# Lattice walks confined to the positive octant

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# The question

Let $\mathcal{S} \subset \mathbb{Z}^3$ be a finite set of steps in the 3D space.

Example: $\mathcal{S} = \{00\bar{1}, 0\bar{1}0, \bar{1}00, 111\}$

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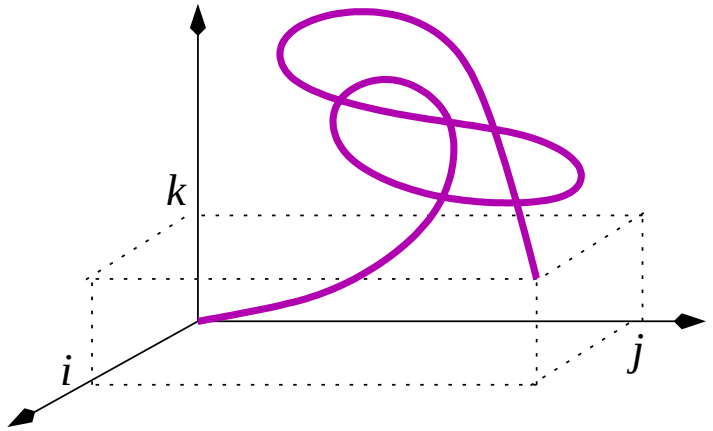
Consider walks that start from $(0,0,0)$, take their steps in $\mathcal{S}$, and are confined to the positive octant $\{(i,j,k) : i \geq 0, j \geq 0, k \geq 0\}$.



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## Questions

- Determine $o(n)$, the number of such walks that have length $n$
- or $o(i,j,k;n)$, the number of such walks that have length $n$ and end at position $(i,j,k)$
- or the associated 4-variable generating function:

$$O(x,y,z;t) = \sum_{i,j,k,n \geq 0} o(i,j,k;n) x^i y^j z^k t^n$$

- or the nature of this generating function.

# A hierarchy of formal power series

• The formal power series $A(t)$ is rational if it can be written

$$A(t) = \frac{P(t)}{Q(t)}$$

where $P(t)$ and $Q(t)$ are polynomials in $t$.

• The formal power series $A(t)$ is algebraic (over $\mathbb{Q}(t)$) if it satisfies a (non-trivial) polynomial equation:

$$P(t, A(t)) = 0.$$

• The formal power series $A(t)$ is D-finite if it satisfies a (non-trivial) linear differential equation:

$$P_k(t)A^{(k)}(t) + \cdots + P_0(t)A(t) = 0.$$

+ extension to several variables + closure properties

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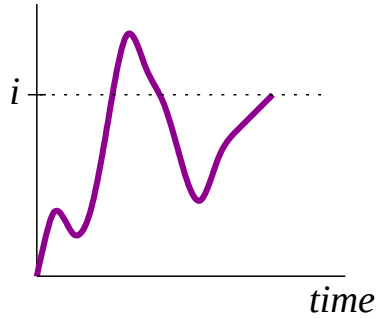
# Lattice paths confined to convex cones

- **Dimension 1**: walks confined to a half-line

  The generating function $H(x;t)$ is always algebraic

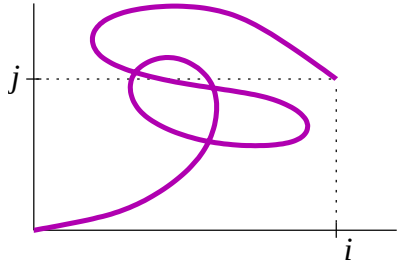



[Gessel 80], [Labelle-Yeh 90], [mbm-Petkovšek 00], [Duchon 00], [Banderier-Flajolet 02]

- **Dimension 2**: walks confined to the positive quadrant

  The generating function $Q(x,y;t)$ is sometimes algebraic, sometimes D-finite, sometimes non-D-finite.

  Complete classification for walks with "small" steps: $\mathcal{S} \subset \{\bar{1},0,1\}^2$

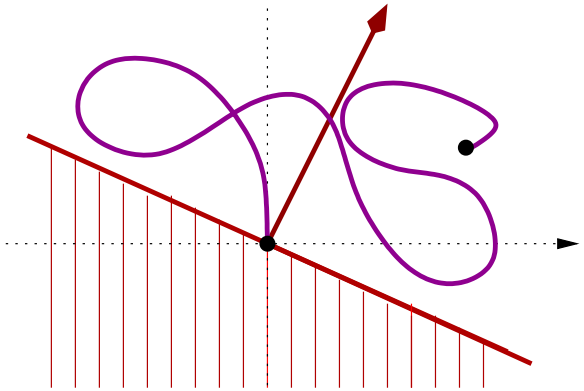



Bernardi, Bostan, MBM, Cori, Denisov, Dulucq, Fayolle, Gessel, Gouyou-Beauchamps, Guy, Janse van Rensburg, Johnson, Kauers, Koutschan, Kratten-thaler, Kurkova, Kreweras, Melczer, Mishna, Niederhausen, Petkovšek, Prell-berg, Raschel, Rechnitzer, Sagan, Salvy, Viennot, Wachtel, Wilf, Yeats, Zeil-berger...

# Why just *positive* cones?

In 2D, a convex rational cone is either:

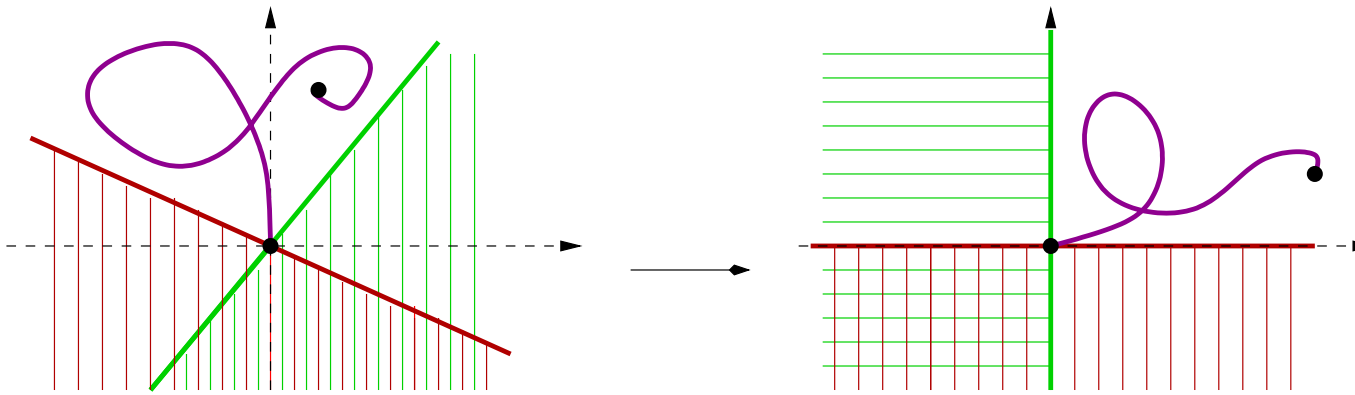
- a half-plane



$\Rightarrow$ equivalent to a (weighted) model of walks on a half-line

$\Rightarrow$ algebraic

- or a linear deformation of the quadrant



$\Rightarrow$ equivalent to a quadrant model

# 3D walks in an octant — Small steps

Let $\mathcal{S} \subset \{\bar{1}, 0, 1\}^3 \setminus \{(0,0,0)\}$ be a set of small steps in the 3D space.

