Prudent self-avoiding walks

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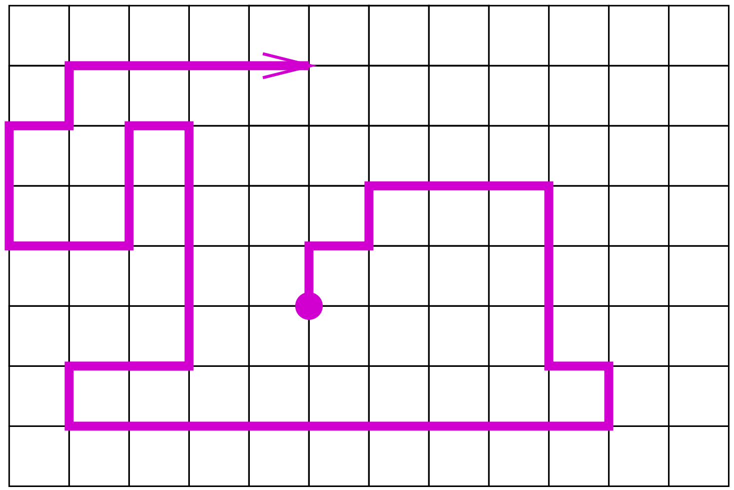
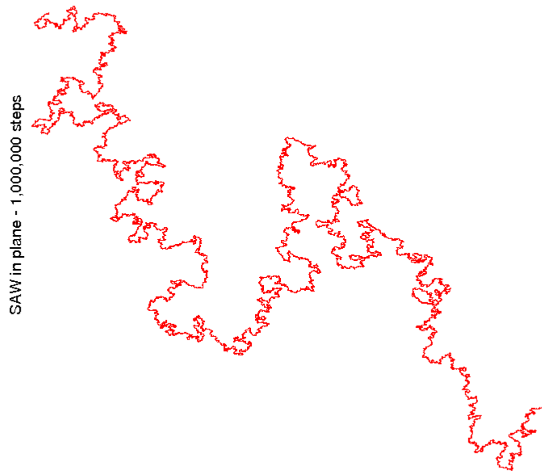
+ ArXiv 2008

Self-avoiding walks (SAW)





Conjectures $(d = 2)$

• Enumeration: The number of n-step SAW is equivalent to $(\kappa)\,\mu^n\, n^{11/32}$ for n large.

• Asymptotic properties: The endpoint lies on average at distance $n^{3/4}$ from the starting point.

• Limit process: The scaling limit of SAW is $\mathrm{SLE}_{8/3}$

(proved under an assumption of conformal invariance [Lawler et al. 02])

This is too hard!

… for exact enumeration

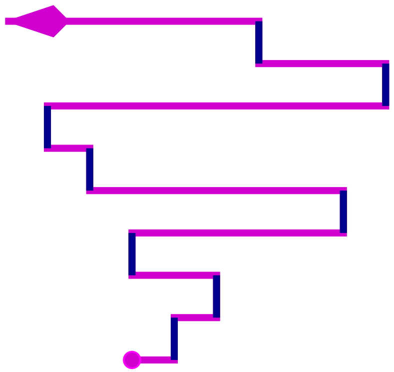
$\Rightarrow$ **Study of toy models**, that should be as general as possible, but still tractable

- develop new techniques in exact enumeration

- solve better and better approximations of real SAW

A toy model: Partially directed walks

- Model: Self-avoiding walks with steps N, W, E



"Markovian with memory 1"

- Enumeration: generating function and asymptotics

$$\sum_n a(n)t^n = \frac{1+t}{1-2t-t^2} \quad \Rightarrow \quad a(n) \sim (1+\sqrt{2})^n \sim (2.41...)^n$$

- Asymptotic properties: coordinates of the endpoint

$$\mathbb{E}(|X_n|) \sim \sqrt{n}, \qquad \mathbb{E}(Y_n) \sim n$$

I. Prudent self-avoiding walks:

Definition, functional equations

Self-directed walks [Turban-Debierre 86]
Exterior walks **[Préa 97]**
Outwardly directed SAW [Santra-Seitz-Klein 01]
Prudent walks **[Duchi 05], [Dethridge, Guttmann, Jensen 07], [mbm 08]**

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A step never points towards a vertex that has been visited before.

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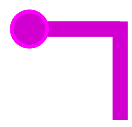

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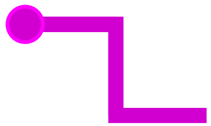

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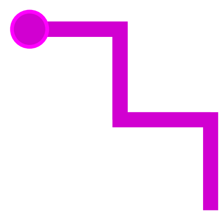

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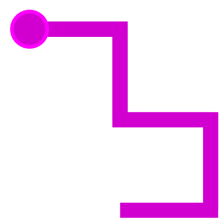

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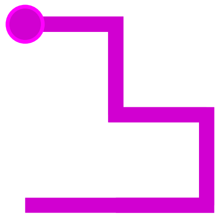

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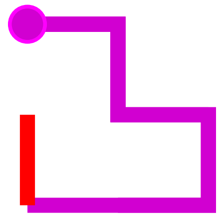

not prudent!

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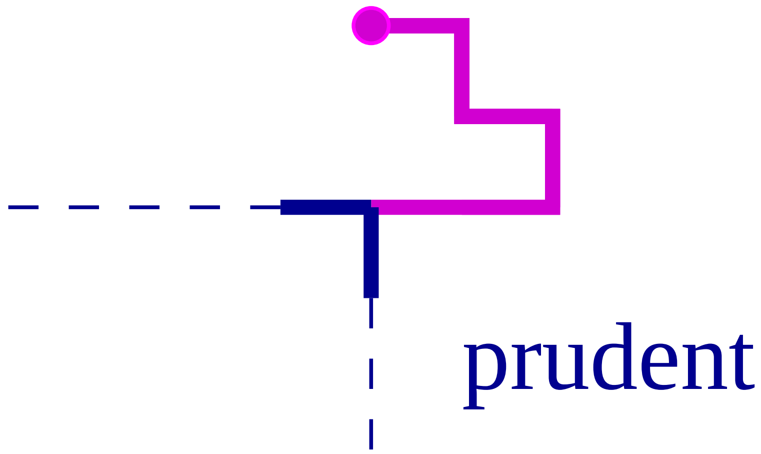

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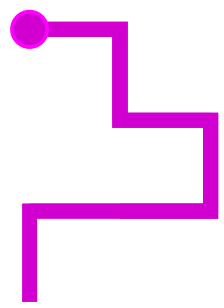

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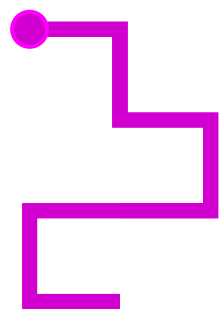

