

# Philippe Flajolet, Founder of Analytic Combinatorics



Mireille Bousquet-Mélou, CNRS, LaBRI, U. Bordeaux 1, France

# Philippe Flajolet and Analytic Combinatorics

Conference in the memory of Philippe Flajolet  
Paris, 14-15-16 December 2011



## Confirmed Speakers

Cyril Banderier  
Frédérique Bassino  
Mireille Bousquet-Mélou  
Nicolas Broutin  
Julien Clément  
Luc Deyroze  
Michael Drmota  
Philippe Dumas  
Mordecai Golin  
Hsien-Kuei Hwang  
Philippe Jacquet  
Jérémy Lumbroso  
Guy Louchard  
Conrado Martinez  
Basile Morcrette  
Cyril Nicaud  
Pierre Nicodème  
Marc Noy  
Andrew Odlyzko  
Nicolas Pouyanne  
Helmut Prodinger  
Bruno Salvy  
Robert Sedgewick  
Michele Soria  
Jean-Marc Steyaert  
Wojciech Szpankowski  
Brigitte Vallée  
Xavier Viennot  
Alfredo Viola  
Marc Ward  
Paul Zimmermann

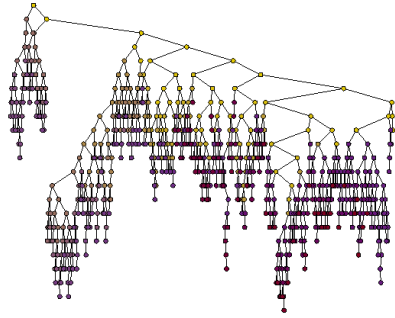
Amphi 25  
4, place Jussieu  
75005 Paris  
France

Contact: [pfac@inria.fr](mailto:pfac@inria.fr)  
<http://algo.inria.fr/pfac>



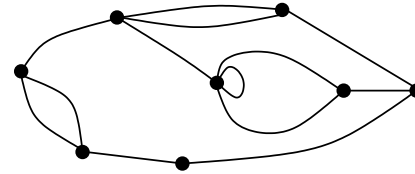
<http://algo.inria.fr/pfac/PFAC/PFAC.html>

# Philippe's favourite topic: discrete random objects



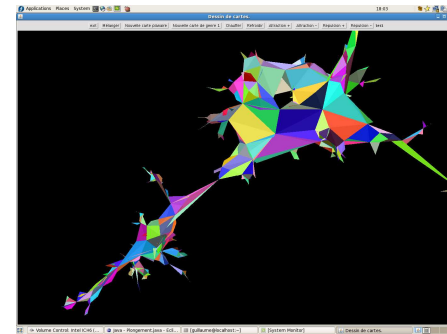
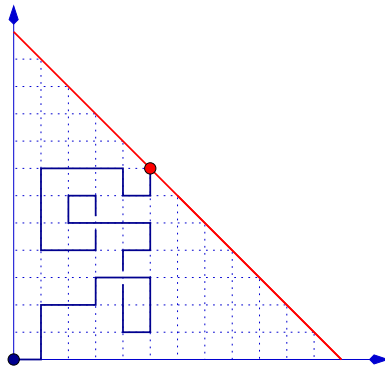
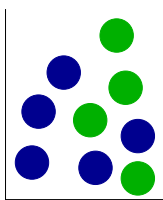
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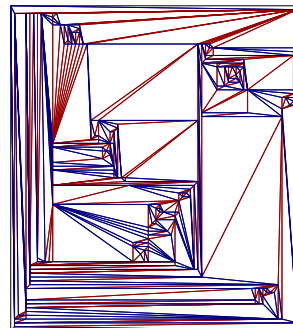


Exact and asymptotic enumeration

Random generation



Analysis of algorithms



Asymptotic properties, limit laws

## The ALEA and AofA communities

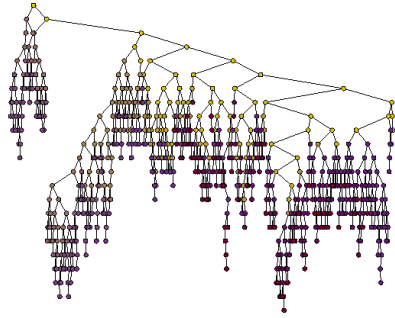


Combinatorics, probability theory, theoretical computer science, statistical physics

★ **ALEA**: randomness (in French); dice (in Latin)

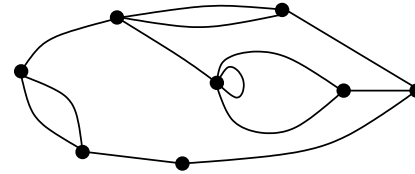
★ **AofA**: Analysis of Algorithms

# Philippe's favourite topic: discrete random objects



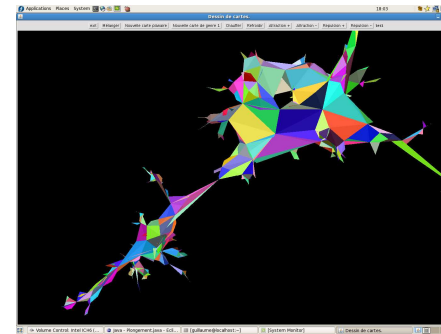
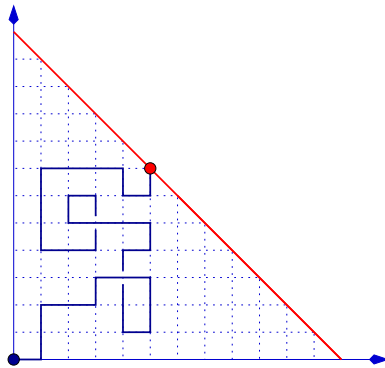
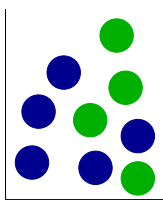
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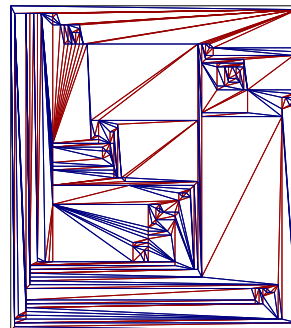


Exact and asymptotic enumeration

Random generation



Analysis of algorithms



Asymptotic properties, limit laws

# Objectives



specific problems

generic approaches

# Objectives



specific problems

generic approaches

Generic methods  
Uniform approaches

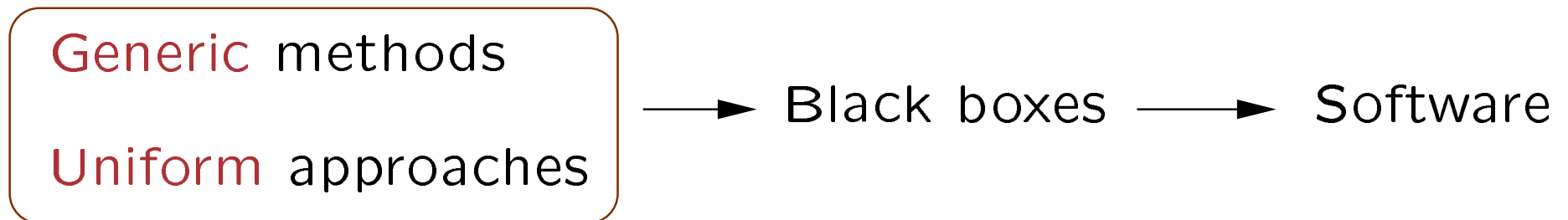
→ Black boxes → Software

# Objectives



specific problems

generic approaches



- Objects with some (recursive) structure (no general SAWs!)
- Extremely useful to many people, large impact