## Questions

- Determine $o(n)$, or $o(i,j,k;n)$ (number of walks)
- or the associated generating function:

$$O(x,y,z;t) = \sum_{i,j,k,n \geq 0} o(i,j,k;n) x^i y^j z^k t^n.$$

There are "only" $2^{26}$ such problems!

~~Specific question~~
~~Ad hoc solution~~ → **Systematic approach**

# The number of interesting distinct models

Remove

- models in which all steps are non-negative (rational GF)
- models in which one positivity condition implies the other two ($\sim$ walks in a half-space $\Rightarrow$ algebraic GF)
- models in which one step is never used

and declare equivalent models that only differ by a permutation of the coordinates.

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Proposition: one is left with $11\,074\,225 \simeq 2^{23.4\cdots}$ distinct models.

With at most 6 steps:

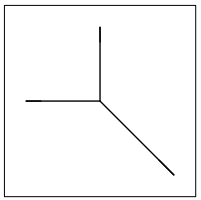
$$73\,u^3 + 979\,u^4 + 6425\,u^5 + 28071\,u^6 + O(u^7)$$

We focus on the 35 548 models with at most 6 steps.

# I. Some useful tools

## I.1. A step-by-step construction, and the associated functional equation

# A functional equation: example in 2D



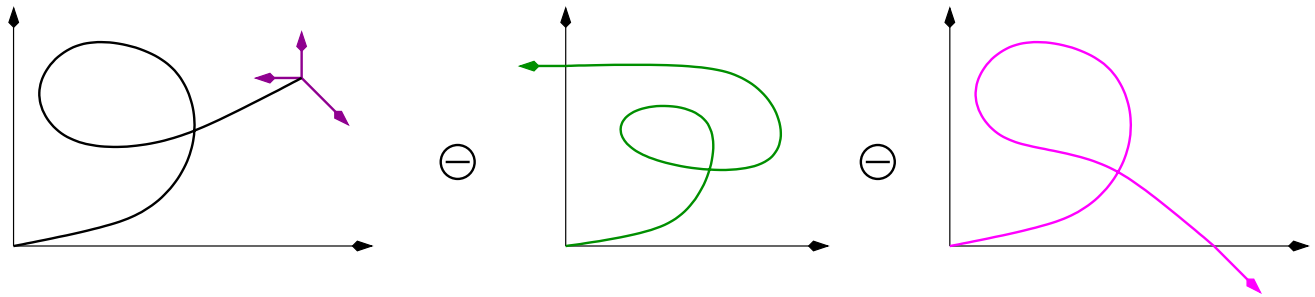
Example: $\mathcal{S} = \{\mathsf{N}, \mathsf{W}, \mathsf{SE}\}$

Generating function:

$$Q(x, y; t) = \sum_{i,j,n \geq 0} q(i, j; n) x^i y^j t^n$$

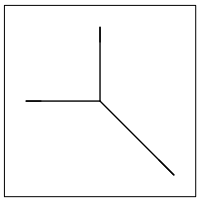
Functional equation:

$$Q(x, y; t) \equiv Q(x, y) = 1 + tyQ(x, y) + t\,\frac{Q(x, y) - Q(0, y)}{x} + tx\,\frac{Q(x, y) - Q(x, 0)}{y}$$



A linear discrete partial differential equation

# A functional equation: example in 2D



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or, with $\bar{x} = 1/x$ and $\bar{y} = 1/y$,

$$\Big(1 - t(y + \bar{x} + x\bar{y})\Big)Q(x, y) = 1 - t\bar{x}Q(0, y) - tx\bar{y}Q(x, 0),$$

or

$$\boxed{\Big((1 - t(y + \bar{x} + x\bar{y})\Big)xyQ(x, y) = xy - tyQ(0, y) - txQ(x, 0)}$$

We call $K(x, y) = (1 - t(y + \bar{x} + x\bar{y})$ the kernel of the equation.

# A functional equation: example in 3D

- Take $\mathcal{S} = \{\bar{1}\bar{1}\bar{1}, \bar{1}\bar{1}1, \bar{1}10, 100\}$. The functional equation reads

$$\begin{aligned} O(x,y,z) =& 1 + t(\bar{x}\bar{y}\bar{z} + \bar{x}\bar{y}z + \bar{x}y + x)O(x,y,z) \\ & - t\bar{x}(y + \bar{y}z + \bar{y}\bar{z})O(0,y,z) - t\bar{x}\bar{y}(z+\bar{z})O(x,0,z) - t\bar{x}\bar{y}\bar{z}O(x,y,0) \\ & + t\bar{x}\bar{y}(z+\bar{z})O(0,0,z) + t\bar{x}\bar{y}\bar{z}O(0,y,0) + t\bar{x}\bar{y}\bar{z}O(x,0,0) \\ & - t\bar{x}\bar{y}\bar{z}O(0,0,0). \end{aligned}$$

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- Equivalently,

$$\begin{aligned} K(x,y,z)xyzO(x,y,z) = xyz - tyz(y + \bar{y}z + \bar{y}\bar{z})O(0,y,z) - tz(z+\bar{z})O(x,0,z) \\ - tO(x,y,0) + tz(z+\bar{z})O(0,0,z) + tO(0,y,0) + tO(x,0,0) - tO(0,0,0) \end{aligned}$$

where the kernel is

$$K(x,y,z) = 1 - t(\bar{x}\bar{y}\bar{z} + \bar{x}\bar{y}z + \bar{x}y + x).$$

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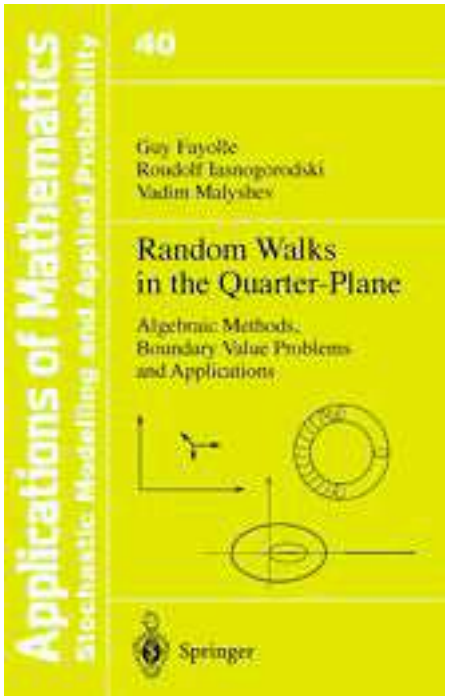
$\Rightarrow$ Determine $O(x,y,z;t)$ up to a large order (in $t$) and try to guess if it is algebraic or D-finite (order $\simeq 50$ and degree $\simeq 3000$ is not unusual).