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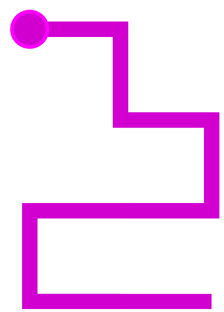

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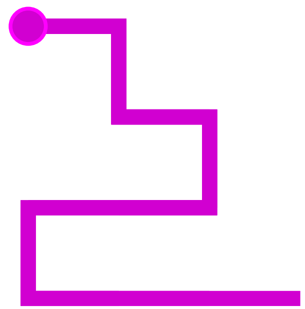

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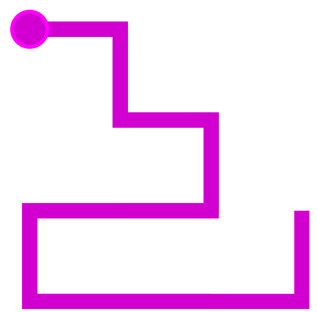

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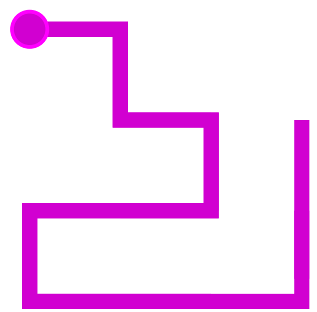

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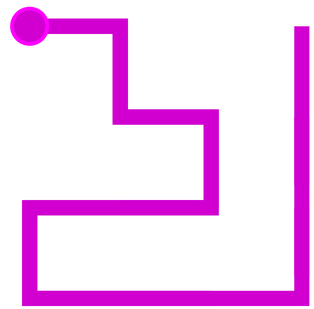

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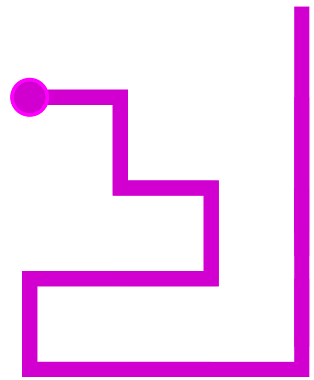

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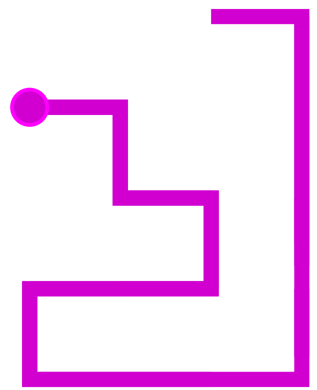

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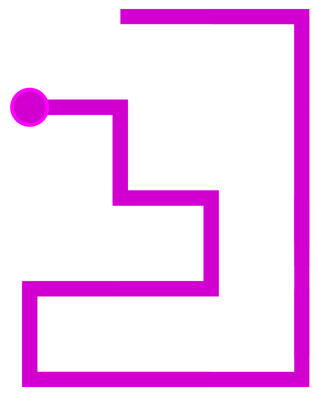

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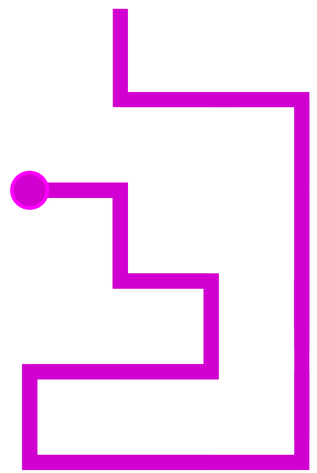

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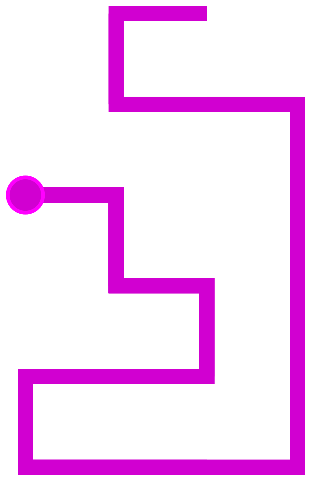

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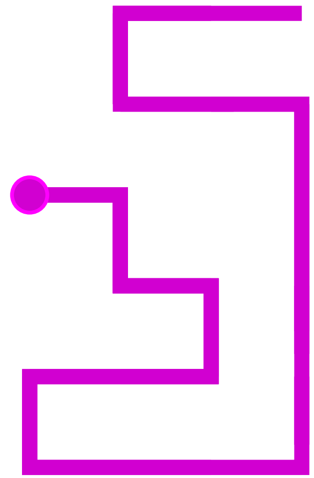

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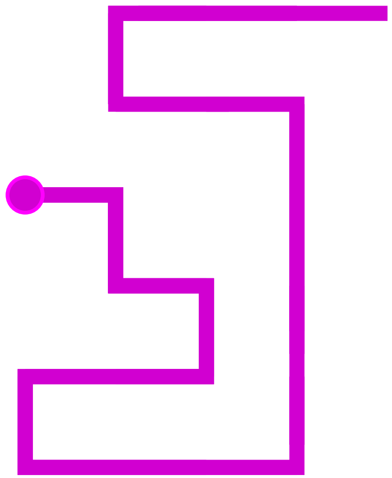

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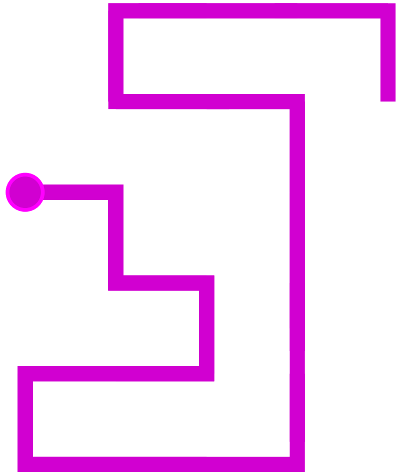

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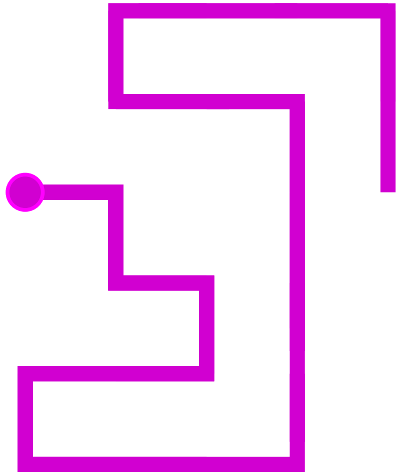

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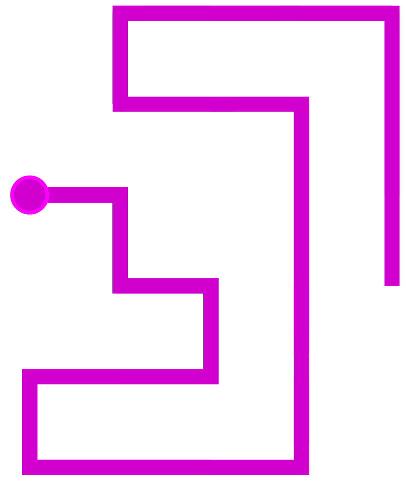