# Objectives



specific problems

generic approaches

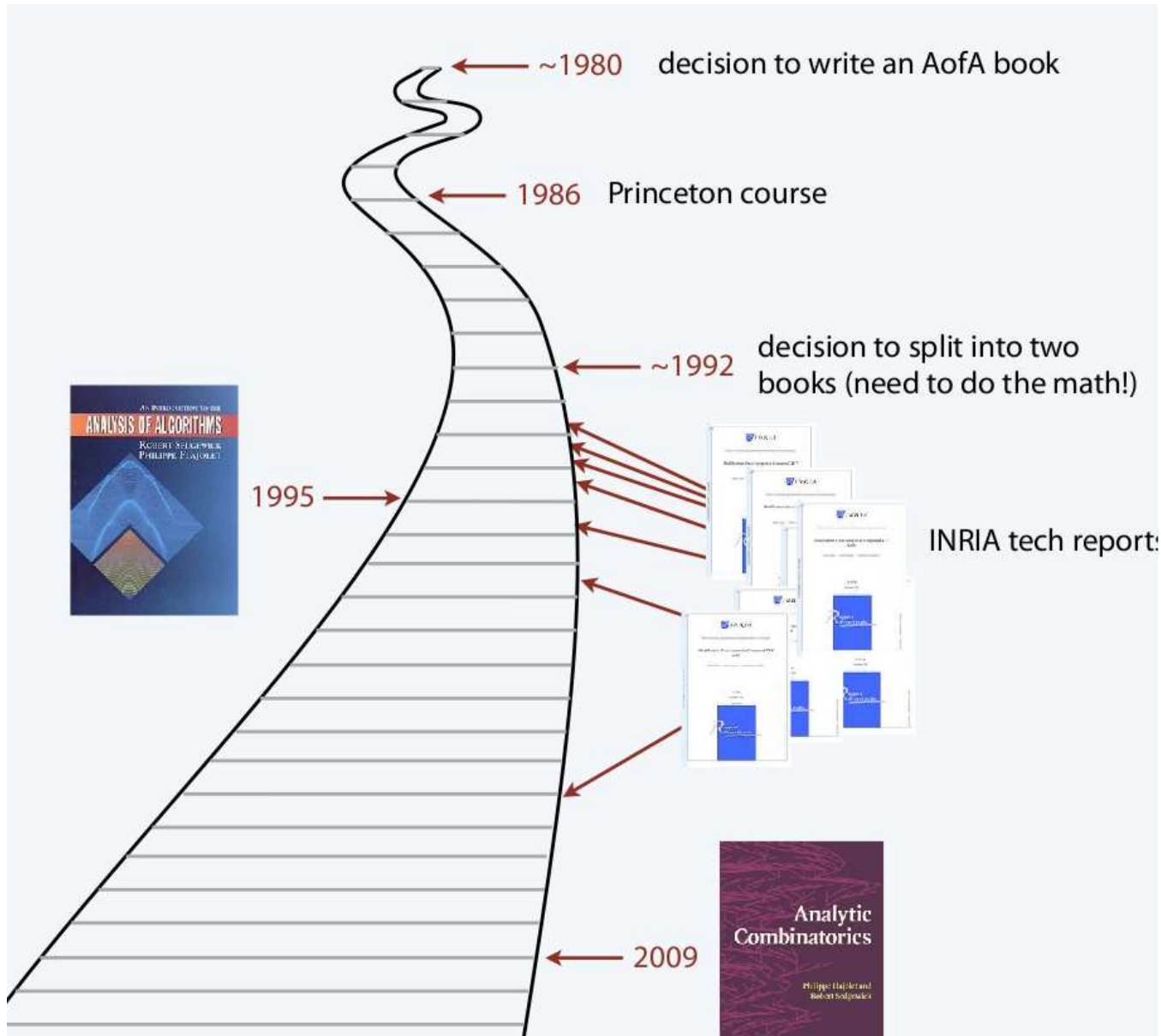
Generic methods

Uniform approaches

→ Black boxes → Software

- Objects with some (recursive) structure (no general SAWs!)
- Extremely useful too many people, large impact

# Analysis of Algorithms and Analytic Combinatorics [Ph. Flajolet, R. Sedgewick]



# Analytic Combinatorics: Contents

## Part A. Symbolic methods

- Combinatorial structures and ordinary generating functions
- Labelled structures and exponential generating functions
- Combinatorial parameters and multivariate generating functions

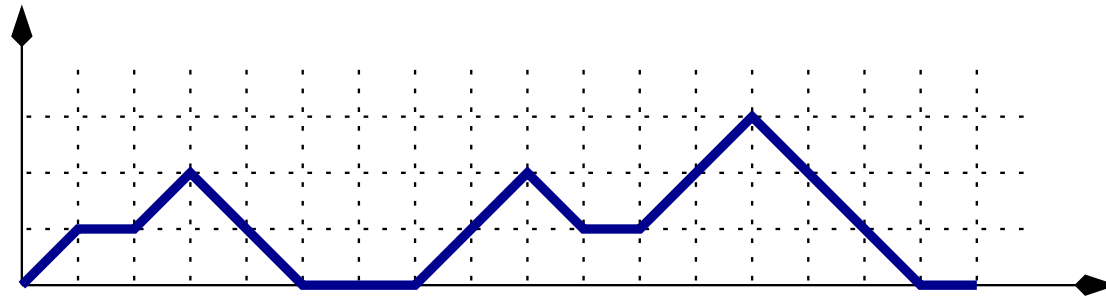
## Part B. Complex asymptotics

- Complex analysis, rational and meromorphic asymptotics
- Applications of rational and meromorphic asymptotics
- Singularity analysis of generating functions
- Applications of singularity analysis
- Saddle-point asymptotics