# I. Some useful tools

## I.2. The group of the walk

[Fayolle et al. 99]



# What is this group? A 2D example

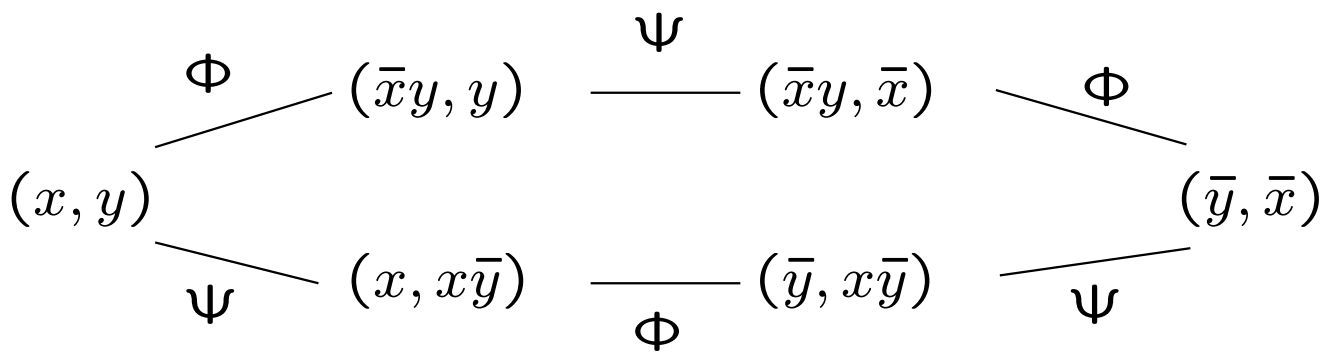
When $\mathcal{S} = \{\mathsf{N}, \mathsf{W}, \mathsf{SE}\}$, the kernel reads

$$K(x,y) = 1 - tS(x,y) \qquad \text{with} \qquad S(x,y) = \bar{x} + y + x\bar{y}.$$

Observation: $S(x,y)$ (and thus $K(x,y)$) is left unchanged by the rational transformations

$$\Phi : (x,y) \mapsto (\bar{x}y, y) \qquad \text{and} \qquad \Psi : (x,y) \mapsto (x, x\bar{y})\,.$$

They are involutions, and generate a finite group $G$:

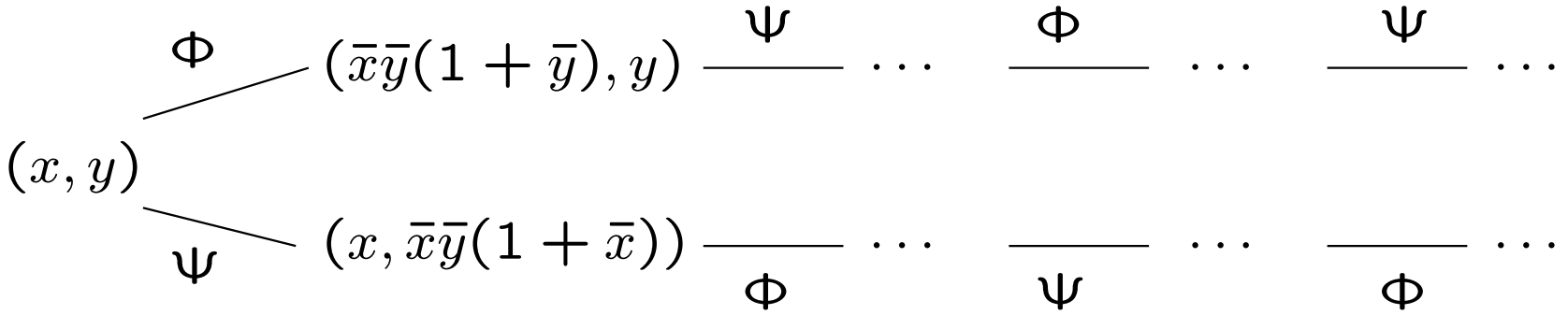


# The group is not always finite

- If $\mathcal{S} = \{\mathsf{S}, \mathsf{W}, \mathsf{SW}, \mathsf{NE}\}$:

$$\Phi : (x, y) \mapsto (\bar{x}\bar{y}(1 + \bar{y}), y) \quad \text{and} \quad \Psi : (x, y) \mapsto (x, \bar{x}\bar{y}(1 + \bar{x}))$$

seem to generate an infinite group



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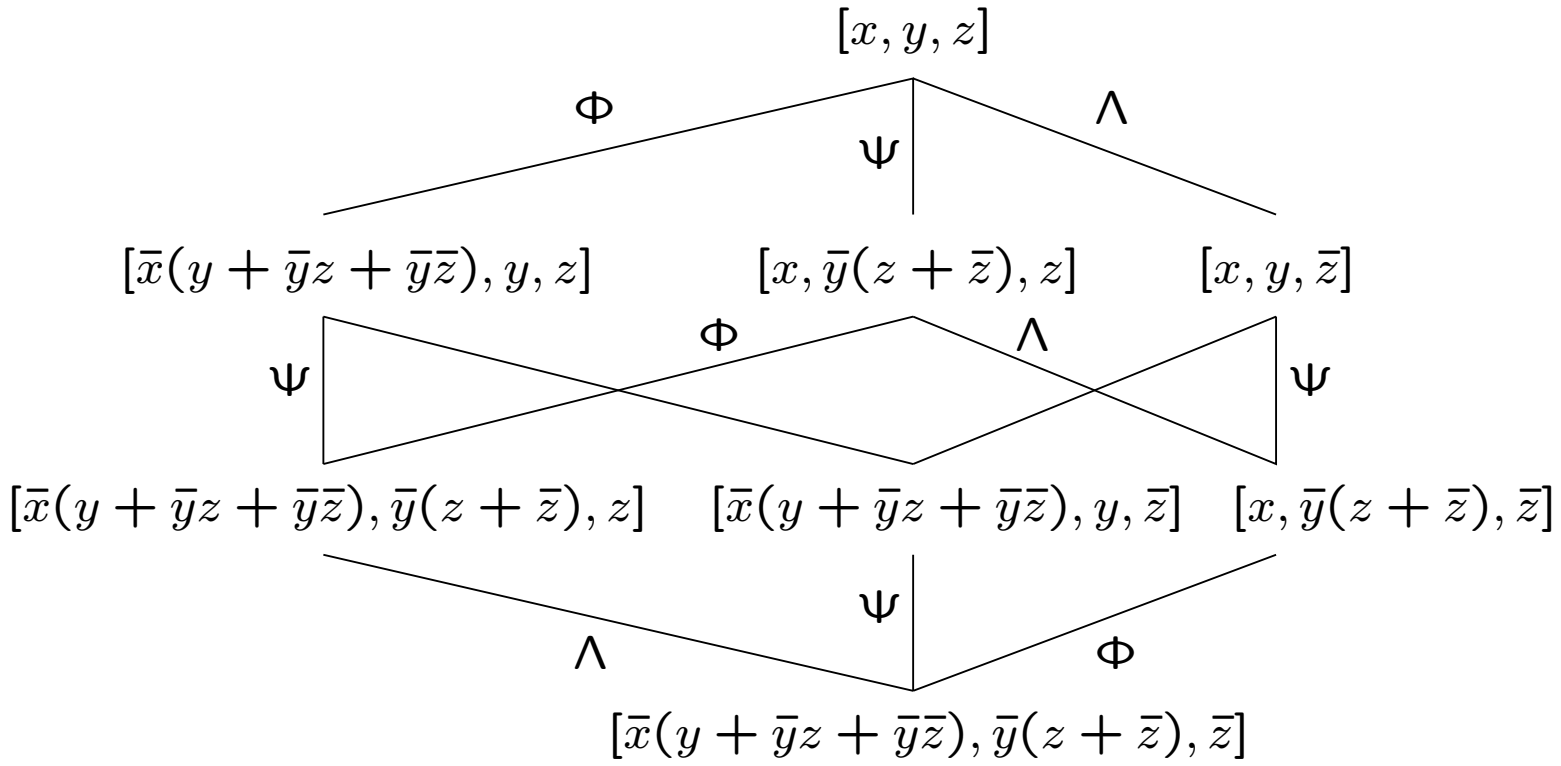
- Take $\mathcal{S} = \{\bar{1}\bar{1}\bar{1}, \bar{1}\bar{1}1, \bar{1}10, 100\}$. The kernel is