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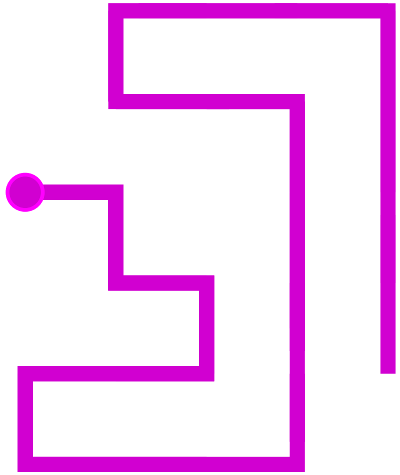

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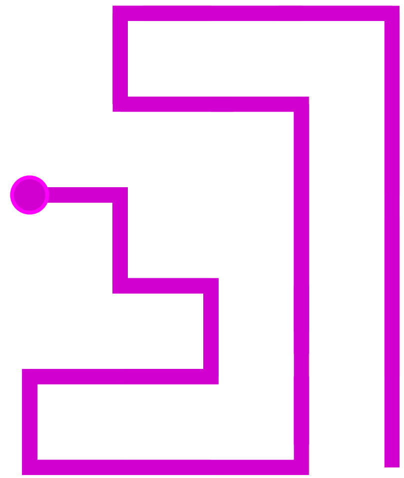

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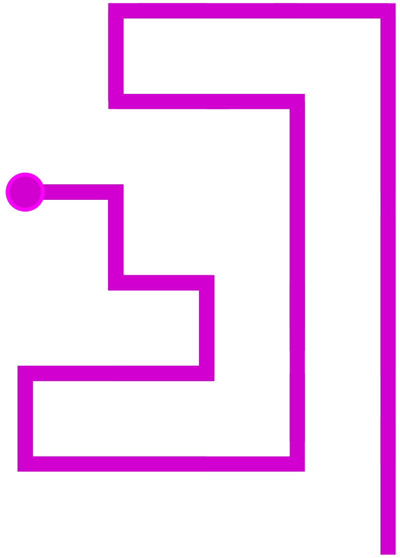

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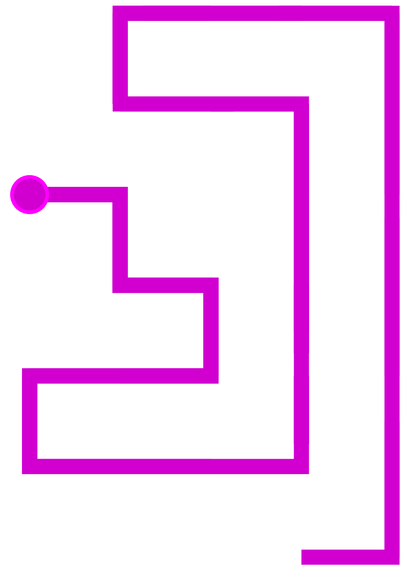

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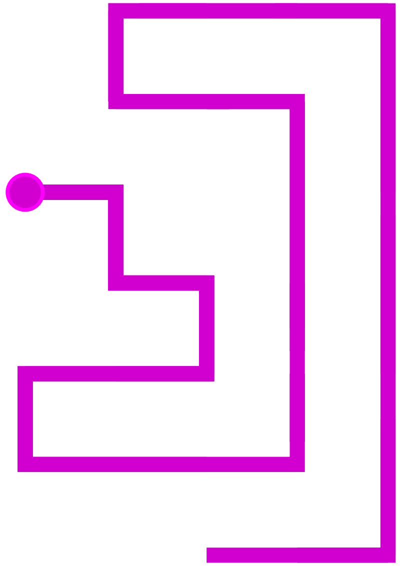

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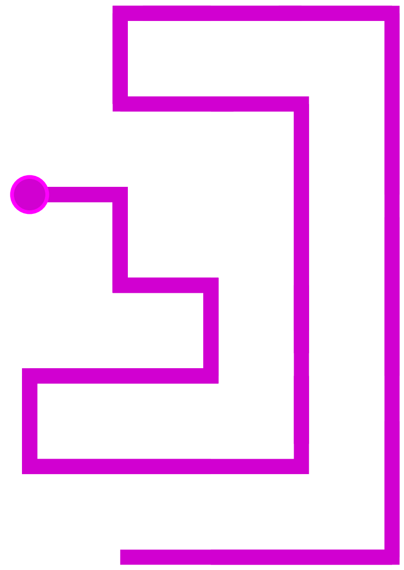

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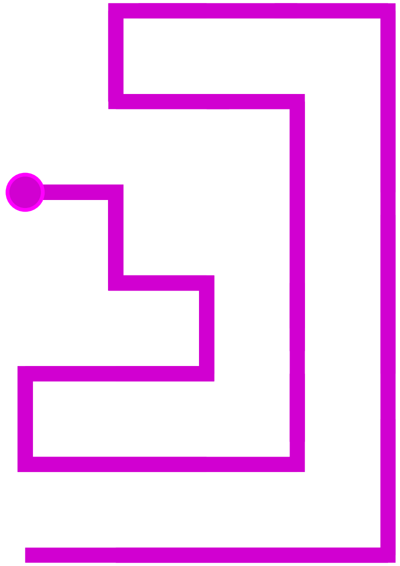

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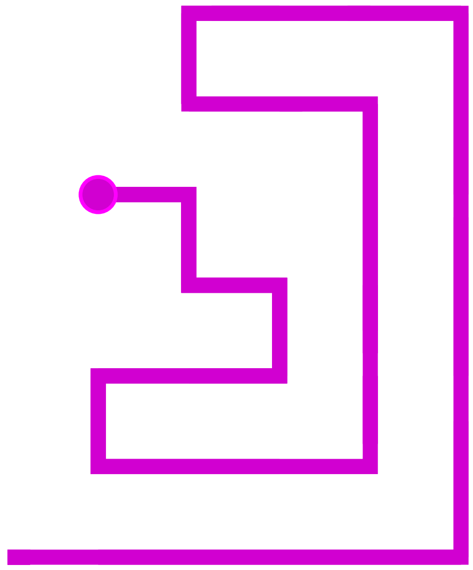