## Part C. Random structures

- Multivariate asymptotics and limit laws

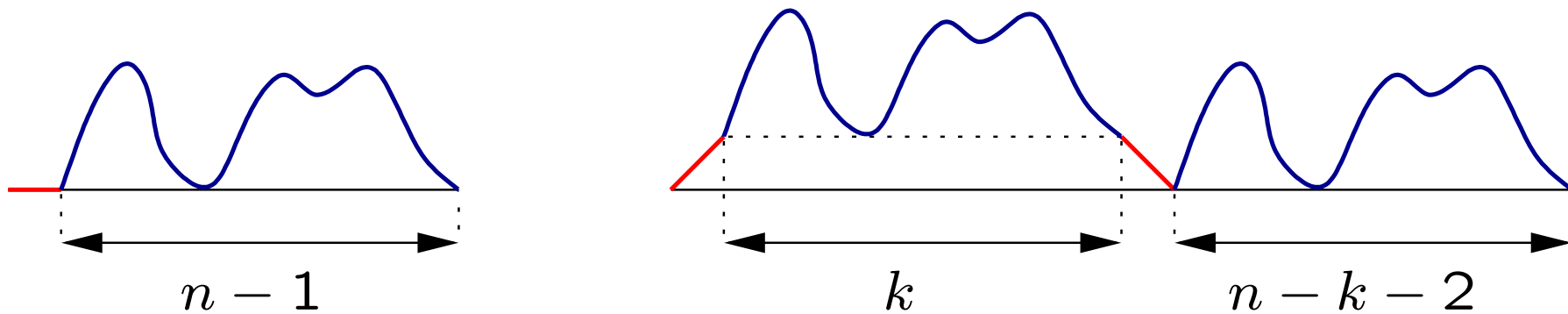
## An example: Enumeration of $\{0, \pm 1\}$ -excursions in the 1970s



A recurrence relation. Let  $a(n)$  be the number of excursions of length  $n$ . Then:

$$a(n) = a(n-1) + \sum_{k=0}^{n-2} a(k)a(n-k-2)$$

with  $a(0) = 1$ .



## An example: Enumeration of $\{0, \pm 1\}$ -excursions in the 1970s

Recurrence relation

$$\begin{cases} a(n) = a(n-1) + \sum_{k=0}^{n-2} a(k)a(n-k-2) \\ a(0) = 1 \end{cases}$$

Generating function

$$A(z) = \sum_{n \geq 0} a(n)z^n$$

Functional equation

$$A(z) = 1 + zA(z) + z^2A(z)^2$$

Expression of the GF

$$A(z) = \frac{1 - z - \sqrt{(1+z)(1-3z)}}{2z^2}$$

Expression of the numbers

$$a(n) = \sum_{k=0}^{n/2} \frac{n!}{k!(k+1)!(n-2k)!}$$

Asymptotic study of the sum

$$a(n) \sim \frac{3\sqrt{3}}{2\sqrt{\pi}} 3^n n^{-3/2}$$

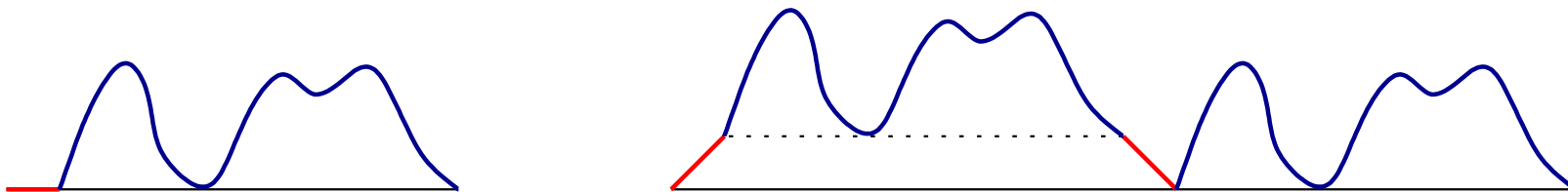
## An example: Enumeration of $\{0, \pm 1\}$ -excursions nowadays

Generating function  $A(z) = \sum_{n \geq 0} a(n)z^n$

Functional equation  $A(z) = 1 + zA(z) + z^2A(z)^2$

Expression of the GF  $A(z) = \frac{1 - z - \sqrt{(1+z)(1-3z)}}{2z^2}$

Singularity analysis  $a(n) \sim \frac{3\sqrt{3}}{2\sqrt{\pi}} 3^n n^{-3/2} \left(1 - \frac{39}{16n} + \frac{2665}{512n^2} + O(1/n^3)\right)$



## Let a computer do the job!

- The Algolib library [Maple]

- The package `comstruct`: enumeration and random generation of combinatorial structures

- The `equivalent` function: asymptotics of the coefficients from the generating function

<http://algo.inria.fr/libraries/>

## **Part A. Symbolic methods**



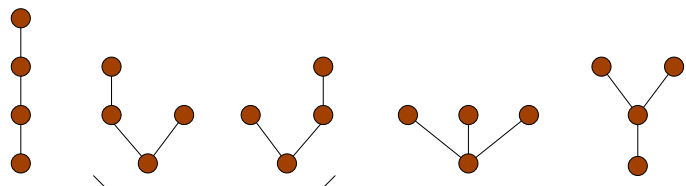
## Write directly equations from the recursive description of the objects

Construction	Generating function
Union $\mathcal{A} = \mathcal{B} \sqcup \mathcal{C}$	$A(t) = B(t) + C(t)$
Product $\mathcal{A} = \mathcal{B} \times \mathcal{C}$  $ (\beta, \gamma)  =  \beta  +  \gamma $	$A(t) = B(t) \cdot C(t)$
Sequence $\mathcal{A} = \text{Seq}(\mathcal{B})$	$A(t) = \frac{1}{1-B(t)}$
Set $\mathcal{A} = \text{Set}(\mathcal{B})$	$A(t) = \exp\left(B(z) - B(z^2)/2 + B(z^3)/3 - \dots\right)$
Multiset $\mathcal{A} = \text{MSet}(\mathcal{B})$	$A(t) = \exp\left(B(z) + B(z^2)/2 + B(z^3)/3 + \dots\right)$
Cycle $\mathcal{A} = \text{Cyc}(\mathcal{B})$	$A(t) = \log \frac{1}{1-B(z)} + \frac{1}{2} \log \frac{1}{1-B(z^2)} + \dots$

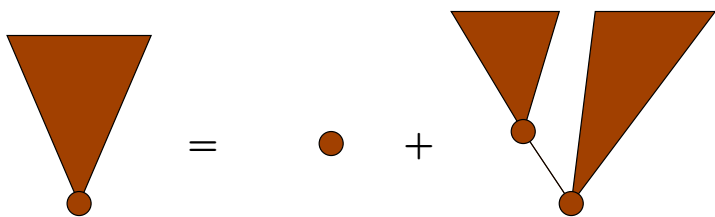
**Constructible objects**

# An example: rooted trees

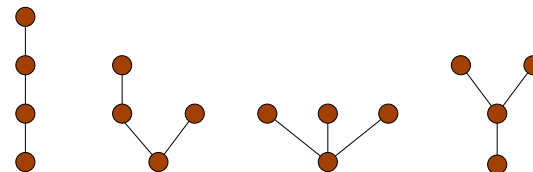
Ordered (plane)