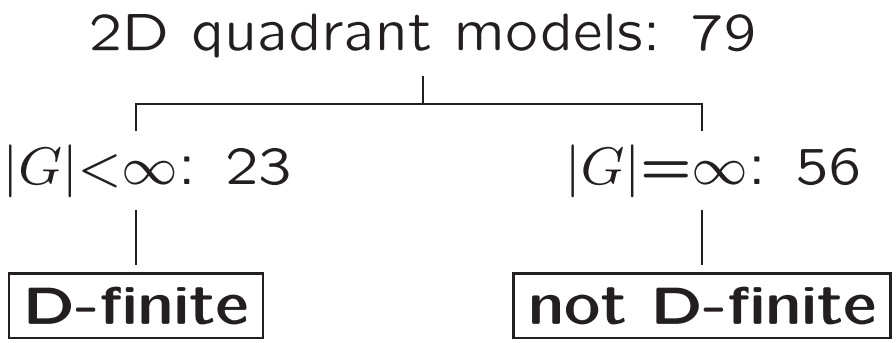
$$K(x, y, z) = 1 - t(\bar{x}\bar{y}\bar{z} + \bar{x}\bar{y}z + \bar{x}y + x).$$

- The group is generated by

$$[x, y, z] \overset{\Phi}{\mapsto} [\bar{x}(y + \bar{y}z + \bar{y}\bar{z}), y, z]\,, \quad [x, y, z] \overset{\Psi}{\mapsto} [x, \bar{y}(z + \bar{z}), z]\,, \quad [x, y, z] \overset{\Lambda}{\mapsto} [x, y, \bar{z}]\,.$$



# Relevance of the group size:
# Classification of quadrant walks with small steps



# I. Some useful tools

## I.3. Models with a finite group: The orbit equation

# The orbit equation: when $\mathcal{S} = \{\mathsf{N}, \mathsf{W}, \mathsf{SE}\}$

- The equation reads

$$K(x,y)\ xyQ(x,y) = xy - txQ(x,0) - tyQ(0,y) \qquad \text{with } K(x,y) = 1 - t(y + \bar{x} + x\bar{y}).$$

- The orbit of $(x,y)$ under $G$ is

$$(x,y) \overset{\Phi}{\longleftrightarrow} (\bar{x}y,y) \overset{\Psi}{\longleftrightarrow} (\bar{x}y,\bar{x}) \overset{\Phi}{\longleftrightarrow} (\bar{y},\bar{x}) \overset{\Psi}{\longleftrightarrow} (\bar{y},x\bar{y}) \overset{\Phi}{\longleftrightarrow} (x,x\bar{y}) \overset{\Psi}{\longleftrightarrow} (x,y).$$

- All transformations of $G$ leave $K(x,y)$ invariant. Hence

$$\begin{array}{rcccccc} K(x,y)\ xyQ(x,y) & = & xy & - & txQ(x,0) & - & tyQ(0,y) \\ K(x,y)\ \bar{x}y^2Q(\bar{x}y,y) & = & \bar{x}y^2 & - & t\bar{x}yQ(\bar{x}y,0) & - & tyQ(0,y) \end{array}$$

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\cdots & = & \cdots & & & & \\
K(x,y)\ x^2\bar{y}Q(x,x\bar{y}) & = & x^2\bar{y} & - & txQ(x,0) & - & tx\bar{y}Q(0,x\bar{y}).
\end{array}$$

# The orbit equation: when $\mathcal{S} = \{\mathsf{N}, \mathsf{W}, \mathsf{SE}\}$

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\end{array}$$

$\Rightarrow$ Form the alternating sum of the equation over all elements of the orbit:

$$K(x,y)\Big(xyQ(x,y) - \bar{x}y^2Q(\bar{x}y,y) + \bar{x}^2yQ(\bar{x}y,\bar{x}) - \bar{x}\bar{y}Q(\bar{y},\bar{x}) + x\bar{y}^2Q(\bar{y},x\bar{y}) - x^2\bar{y}Q(x,x\bar{y})\Big) = xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}.$$

# The orbit equation in general

For all models with a finite group,

$$\sum_{g\in G} \mathrm{sign}(g)\, g(xyQ(x,y;t)) = \frac{1}{K(x,y;t)} \sum_{g\in G} \mathrm{sign}(g)\, g(xy),$$

where

$$g(A(x,y)) := A(g(x,y))$$

- The right-hand side is a rational series, called the orbit sum.

Example:

$$xyQ(x,y) - \bar{x}y^2Q(\bar{x}y,y) + \bar{x}^2yQ(\bar{x}y,\bar{x}) - \bar{x}\bar{y}Q(\bar{y},\bar{x}) + x\bar{y}^2Q(\bar{y},x\bar{y}) - x^2\bar{y}Q(x,x\bar{y}) = \frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})}.$$

# I. Some useful tools

## I.4. Models with a finite group: Extracting the main generating function in the orbit equation

# A 2D example: $\mathcal{S} = \{\mathsf{N}, \mathsf{W}, \mathsf{SE}\}$

$$xyQ(x,y) - \bar{x}y^2Q(\bar{x}y,y) + \bar{x}^2yQ(\bar{x}y,\bar{x}) - \bar{x}\bar{y}Q(\bar{y},\bar{x}) + x\bar{y}^2Q(\bar{y},x\bar{y}) - x^2\bar{y}Q(x,x\bar{y}) = \frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})}$$

- Both sides are power series in $t$, with coefficients in $\mathbb{Q}[x, \bar{x}, y, \bar{y}]$.