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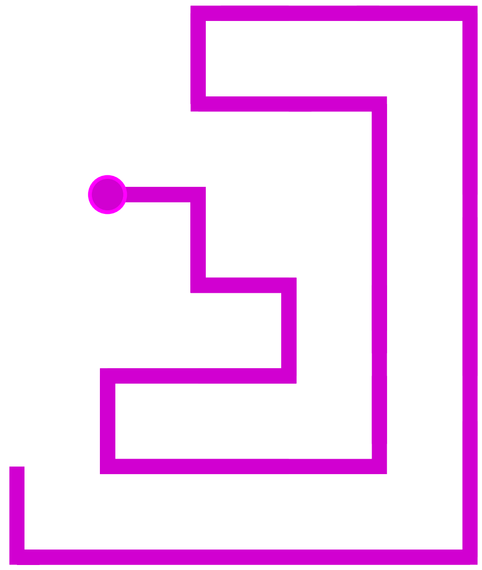

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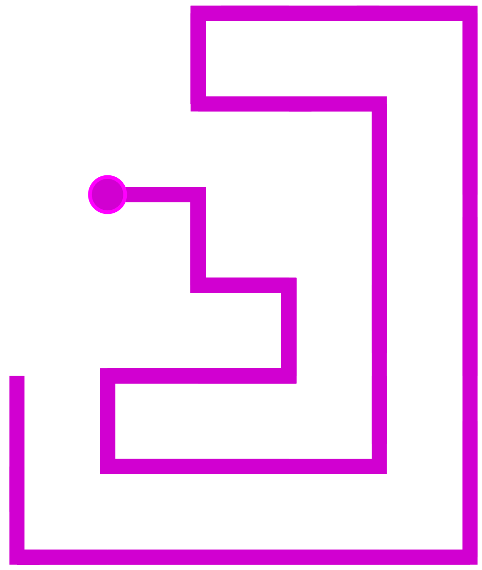

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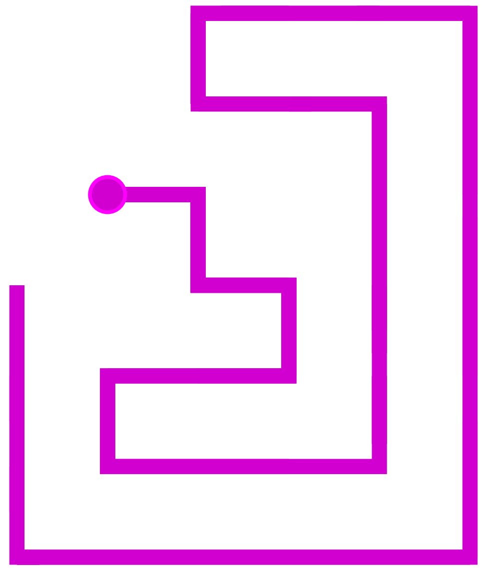

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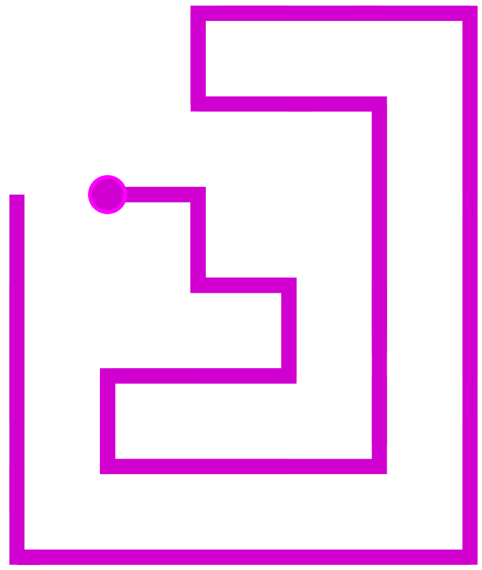

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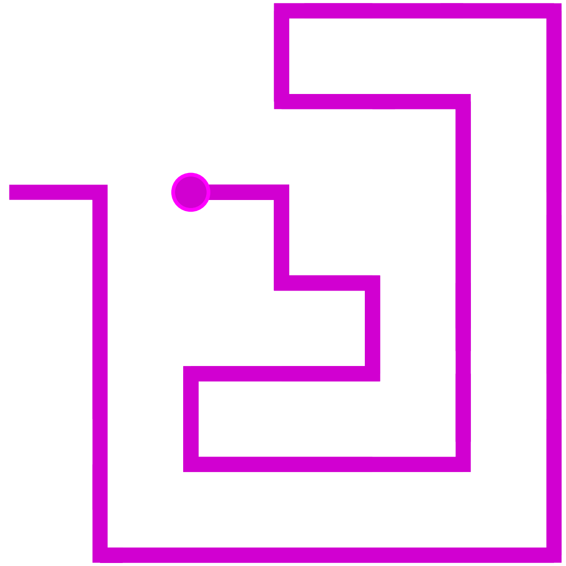

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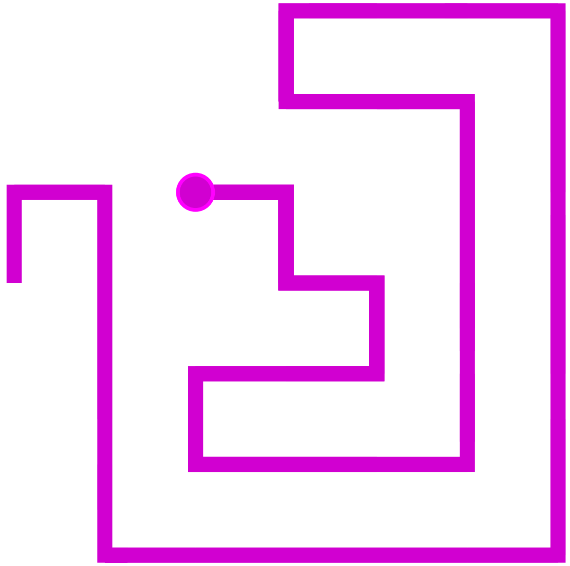

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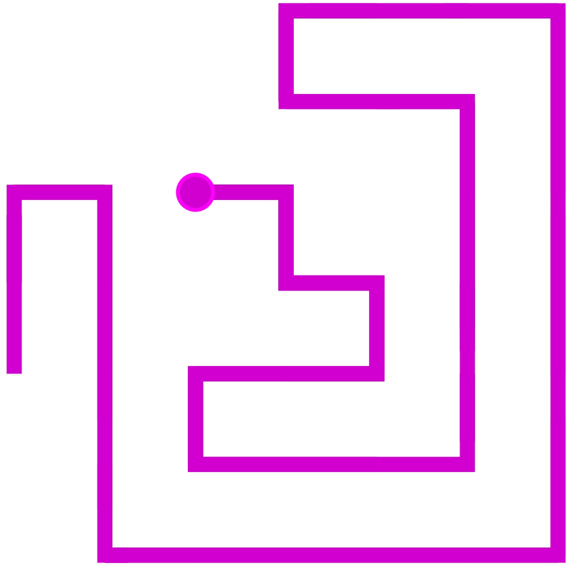