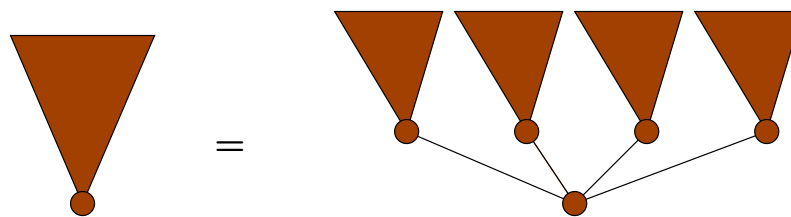
$$T(z) = z + T(z)^2$$



Non-ordered



$$T(z) = z \exp \left( T(z) + T(z^2)/2 + T(z^3)/3 + \dots \right)$$



## An idea in the spirit of times

Flajolet: **symbolic combinatorics**

Bender-Goldman: prefabs (1971)

Foata: composé partitionnel (1978)

Formal languages in theoretical computer science

Joyal: **theory of species** (1980)

## **Part B. Analytic methods**

# Singularity analysis of generating functions

[Flajolet, Odlyzko 90]

A general correspondence between

the singular expansion of a series  $A(z)$  near its dominant singularities

and

the asymptotic expansion of the  $n$ th coefficient  $a(n)$  of this series

**Example:** If the dominant singularity has modulus  $\rho$ , then the coefficients satisfy

$$\limsup a(n)^{1/n} = 1/\rho.$$

## Singularity analysis of generating functions

- **Asymptotic expansions** for the coefficients of some “simple” series of a standard scale, for instance

$$(1 - z)^{-\alpha} \left( \frac{1}{z} \log \frac{1}{1 - z} \right)^\beta$$

(Example:  $n^{\alpha-1}/\Gamma(\alpha)$  when  $\beta = 0$ )

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- **Transfer theorem:** under certain hypotheses, if  $A(z)$  has a unique dominant singularity at 1, and, as  $z \rightarrow 1$ ,

$$A(z) = S(z) + O(R(z)),$$

then, with obvious notation ( $a(n) = [z^n]A(z)$ , etc.),

$$a(n) = s(n) + O(r(n)).$$

Proof: Cauchy's formula and Hankel contours

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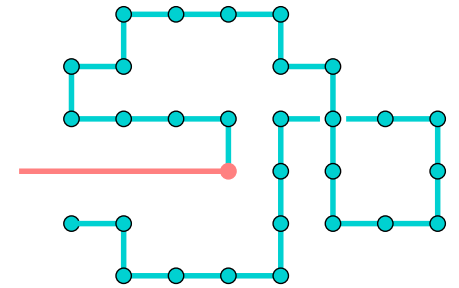
Proof: Cauchy's formula and Hankel contours

- Several singularities: add up the contributions



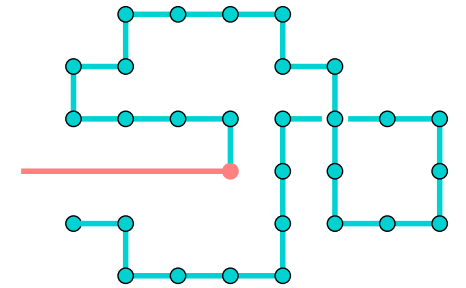
## Example: walks on the slit plane

$$A(z) = \frac{(1 + \sqrt{1 + 4z})^{1/2} (1 + \sqrt{1 - 4z})^{1/2}}{2(1 - 4z)^{3/4}}$$



## Example: walks on the slit plane

$$A(z) = \frac{(1 + \sqrt{1 + 4z})^{1/2} (1 + \sqrt{1 - 4z})^{1/2}}{2(1 - 4z)^{3/4}}$$



- As  $z \rightarrow \pm 1/4$ ,

$$A(z) = A_{\pm}(z) + O\left((1 \pm 4z)^{3/4}\right)$$

where

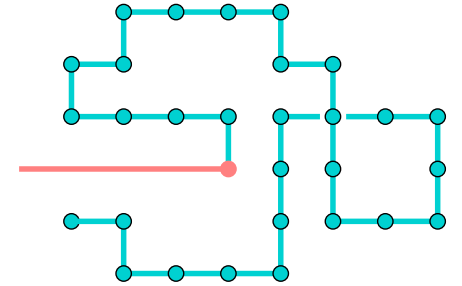
$$A_+(z) := \frac{\sqrt{1 + \sqrt{2}}}{2(1 - 4z)^{3/4}} + \frac{\sqrt{1 + \sqrt{2}}}{4(1 - 4z)^{1/4}} - \frac{(2\sqrt{2} + 1)(1 - 4z)^{1/4}}{16\sqrt{1 + \sqrt{2}}}$$

and

$$A_-(z) := \frac{\sqrt{1 + \sqrt{2}}}{2^{7/4}} + \frac{\sqrt{1 + \sqrt{2}}(1 + 4z)^{1/2}}{2^{11/4}}.$$

## Example: walks on the slit plane

$$A(z) = \frac{(1 + \sqrt{1 + 4z})^{1/2}(1 + \sqrt{1 - 4z})^{1/2}}{2(1 - 4z)^{3/4}}$$



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and

$$A_-(z) := \frac{\sqrt{1 + \sqrt{2}}}{2^{7/4}} + \frac{\sqrt{1 + \sqrt{2}}(1 + 4z)^{1/2}}{2^{11/4}}.$$

- Consequently,

$$\begin{aligned} [z^n]A(z) &= [z^n]A_1(z) + [z^n]A_2(z) + O(4^n n^{-7/4}) \\ &= \frac{\sqrt{1 + \sqrt{2}}}{2\Gamma(3/4)} \frac{4^n}{n^{1/4}} \left( 1 + \frac{\sqrt{2}\Gamma(3/4)^2}{4\pi\sqrt{n}} - \frac{\sqrt{2}}{32n} - \frac{\Gamma(3/4)(-1)^n}{2^{11/4}\sqrt{\pi}n^{5/4}} + O\left(\frac{4^n}{n^{7/4}}\right) \right) \end{aligned}$$

## Bumps in the road

- When there is a functional equation for  $A(z)$ , but no explicit expression (implicit functions)
- Checking the assumptions may be tricky
- When the function is entire ( $\rightarrow$  saddle-point methods)
- When there is no singular behaviour of the form

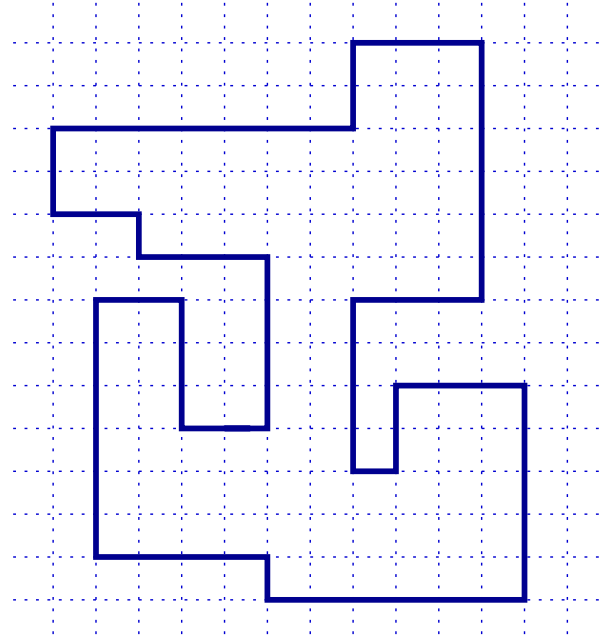
$$A(z) = S(z) + O(R(z)),$$

with  $S$  and  $R$  “simple”.