- Extract the part with positive powers of $x$ and $y$:

$$xyQ(x,y) = [x^{>0}y^{>0}] \, \frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})}$$

is a D-finite series.

[Lipshitz 88]

# Extraction for quadrant models

- For 23 models, we found a finite group and wrote the orbit equation:

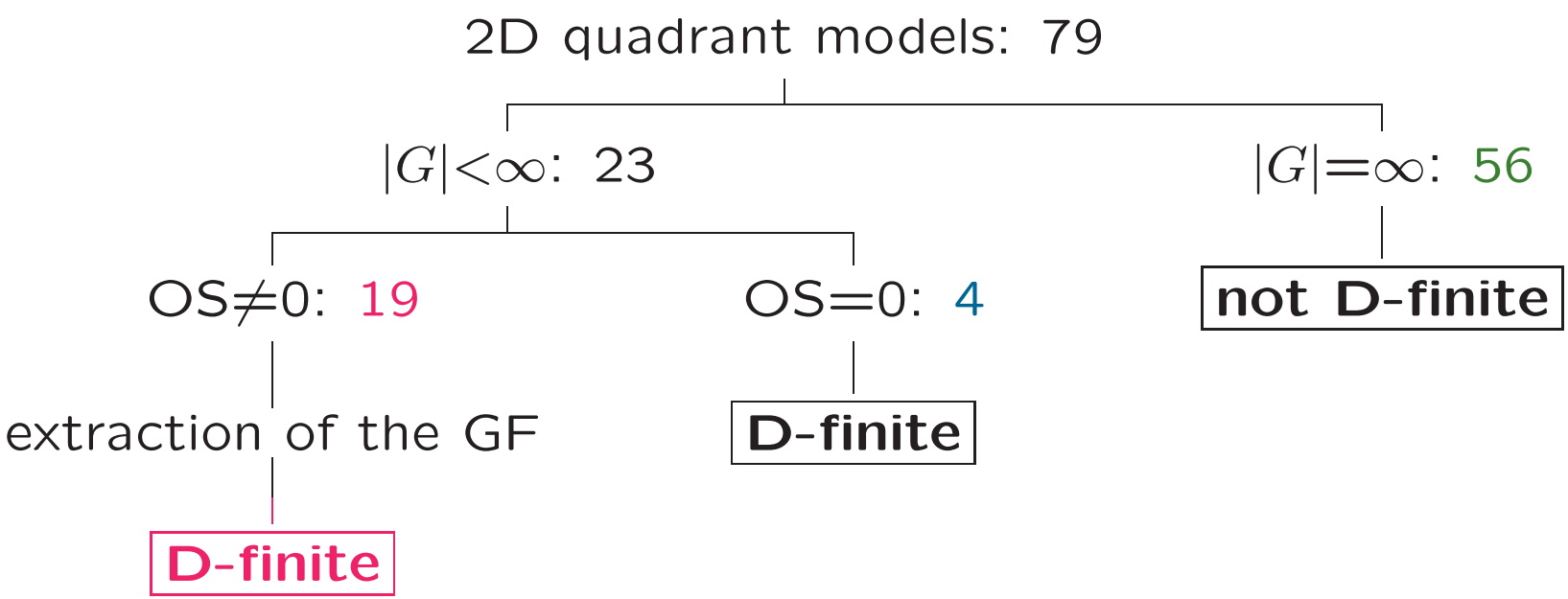
$$\sum_{g\in G} \mathrm{sign}(g)g(xyQ(x,y;t)) = \frac{1}{K(x,y;t)} \sum_{g\in G} \mathrm{sign}(g)g(xy),$$

- For the 19 models where the orbit sum is non-zero,

$$xyQ(x,y;t) = [x^{>0}y^{>0}]\frac{1}{K(x,y;t)} \sum_{g\in G} \mathrm{sign}(g)g(xy)$$

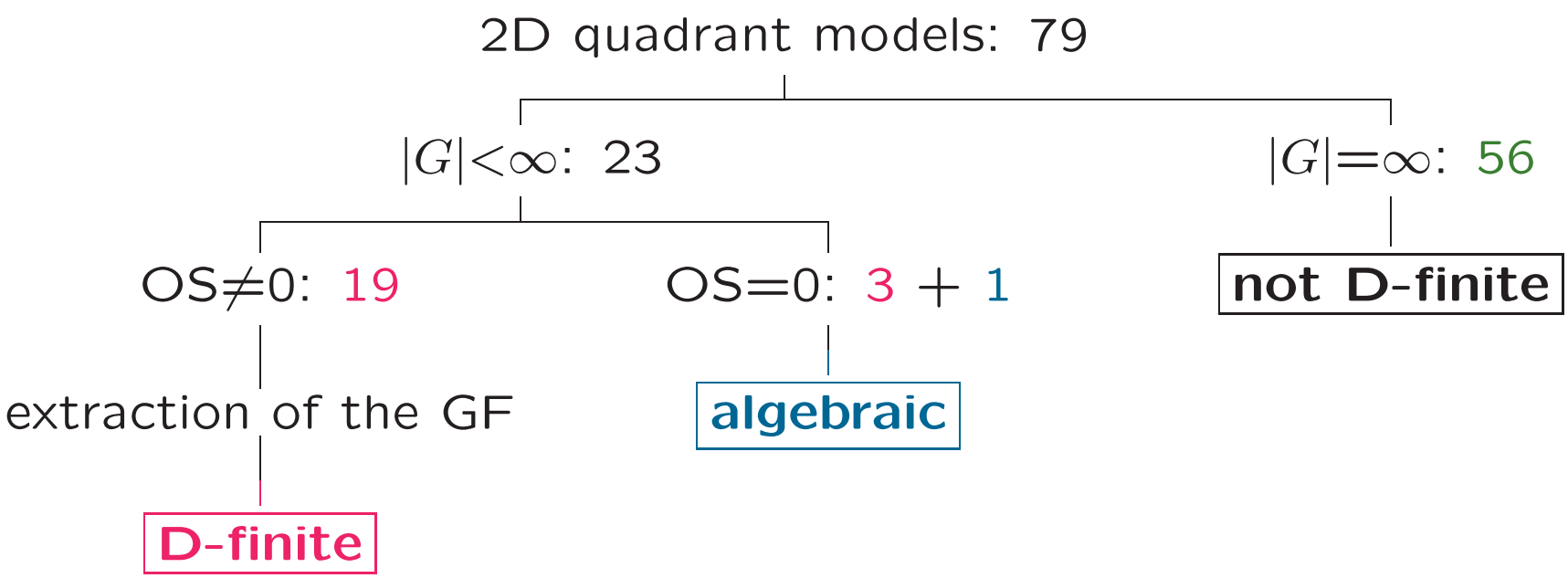
is a D-finite series.