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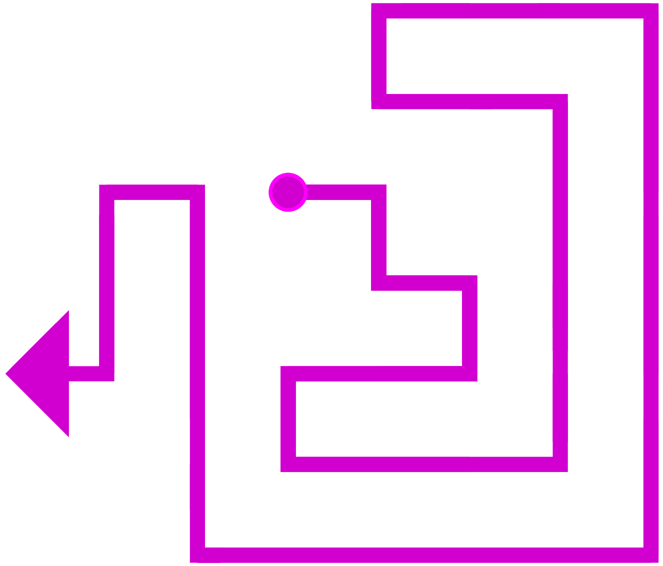
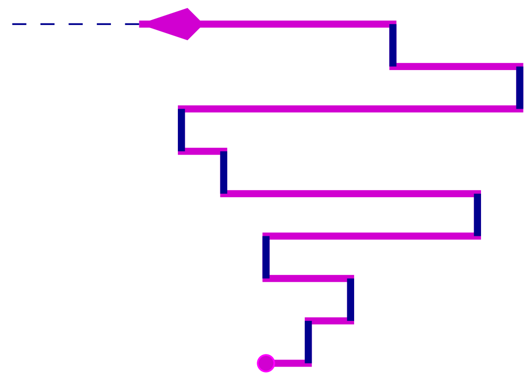


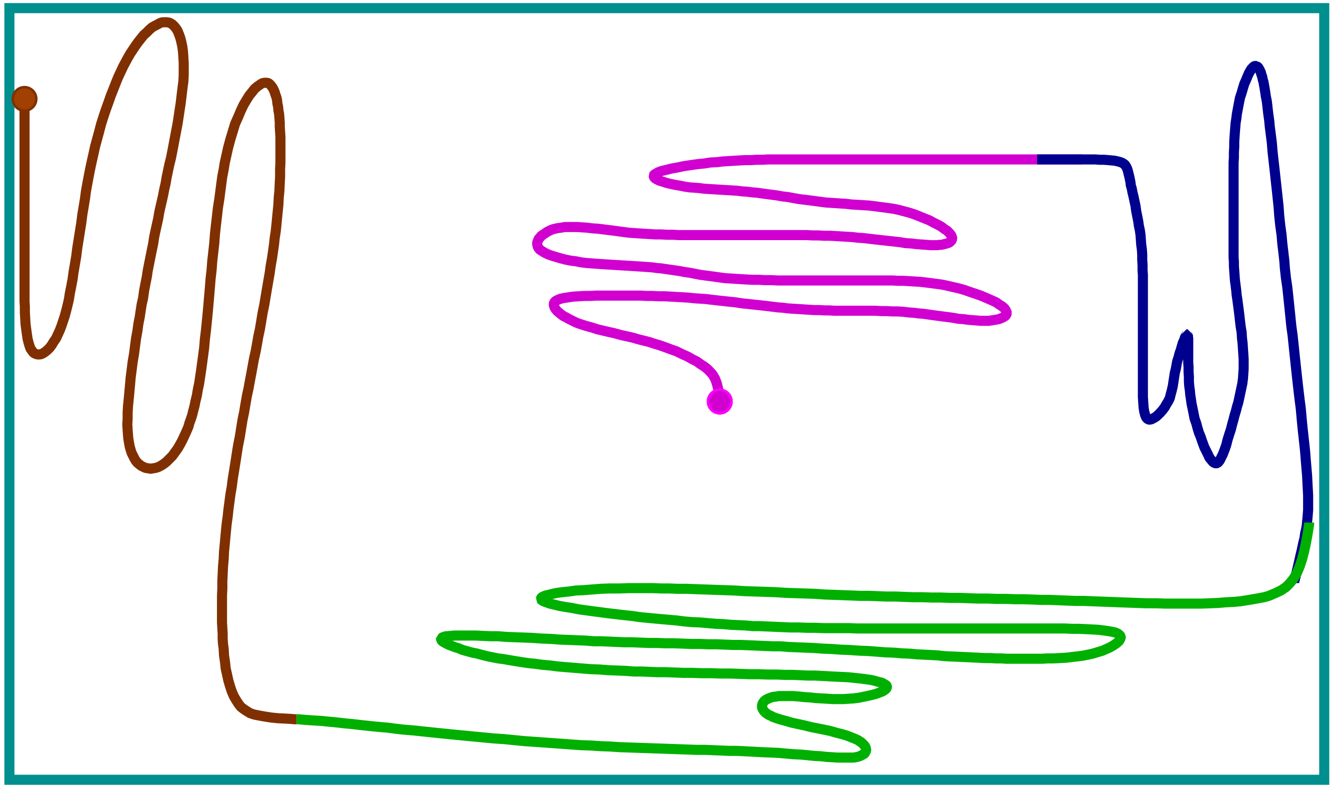
Remark: Partially directed walks are prudent