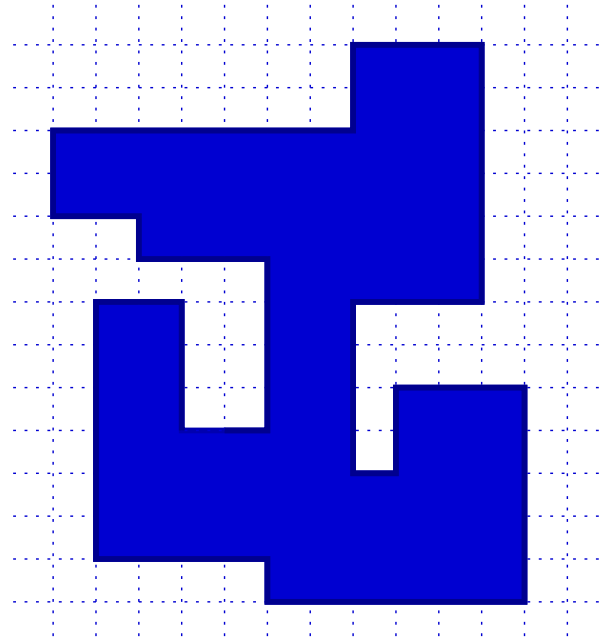


An example: 3-sided prudent self-avoiding polygons  
[Beaton, Flajolet, Guttmann 11]

# Self-avoiding polygons



## Self-avoiding polygons



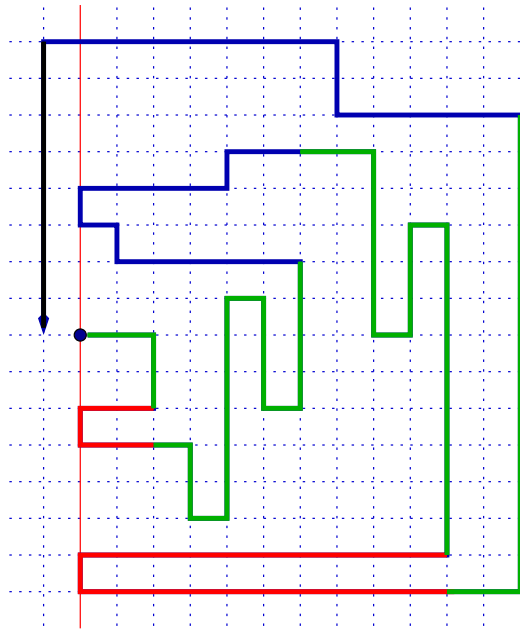
- Awfully hard to count
- Conjectured asymptotics: for polygons of area  $n$ ,

$$p_n \sim \kappa \mu^n n^0,$$

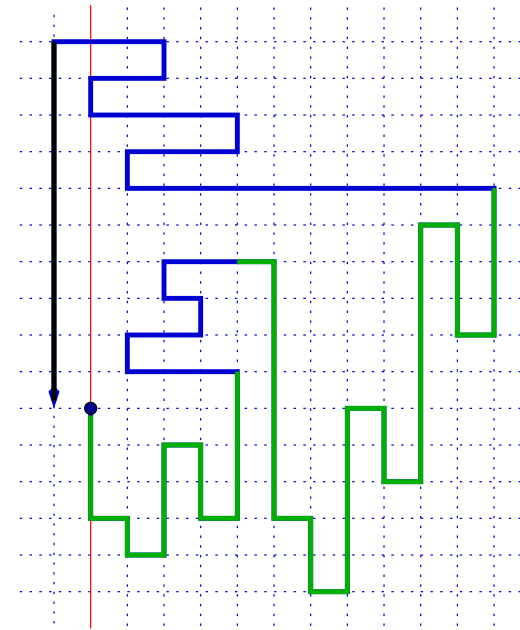
where the exponent depends only on the dimension

## An easier family: 3-sided prudent polygons

Prudent walks: a step never points towards a vertex it has already visited



Prudent polygon



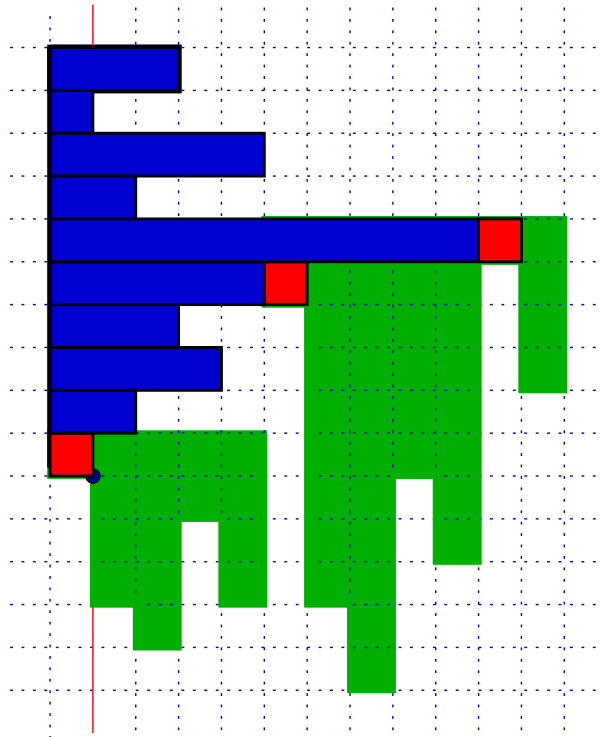
Three-sided prudent polygon

[Turban-Debierre 86], [Préa 97], [Santra-Seitz-Klein 01], [Duchi 05], [Dettridge, Guttmann, Jensen 07], [mbm 08], [Beffara, Friedli, Velenik 10], [Schwerdtfeger 08], [Beaton, Flajolet, Guttmann 11]...