# Classification of quadrant walks with small steps




[mbm-Mishna 10]

# Classification of quadrant walks with small steps




[mbm-Mishna 10], [Bostan-Kauers 10], [Bostan-Kurkova-Raschel 13(a)]

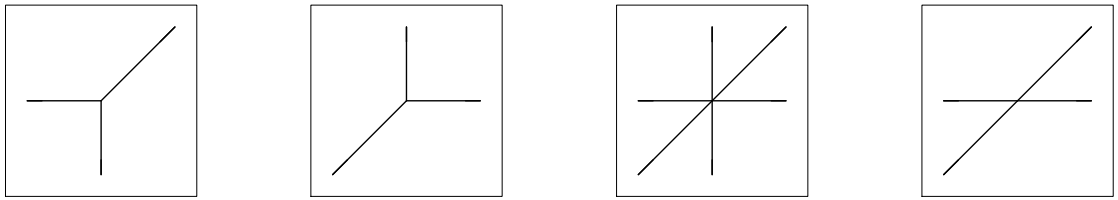
# When the orbit sum is zero

In 4 cases, the orbit sum vanishes:

$$\sum_{g\in G} \mathrm{sign}(g)g(xyQ(x,y;t)) = \frac{1}{K(x,y;t)} \sum_{g\in G} \mathrm{sign}(g)g(xy) = 0.$$

This happens:

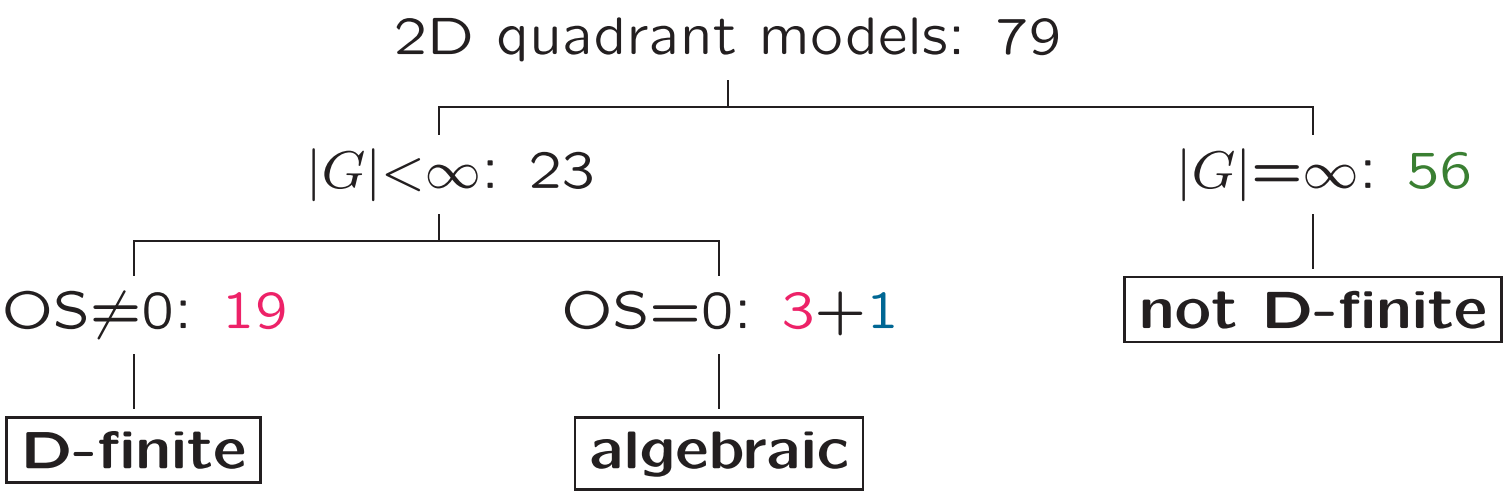
- for the Kreweras trilogy, $\mathcal{S} = \{\mathsf{W},\mathsf{S},\mathsf{NE}\},\ \{\mathsf{E},\mathsf{N},\mathsf{SW}\},\ \{\mathsf{W},\mathsf{S},\mathsf{E},\mathsf{N},\mathsf{NE},\mathsf{SW}\}$,
- for Gessel's walks, $\mathcal{S} = \{\mathsf{E},\mathsf{W},\mathsf{NE},\mathsf{SW}\}$.



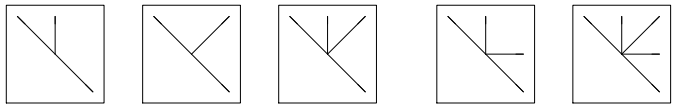
In all 4 cases, $Q(x,y;t)$ is algebraic.

[mbm-Mishna 10], [Bostan-Kauers 10], [Bostan-Kurkova-Raschel 13(a)]

# Classification of quadrant walks




- D-finiteness and algebraicity: [mbm-Mishna 10], [Bostan-Kauers 10], [Bostan-Kurkova-Raschel 13(a)]
- Non-D-finiteness:
  - 5 singular cases: $Q(1,1;t)$ is not D-finite [Mishna-Rechnitzer 07], [Melczer-Mishna 13]



  - 51 non-singular cases: $Q(0,0;t)$ is not D-finite [Bostan-Raschel-Salvy 14], and $Q(x,0;t)$ is not D-finite in $x$ [Kurkova-Raschel 12]

# II. Application to octant models

# The dimension of a model

Definition. A model has dimension (at most) $d$ if $d$ of the positivity conditions imply the remaining $3-d$ ones.

We have already discarded models of dimension 0 or 1.

Proposition. Among the 35 548 models with at most 6 steps, 14 744 have dimension two and 20 804 have dimension three.
(integer linear programming)

# Our toolbox

- Write a functional equation for $O(x, y, z; t) = \sum_{i,j,k,n} o(i, j, k; n) x^i y^j z^k t^n$
- Determine if the group of the walk is finite
- If it is, form the orbit equation
- and *try to* extract the generating function $O(x, y, z; t)$