A property of prudent walks

A property of prudent walks

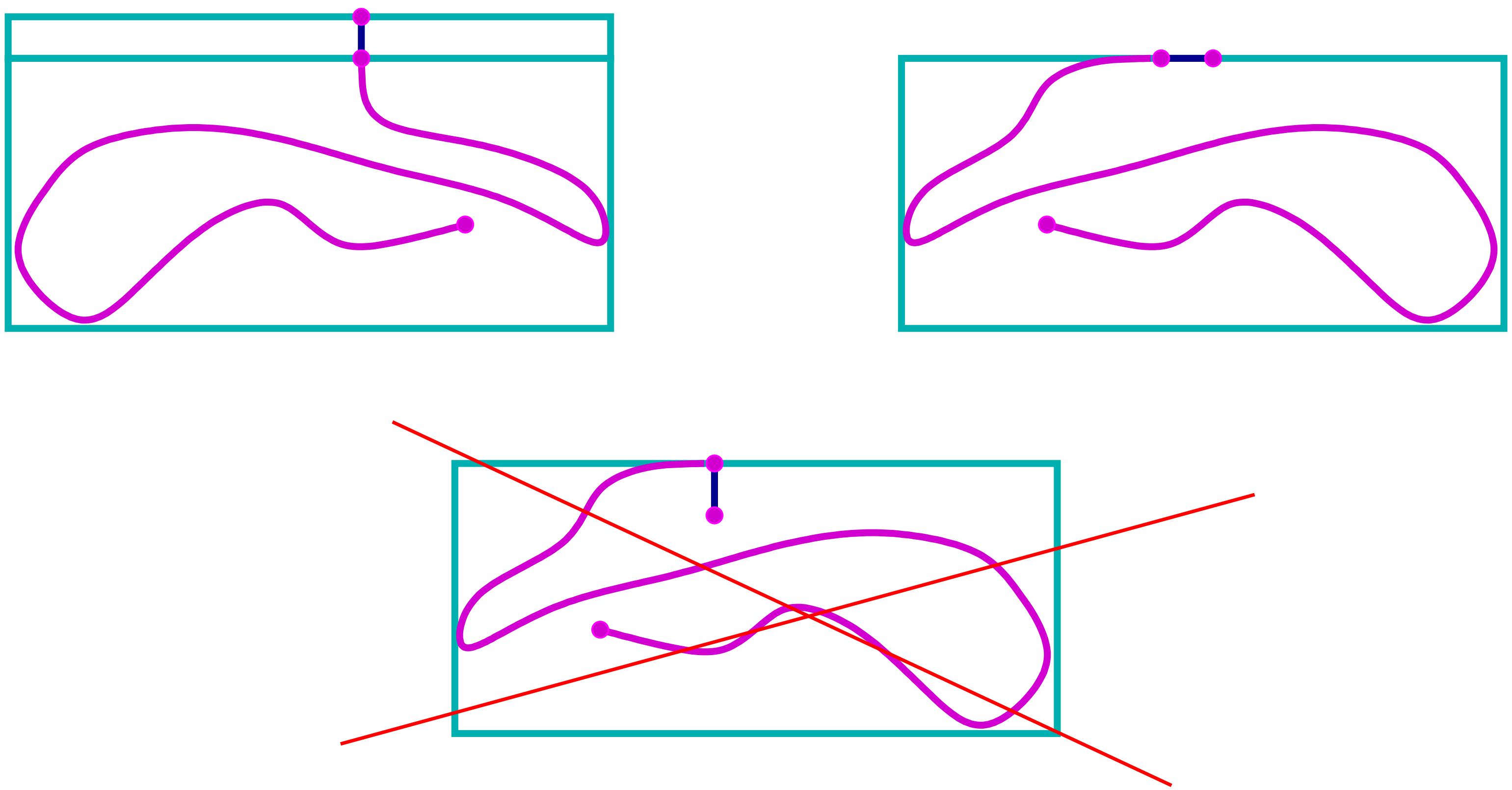
The box of a prudent walk



The endpoint of a prudent walk is always on the border of the box

Recursive construction of prudent walks

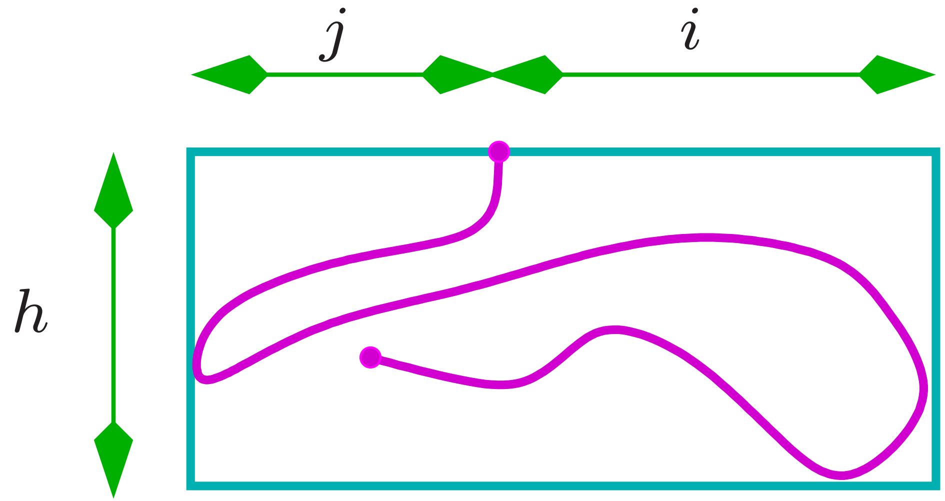
Each new step either inflates the box or walks (prudently) along the border.



Recursive construction of prudent walks: Where is the endpoint?

- Three more parameters

(*catalytic* parameters)



- Generating function of prudent walks ending on the top of their box:

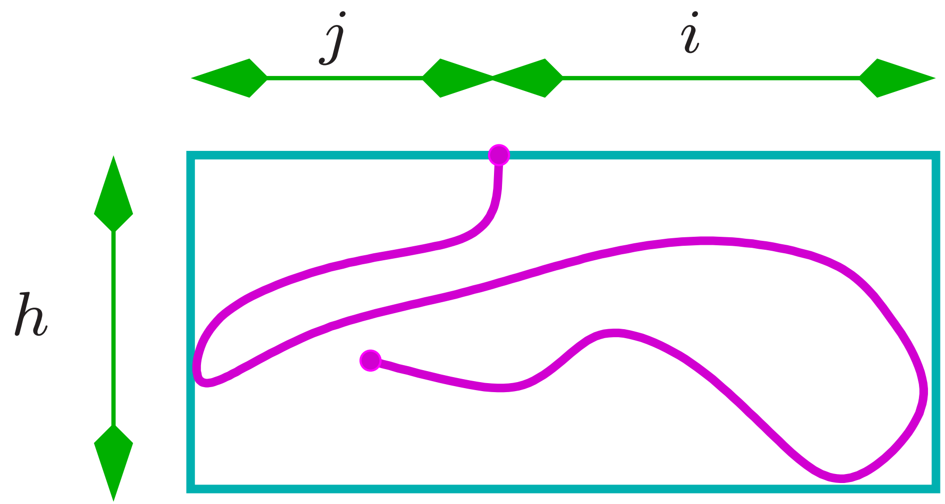
$$T(t; u, v, w) = \sum_{w} t^{|w|} u^{i(w)} v^{j(w)} w^{h(w)}$$

Series with three catalytic variables u, v, w

Recursive construction of prudent walks: Where is the endpoint?

