t) \equiv \mathrm{R} = t - 3t^2 - 12t^3 - 105t^4 - 1206t^5 - \cdots$, be the unique formal power series with constant term 0 satisfying

$$t = \sum_{n \geq 0} \frac{1}{n+1} \binom{2n}{n} \binom{3n}{n} \mathrm{R}^{n+1}.$$

• Then the generating function of 4-valent Eulerian orientations, counted by vertices, is

$$\begin{aligned} \mathrm{Q}(t) &= \frac{1}{3t^2} \left(t - 3t^2 - \mathrm{R}(t) \right) \\ &= 4t + 35t^2 + 402t^3 + \cdots . \end{aligned}$$

• Asymptotics:

$$q_n \sim \frac{1}{18} \frac{(4\sqrt{3}\,\pi)^{n+2}}{n^2 (\log n)^2}$$

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How can one *guess* that?

Guess and check: the story

Let $\mathrm{R} = t - 3t^2 + \cdots$ satisfy:

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- about 100 coefficients q_n of $Q(t)$
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Luck and Andrew EP

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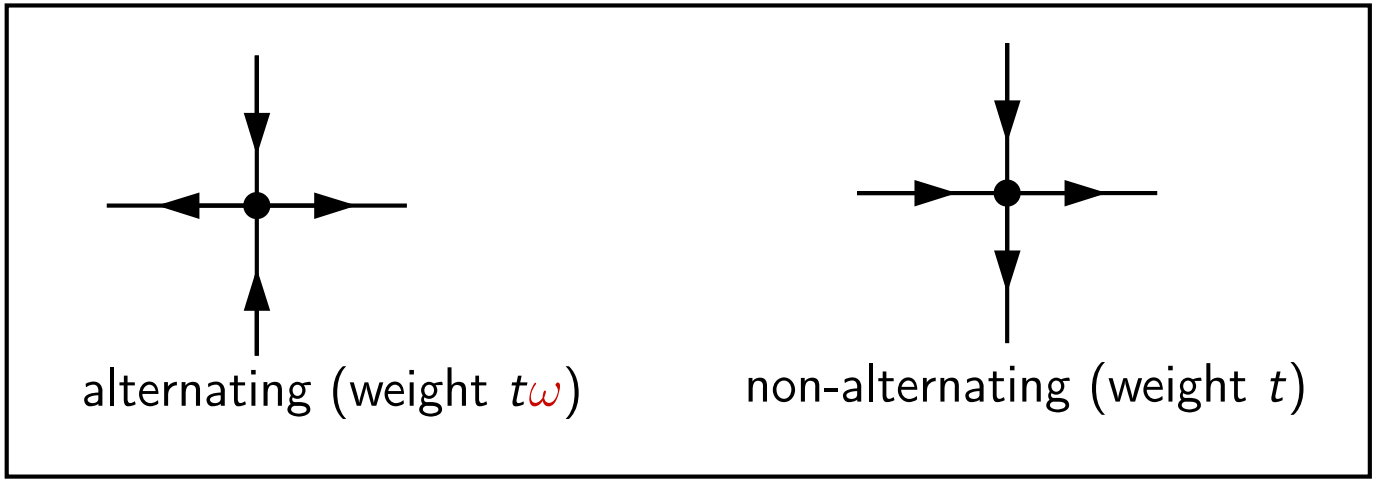
VI. The 6-vertex model

An additional weight ω on alternating vertices of 4-valent maps



This was already solved... [Kostov, Nucl. Physics B 00]

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Recall: solution when $\omega = 1$

Let $\mathrm{R}(t) \equiv \mathrm{R} = t - 3t^2 - 12t^3 - 105t^4 - 1206t^5 - \cdots$, be the unique formal power series with constant term 0 satisfying

$$t = \sum_{n \geq 0} \frac{1}{n+1} \binom{2n}{n} \binom{3n}{n} \mathrm{R}^{n+1}.$$

Then the generating function of 4-valent Eulerian orientations, counted by vertices, is

$$\mathrm{Q}(t) = \frac{1}{3t^2} \left(t - 3t^2 - \mathrm{R}(t) \right).$$

This was already solved... [Kostov, Nucl. Physics B 00]

(Once digested/corrected by Paul Zinn-Justin)
Write $\omega = -2\cos(2\alpha)$.

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Write $\omega = -2\cos(2\alpha)$. Let $q(t,\omega) \equiv q = t + (6\,\omega + 6)\,t^2 + \cdots$ satisfy:

$$t = \frac{\cos\alpha}{64\sin^3\alpha}\left(-\frac{\vartheta(\alpha,q)\vartheta'''(\alpha,q)}{\vartheta'(\alpha,q)^2} + \frac{\vartheta''(\alpha,q)}{\vartheta'(\alpha,q)}\right)$$

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$$\vartheta(z,q) = 2\sin(z)q^{1/8}\prod_{n=1}^{\infty}(1-2\cos(2z)q^n+q^{2n})(1-q^n).$$

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- Then the generating function of quartic rooted planar Eulerian orientations, counted by vertices, with a weight ω per alternating vertex is

$$\mathrm{Q}(t,\omega) = \frac{1}{(\omega+2)t^2}\left(t - (\omega+2)t^2 - \mathrm{R}(t,\omega)\right).$$

Two questions

- Relate this to the MBM-EP solution when $\omega = 0$ or $\omega = 1$
[Elvey Price & Zinn Justin 19+]

- Find a new proof of Kostov's result via our functional equations
[mbm & Elvey Price 19+]

Complements and perspectives

- record the number of faces in Eulerian orientations (equivalently, the number of oriented faces in 4-valent Eulerian orientations) — done
- record the label of a vertex in a pointed labelled map — in progress. Scaling and limit law?
- asymptotics and phase transitions when ω varies

Something to take home

Let $\mathsf{R} = t - 3t^2 + \cdots$ satisfy:

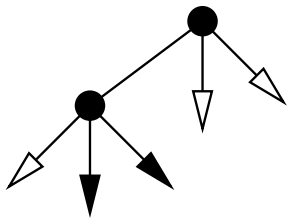
$$t = \sum_{n \geq 0} \frac{1}{n+1} \binom{2n}{n} \binom{3n}{n} \mathsf{R}^{n+1}.$$

The GF of 4-valent Eulerian orientations is

$$\mathsf{Q}(t) = \frac{1}{3t^2} \left(t - 3t^2 - \mathsf{R}(t) \right).$$

Another interpretation [mbm-Courtiel 15]

Consider ternary trees with black and white leaves. Define the charge at a node to be the number of white leaves minus the number of black leaves in the associated subtree. Then $(t - R)/t = 3t(1 + Q)$ counts (by the number of nodes) trees in which the root is the only vertex of charge 1.



Bijection?

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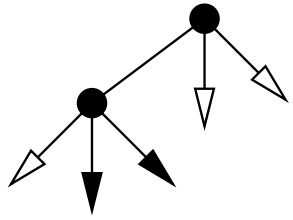

Bijection?