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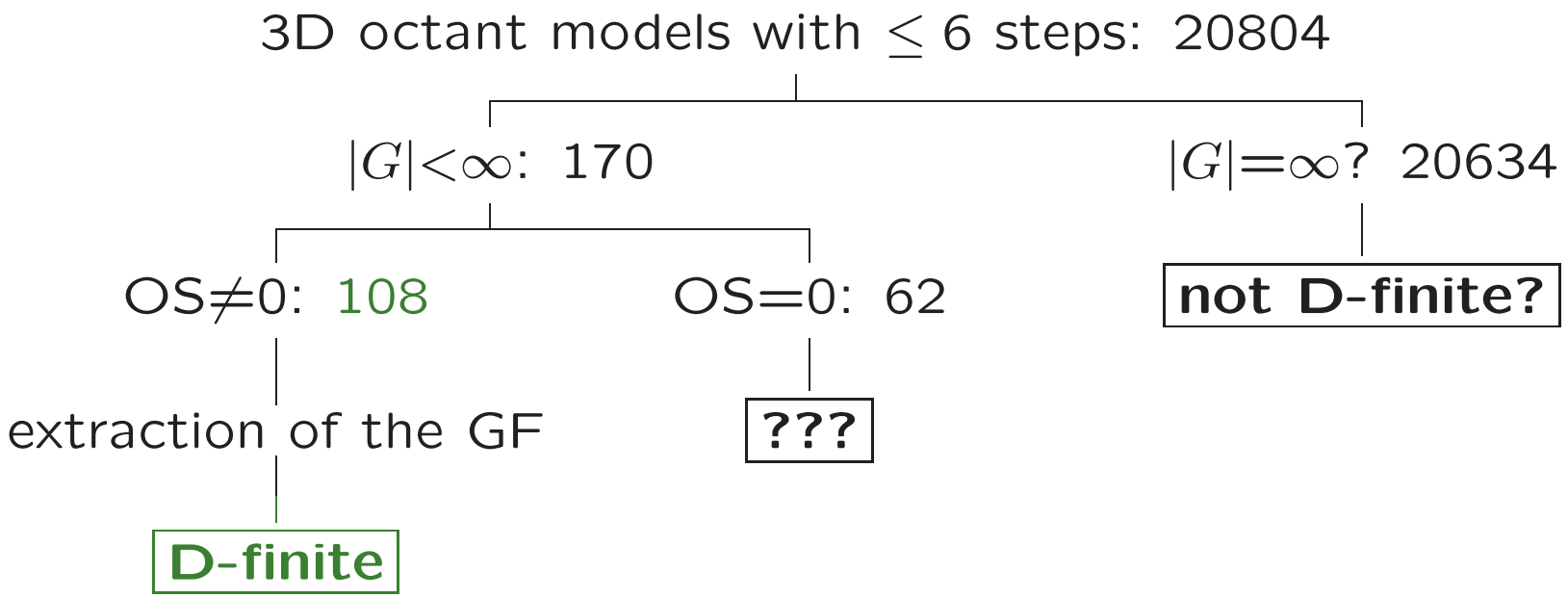
What happens?

# III. Three-dimensional octant models (with at most 6 steps)

# Classification of 3D octant models (with at most 6 steps)




[Bostan-MBM-Kauers-Melczer 14]

## When the orbit sum is zero (62 models)

- For 19 models, including 3D Kreweras, we did not guess any DE

  Example:

$$\mathcal{S} = \{111, \bar{1}00, 0\bar{1}0, 00\bar{1}\}$$

- the remaining 43 models are of Hadamard type and D-finite

## Hadamard models: example

Take $S(x,y,z) = x + (1 + x + \bar{x})(yz + \bar{y} + \bar{z})$. The group has order 12, and the orbit sum is zero.

To construct an octant walk of length $n$ with steps in $\mathcal{S}$:

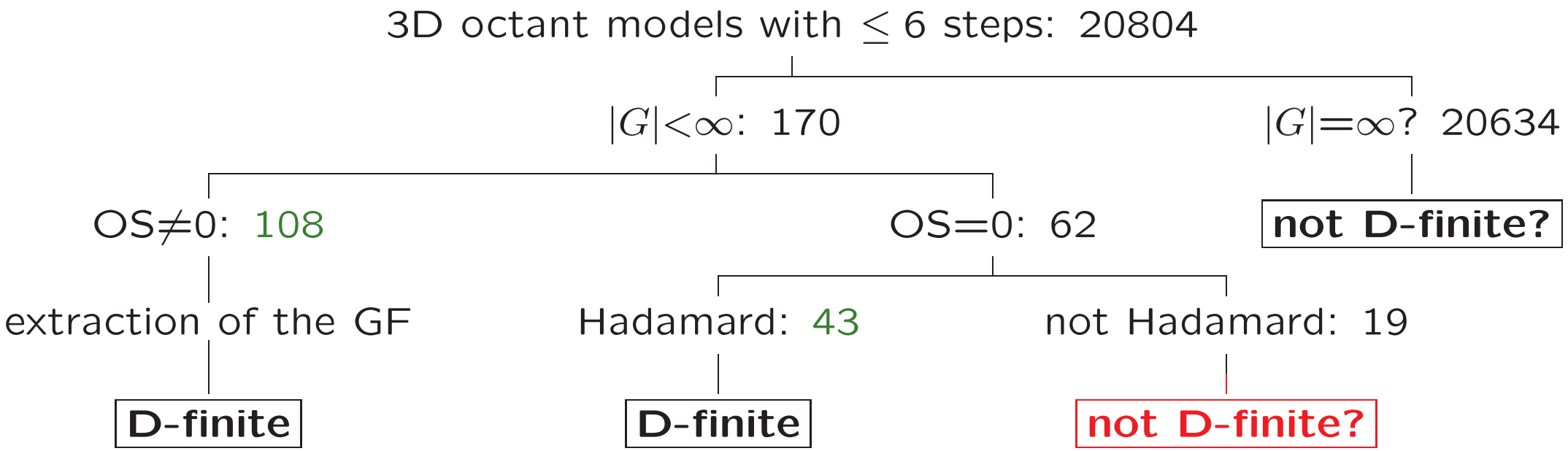
- take a 1D walk $h = h_1 \dots h_n$ with steps in $\{\bar{1}, 0, 1, 1\}$ on the $x$-axis; say it has $\ell$ black steps;
- take a quadrant walk $q = q_1 \dots q_\ell$ with steps in $\{11, \bar{1}0, 0\bar{1}\}$ in the $yz$-plane;
- in $h$, replace $h_i$ by $(h_i, 0, 0)$ if $h_i$ is red, by $(h_i, q_j)$ if $h_i$ is the $j$th black step of $h$.

The D-finiteness of $O(x,y,z;t)$ then follows from the D-finiteness of the GFs $H(x;t)$ and $Q(y,z;t)$ of the two projected walks.

The argument applies with $S(x,y,z) = U(x) + V(x)T(y,z)$ or

$$S(x,y,z) = U(x,y) + V(x,y)T(z).$$

# Classification of 3D octant models




[Bostan-MBM-Kauers-Melczer 14]

# IV. Two-dimensional octant models (with at most 6 steps)

- One positivity condition (say, in $z$) holds automatically if the other two hold.