## An easier family: 3-sided prudent polygons

Three-sided prudent polygons decompose into bars, bargraphs, and separating cells [...] and their area generating function is

$$P(z) = \frac{2z(3 - 10z + 9z^2 - z^3)}{(1 - 2z)^2(1 - z)} + \frac{2z^3(1 - z)^2}{(1 - 2z)^2} \sum_{m \geq 1} \frac{(-1)^{m+1} z^{2m}}{(1 - 2z)^m (1 - z - z^{m+1})} \prod_{k=1}^{m-1} \frac{1 - z - z^k + z^{k+1} - z^{k+2}}{1 - z - z^{k+1}}$$





## Asymptotics

- The series:

$$P(z) = \frac{2z(3 - 10z + 9z^2 - z^3)}{(1 - 2z)^2(1 - z)} + \frac{2z^3(1 - z)^2}{(1 - 2z)^2} \sum_{m \geq 1} \frac{(-1)^{m+1} z^{2m}}{(1 - 2z)^m (1 - z - z^{m+1})} \prod_{k=1}^{m-1} \frac{1 - z - z^k + z^{k+1} - z^{k+2}}{1 - z - z^{k+1}}$$

- A beautiful singularity analysis yields a very unusual asymptotic behaviour in the study of lattice models:

$$P_n = \kappa(\log n) 2^n n^\gamma + O(\log n 2^n n^{\gamma-1})$$

where the exponent  $\gamma$  is irrational

$$\gamma = \log_2 3$$

and  $\kappa(x)$  is a periodic function of  $x$ .

[Beaton, Flajolet, Guttmann 11]

## **Part C. Random structures and limit laws**

## The study of additional statistics

$$\begin{aligned} \text{size} : \mathcal{A} &\rightarrow \mathbb{N} \\ a &\mapsto |a| \end{aligned}$$

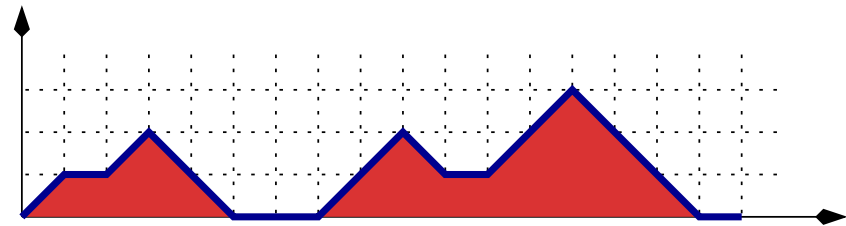
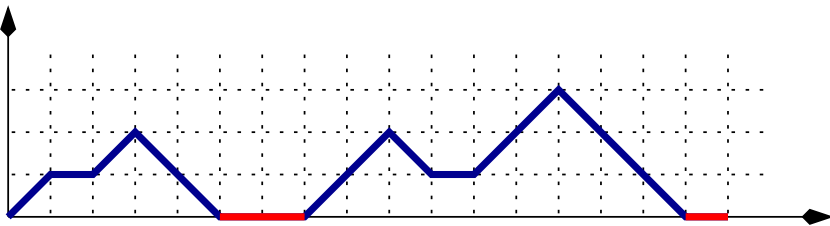
$$\begin{aligned} s : \mathcal{A} &\rightarrow \mathbb{Z} \\ a &\mapsto s(a) \end{aligned}$$

When objects of size  $n$  are taken uniformly at random, the statistic  $s$  becomes a **random variable**  $S_n$ :

$$\mathbb{P}(S_n = k) = \frac{a(n, k)}{a(n)}$$

where  $a(n, k)$  is the number of objects of size  $n$  for which the additional statistic  $s$  equals  $k$ .

**Examples:** the number of contacts in an excursion, or the area under it



## Bivariate generating functions

- Let  $a(n, k)$  be the number of objects of size  $n$  for which the statistic  $s$  is  $k$ , and let

$$A(z, u) = \sum_{n,k} a(n, k) z^n u^k.$$

## Bivariate generating functions

- Let  $a(n, k)$  be the number of objects of size  $n$  for which the statistic  $s$  is  $k$ , and let

$$A(z, u) = \sum_{n, k} a(n, k) z^n u^k.$$

- The  $j^{\text{th}}$  moment of  $S_n$  is

$$\mathbb{E}(S_n^j) = \frac{[z^n] \left( u \frac{\partial}{\partial u} \right)^j A(z, u) \Big|_{u=1}}{[z^n] A(z, 1)}$$

**Idea:** Study  $[z^n] \left( u \frac{\partial}{\partial u} \right)^j A(z, u) \Big|_{u=1}$  for a generic  $j$  using the analytic techniques of Part B. Combine the results with a theorem on the convergence of moments to obtain a limit law for  $S_n$ .

## Bivariate generating functions

- Let  $a(n, k)$  be the number of objects of size  $n$  for which the statistic  $s$  is  $k$ , and let

$$A(z, u) = \sum_{n, k} a(n, k) z^n u^k.$$

- The probability generating function of the random variable  $S_n$  is

$$p_n(u) := \frac{[z^n]A(z, u)}{[z^n]A(z, 1)}$$

**Idea:** Study  $[z^n]A(z, u)$  with  $u$  as a real parameter using the analytic techniques of Part B. Combine the results with continuity theorems on probability generating functions to obtain a limit law for  $S_n$ .

## Example: contacts in a $\{0, \pm 1\}$ -excursion

- GF of excursions by length ( $z$ ) and contacts ( $u$ ):

$$A(z, u) = \frac{1}{1 + z(1 - 2u) + \sqrt{(1 + z)(1 - 3z)}}$$

- Singularity analysis: For  $u \in (0, 2)$ ,

$$[z^n]A(z, u) \sim \frac{3^{3/2}}{2\sqrt{\pi}(2-u)^2} 3^n n^{-3/2}$$

so that

$$p_n(u) := \frac{[z^n]A(z, u)}{[z^n]A(z, 1)} \rightarrow \frac{1}{(2-u)^2} = \sum_{k \geq 0} \frac{k+1}{2^{k+2}} u^k.$$

$\Rightarrow$  Convergence to a discrete limit law (negative binomial)

## Example: area under a $\{0, \pm 1\}$ -excursion

- GF of excursions by length ( $z$ ) and area ( $u$ ):

$$A(z, u) = 1 + zA(z, u) + z^2uA(z, u)A(zu, u)$$

- Repeated differentiations of the equation

$\Rightarrow \frac{\partial^j A}{\partial u^j}(z, 1)$  is always quadratic and

$$\left(u \frac{\partial}{\partial u}\right)^j A(z, u) \Big|_{u=1} = \sum_n z^n a(n) \mathbb{E}(S_n^j) \sim \frac{c_j}{(1-3z)^{(3j-1)/2}}$$

with

$$c_0 = -3\sqrt{3}, \quad c_j = \frac{j(3j-4)}{4\sqrt{3}} c_{j-1} + \frac{1}{6\sqrt{3}} \sum_{i=1}^{j-1} \binom{j}{i} c_i c_{j-i}$$