- Three more parameters

(*catalytic* parameters)



- Generating function of prudent walks ending on the top of their box:

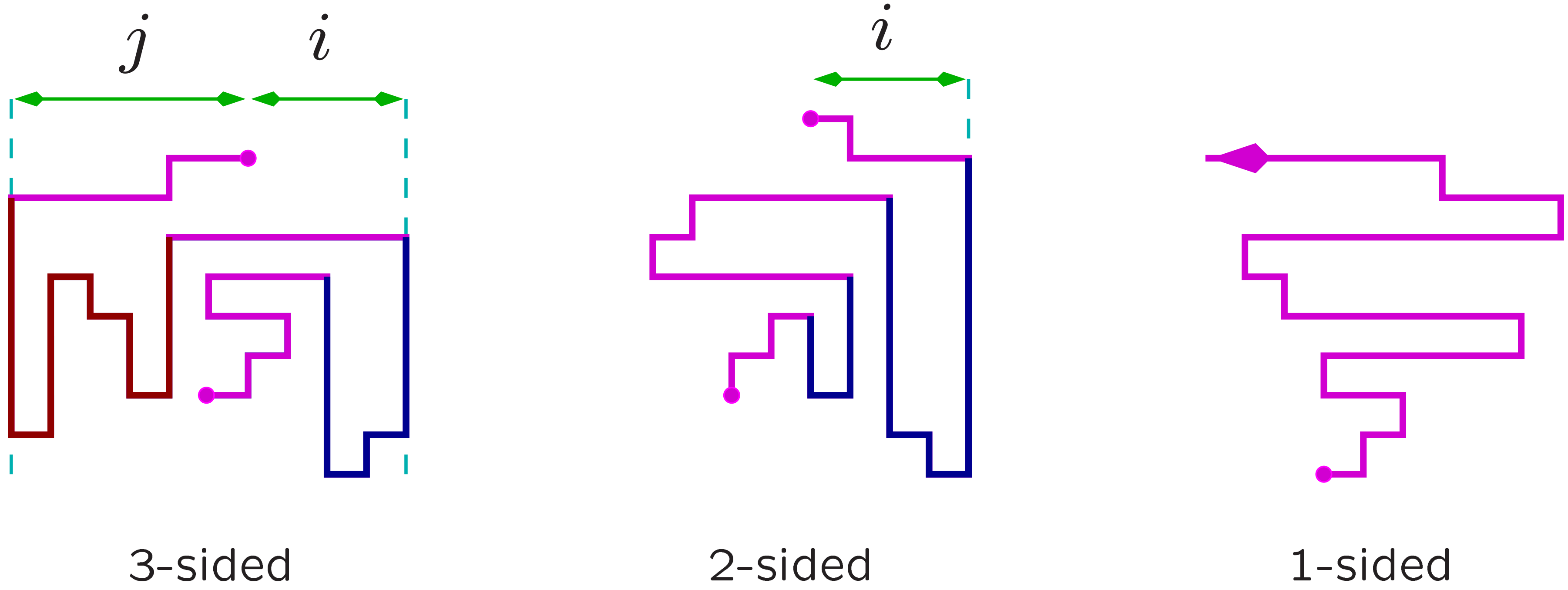
$$\left(1 - \frac{uvwt(1-t^2)}{(u-tv)(v-tu)}\right) T(t;u,v,w) =$$
$$1 + \mathcal{T}(t;w,u) + \mathcal{T}(t;w,v) - tv\frac{\mathcal{T}(t;v,w)}{u-tv} - tu\frac{\mathcal{T}(t;u,w)}{v-tu}$$

with $\mathcal{T}(t;u,v) = tvT(t;u,tu,v)$.

- Generating function of all prudent walks, counted by the length and the half-perimeter of the box:

$$P(t;u) = 1 + 4T(t;u,u,u) - 4T(t;0,u,u)$$

Simpler families of prudent walks [Préa 97]



• The endpoint of a 3-sided walk lies always on the top, right or left side of the box

• The endpoint of a 2-sided walk lies always on the top or right side of the box

• The endpoint of a 1-sided walk lies always on the top side of the box (= partially directed!)

Functional equations for prudent walks: The more general the class, the more additional variables

(Walks ending on the top of the box)

- General prudent walks: three catalytic variables

$$\left(1-\frac{uvwt(1-t^2)}{(u-tv)(v-tu)}\right)T(t;u,v,w)=1+\mathcal{T}(w,u)+\mathcal{T}(w,v)-tv\frac{\mathcal{T}(v,w)}{u-tv}-tu\frac{\mathcal{T}(u,w)}{v-tu}$$

with $\mathcal{T}(u,v)=tvT(t;u,tu,v)$.

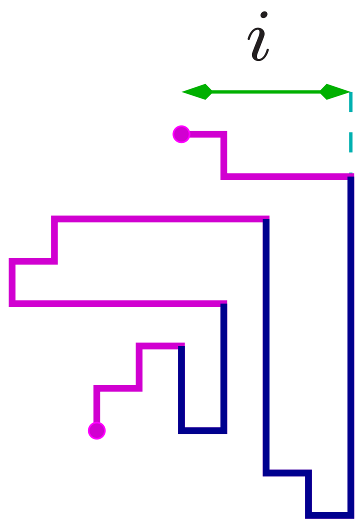
- Three-sided walks: two catalytic variables

$$\left(1-\frac{uvt(1-t^2)}{(u-tv)(v-tu)}\right)T(t;u,v)=1+\cdots-\frac{t^2v}{u-tv}T(t;tv,v)-\frac{t^2u}{v-tu}T(t;u,tu)$$

- Two-sided walks: one catalytic variable

$$\left(1-\frac{tu(1-t^2)}{(1-tu)(u-t)}\right)T(t;u)=\frac{1}{1-tu}+t\,\frac{u-2t}{u-t}\,T(t;t)$$

II. Two-sided prudent walks



Two-sided walks: exact enumeration

Proposition The length generating function of 2-sided walks is:

$$P(t) = \frac{1}{1-2t-2t^2+2t^3}\left(1+t-t^3+t(1-t)\sqrt{\frac{1-t^4}{1-2t-t^2}}\right)$$

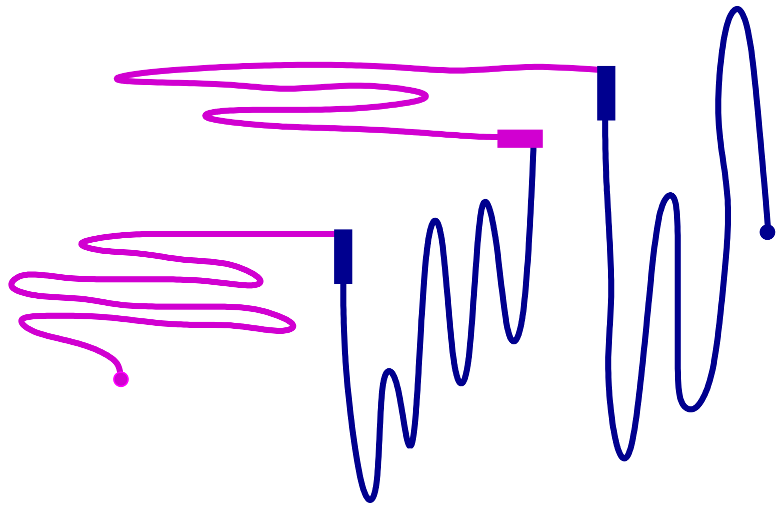
[Duchi 05]

Proofs

- Kernel method applied to

$$\begin{cases} \left((1-tu)(u-t)-tu(1-t^2)\right)T(t;u) & = & u-t+t(u-2t)(1-tu)T(t;t) \\ P(t) & = & 2T(t;1)-T(t;0) \end{cases}$$