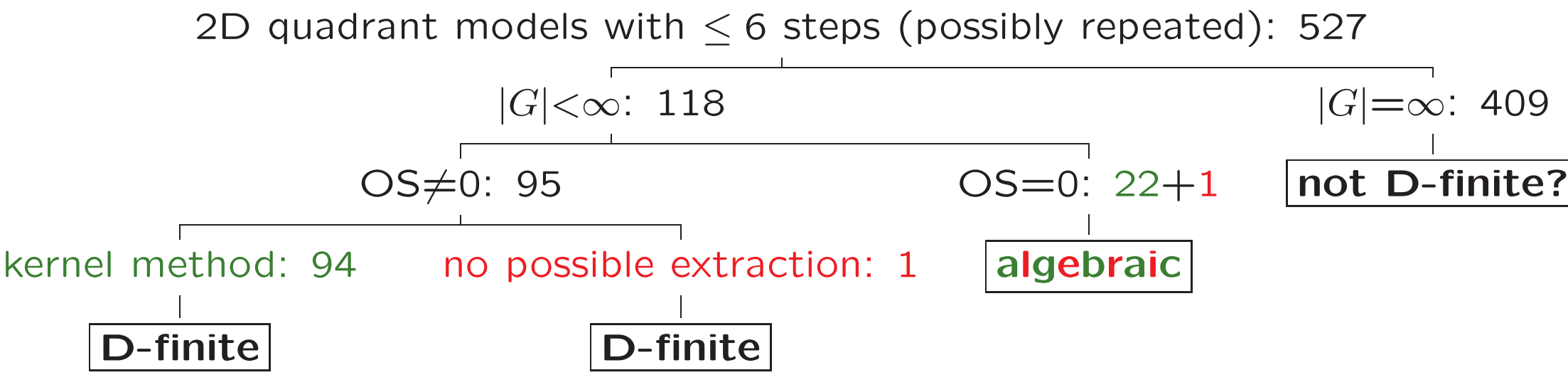
- Study the projection of the walk on the $xy$-plane: multiset of steps

$$\mathcal{S}_1 = \{ij : \exists k \text{ such that } ijk \in \mathcal{S}\} \setminus \{00\}$$

Main series:

$$Q(x, y; t) := O(x, y, 1; t)$$

# Classification of 2D quadrant models with (possibly) repeated steps




[Bostan-MBM-Kauers-Melczer 14]

# Two remarkable quadrant models with repeated steps

Proposition

• For the quadrant model $\mathcal{S} = \{\bar{1}\bar{1}, \bar{1}0, 1\bar{1}, 10, 10, 11\}$ (with an East repeated step), the generating function $Q(x, y; t)$ is algebraic. Moreover some coefficients are nice:

$$Q(0, y; t) = \sum_{n \geq j \geq 0} \frac{6(2j+1)!(6n+1)!(2n+j+1)!}{j!^2(3n)!(4n+2j+3)!(n-j)!(n+1)} y^j t^{2n}.$$

(Group of order 6, zero orbit sum)

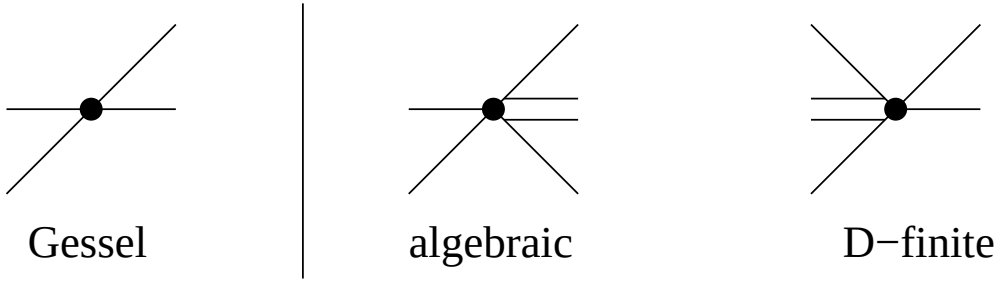
• For the reverse model $\overline{\mathcal{S}} = \{11, 10, \bar{1}1, \bar{1}0, \bar{1}0, \bar{1}\bar{1}\}$ (with a West repeated step), the series $Q(x, y; t)$ is D-finite.

In both cases, we only have computer algebra proofs
(in the spirit of [Bostan & Kauers 10])

# Questions

- Explore the 19 three-dimensional models with zero orbit sum that are not Hadamard: are they D-finite?

- Find bare hands proofs for



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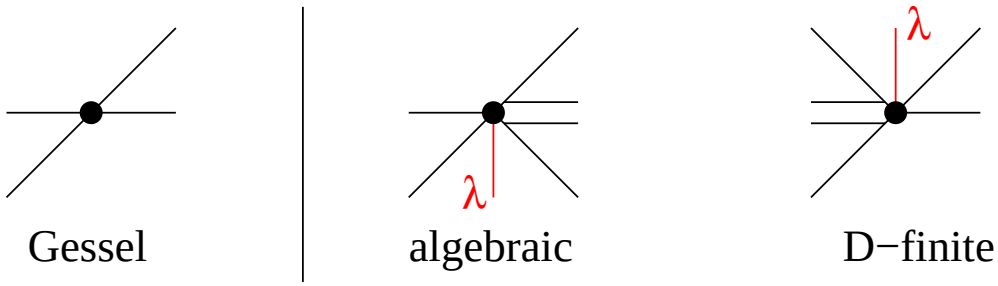
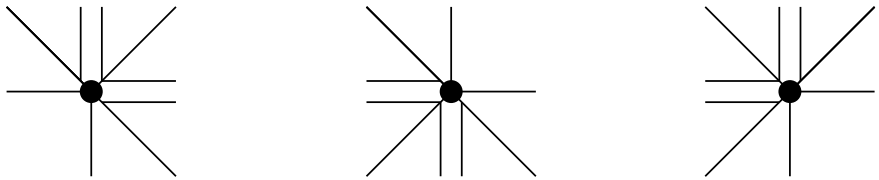



[Kauers, Yatchak 14(a)]

- Find any proof for the (conjecturally algebraic) models [Kauers, Yatchak 14(a)]



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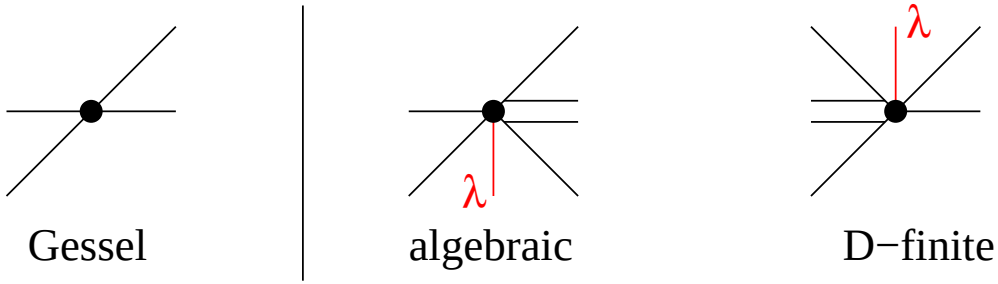


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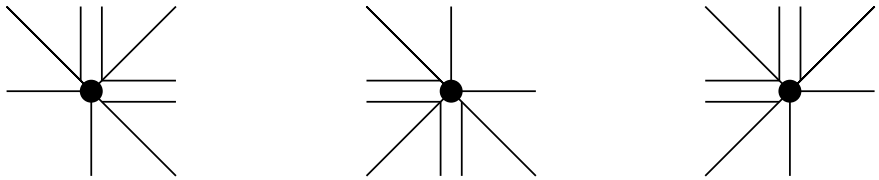

- larger steps, arbitrary dimensions...