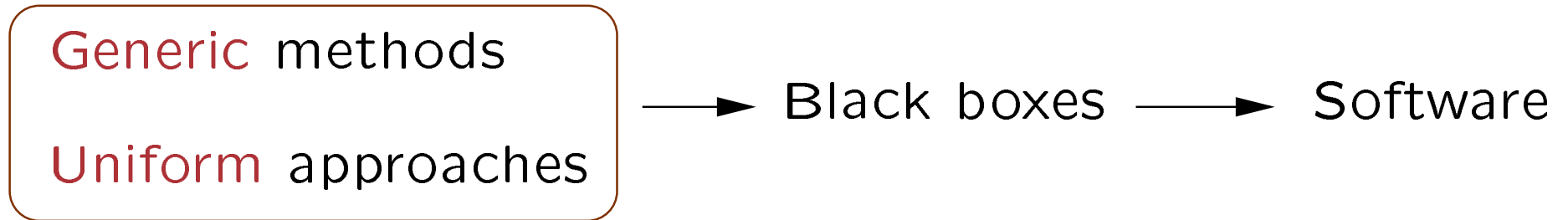
- Singularity analysis

$$\mathbb{E} \left( \left( \frac{S_n}{n^{3/2}} \right)^j \right) \sim \frac{c_j}{c_0} \frac{\Gamma(-1/2)}{\Gamma((3j-1)/2)}$$

Moments of the Airy distribution  $\Rightarrow$  Convergence in law



## Objectives



specific problems

generic approaches

## The quasi-power theorem [Hwang 98]

- Assume that, uniformly in a neighborhood of  $u = 1$ ,

$$p_n(u) := \frac{[z^n]A(z, u)}{[z^n]A(z, 1)} = A(u)B(u)^{\beta_n} (1 + O(1/\kappa_n))$$

where  $A(u)$  and  $B(u)$  are analytic at  $u = 1$ ,  $A(1) = B(1) = 1$  and  $\beta_n, \kappa_n \rightarrow \infty$ . Assume moreover

$$v_B := B''(1) + B'(1) - B'(1)^2 \neq 0.$$

- Then

$$\mathbb{E}(S_n) = \beta_n m_B + m_A + O(1/\kappa_n), \quad \mathbb{V}(S_n) = \beta_n v_B + v_A + O(1/\kappa_n),$$

where  $m_A = A'(1)$ , and the normalized random variable

$$\frac{S_n - \mathbb{E}(S_n)}{\sqrt{\mathbb{V}(S_n)}}$$

converges to a standard normal distribution.

## One of Philippe's tables

$A(z, u)$ for $u \sim 1$	type of law	methods and schemas
sing. & exp. fixed	discrete	subcritical composition subcritical sequence, set...
sing. moved, exp. fixed	Gaussian $(n, n)$	supercritical composition meromorphic perturbation (rational functions) sing. analysis perturbation (algebraic, implicit functions)
sing. fixed, exp. moves	Gaussian $(\log n, \log n)$	(exp-log structures) (diff. equations)
sing. & exp. move	Gaussian	[Gao-Richmond]
essential singularity	often Gaussian	saddle-point perturbation
discontinuous type	non-Gaussian stable	various cases critical composition

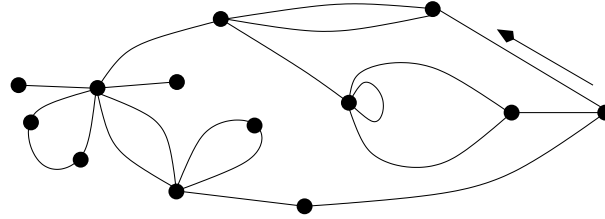
**Fig. IX.4.** A rough typology of bivariate generating functions  $A(z, u)$  and limit laws studied in this chapter, based on the way singularities and exponents evolve for  $u \sim 1$ .

## One of Philippe's tables

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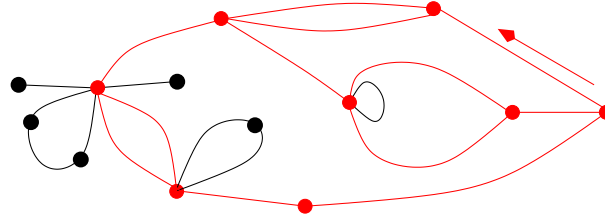
**Fig. IX.4.** A rough typology of bivariate generating functions  $A(z, u)$  and limit laws studied in this chapter, based on the way singularities and exponents evolve for  $u \sim 1$ .

## The core of a random planar map

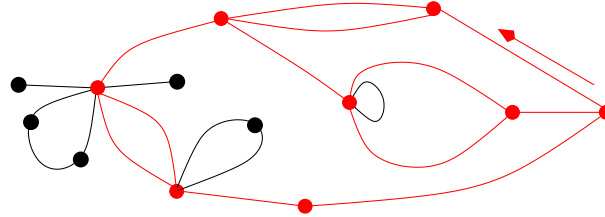


[Banderier, Flajolet, Schaeffer, Soria 01]

# The core of a random planar map



## The core of a random planar map



- **Generating functions:** Let  $M(z)$  be the generating function of planar maps:

$$M(z) = \frac{(1 - 12z)^{3/2} - 1 + 18z}{54z^2}.$$

Define  $C(z)$  by

$$M(z, 1) = C(zM(z, 1)^2)$$

Then the generating function of planar maps counted by edges ( $z$ ) and the size of the core ( $u$ ) is

$$M(z, u) = C(zuM(z, 1)^2).$$

- **Critical composition**  $\Rightarrow$  non-gaussian, bimodal limit law

## Bimodal limit law

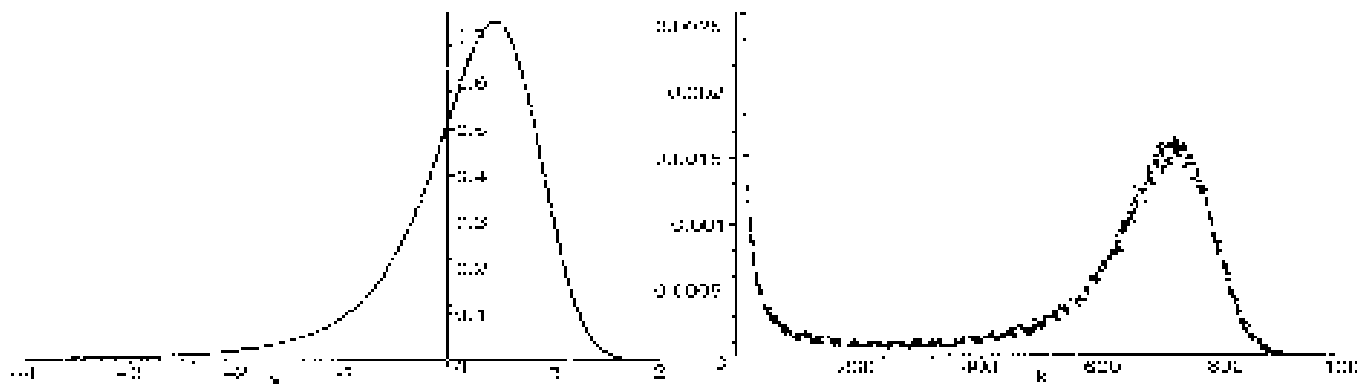
- Left region,  $k$  fixed: **discrete limit law** (total probability  $1/3$ ):

$$\mathbb{P}(S_n = k) \rightarrow p(k) \text{ as } n \rightarrow \infty$$

- Central region,  $k = 2n/3 + xn^{2/3}$ : **continuous limit law** (total probability  $2/3$ ; local version)

$$\mathbb{P}(S_n = k) \sim \kappa n^{-2/3} \mathcal{A}(3x/2)$$

where  $\mathcal{A}$  is explicit, related to the Airy function.



**Figure IX.21.** Left: The standard "Airy map distribution". Right: Observed frequencies of core-sizes  $k \in [20; 1000]$  in 50,000 random maps of size 2,000, showing the bimodal character of the distribution.



## **Part D. Uniform random generation**

## Recursive approaches for constructible objects

- Exact size: an excursion of size 1000 in 7 seconds  
[Flajolet, Zimmermann, Van Cutsem, 94]

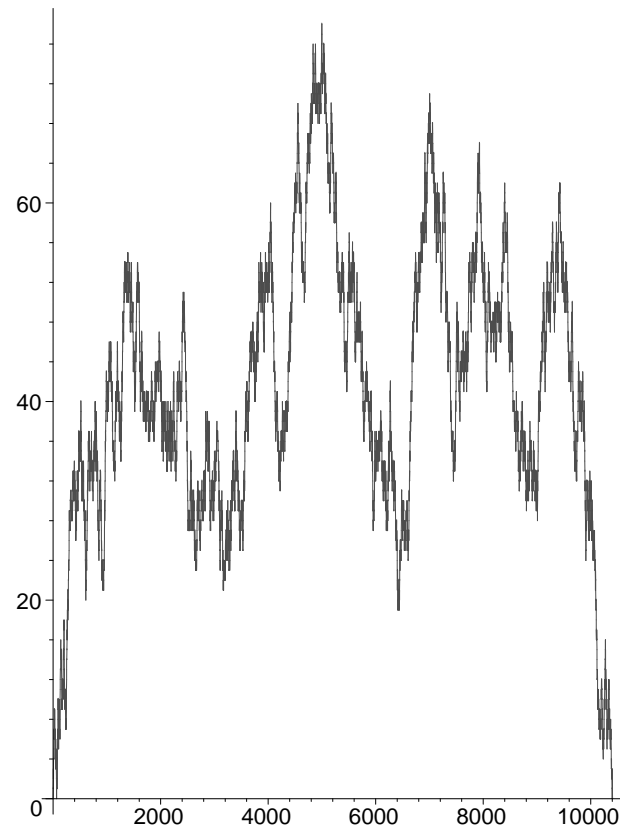
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[Flajolet, Zimmermann, Van Cutsem, 94]

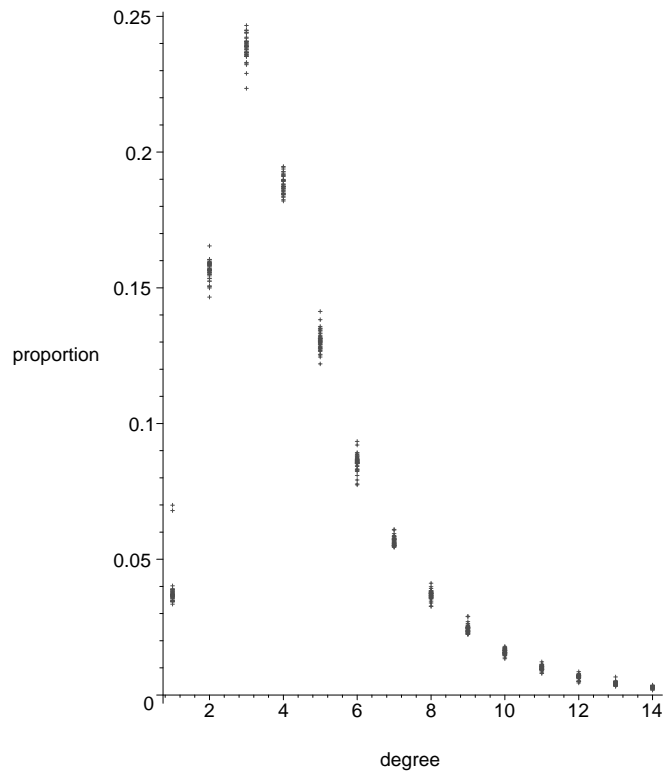
- Approximate size (“Boltzmann generation”): an excursion of size  $> 10000$  in 3 seconds

[Duchon, Flajolet, Louchard, Schaeffer 04]

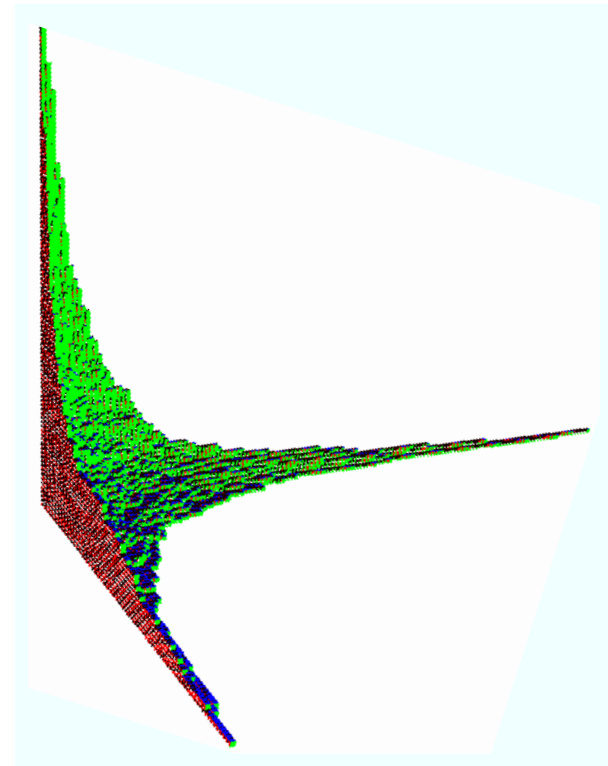


## Applications of Boltzmann sampling

- Uniform random sampling of **planar graphs** in linear time (approximate size) [Fusy 09] (100,000 vertices in a few seconds)
- **Plane partitions** in time  $O(n \log n^3)$  (approximate size) and  $O(n^{4/3})$  (exact size) [Bodini, Fusy, Pivoteau 10] (10,000 cubes in a few seconds)



Vertex degrees in random planar graphs



Plane partition of size  $\sim 15,000$

# Part E. Software

## Let a computer do the job!

- The Algolib library [Maple]

- The **equivalent** function: asymptotics of the coefficients from the generating function [Salvy]

- The package **combstruct**: enumeration and random generation of combinatorial structures

- **gfun**: guess and handle holonomic generating functions [Salvy, Zimmermann, Murray]

<http://algo.inria.fr/libraries/>

Please ask questions...

but read the book!