- Context-free grammar



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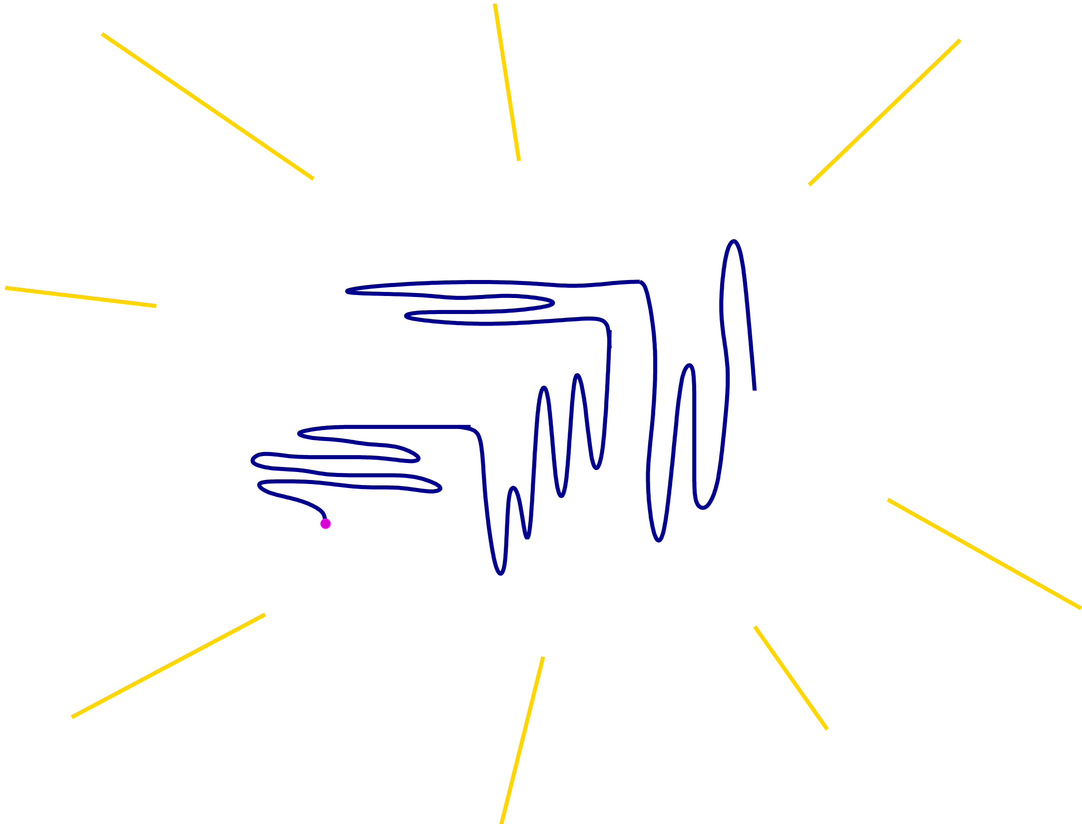


But there He came...

Asymptotic enumeration

Properties of large random objects

Random generation

Two-sided walks: asymptotic enumeration

- The length generating function of 2-sided walks is

$$P(t) = \frac{1}{1 - 2t - 2t^2 + 2t^3} \left(1 + t - t^3 + t(1-t)\sqrt{\frac{1 - t^4}{1 - 2t - t^2}} \right)$$

- Dominant singularity: a simple pole for $1 - 2t - 2t^2 + 2t^3 = 0$, that is, $t_c = 0.40303...$. Asymptotically,

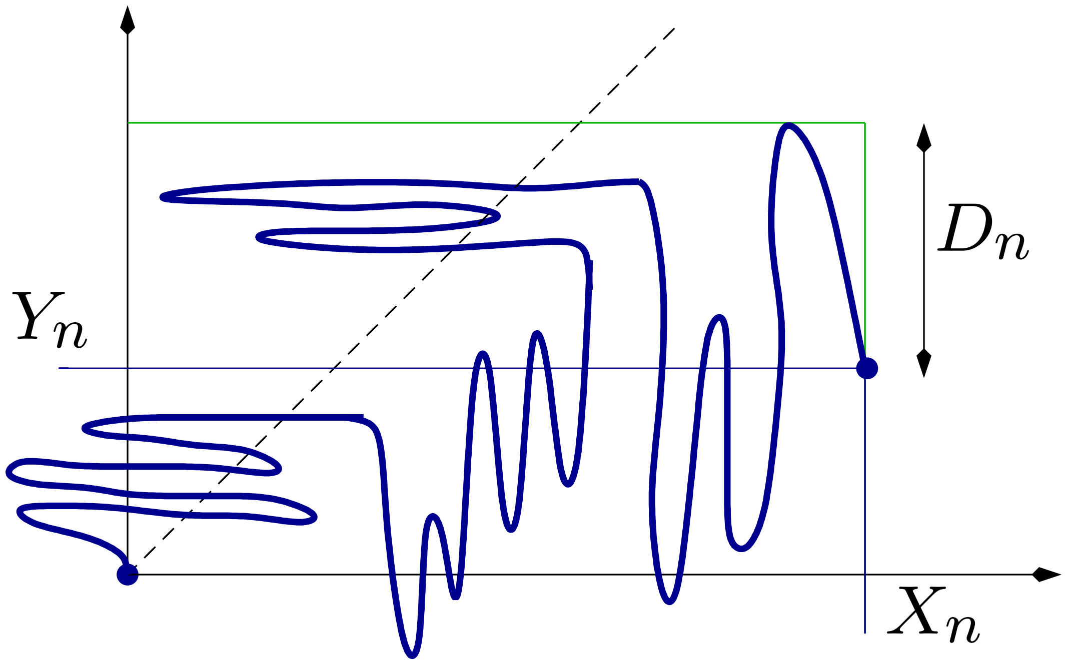
$$p(n) \sim \kappa \, (2.48...)^n$$

Compare with 2.41... for partially directed walks.

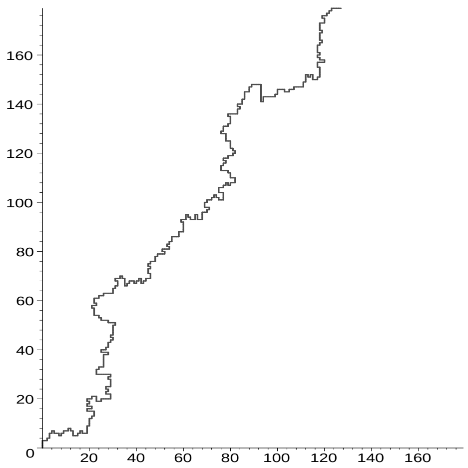
Two-sided walks: properties of large random walks (uniform distribution)

- The random variables X_n, Y_n and D_n satisfy

$$\mathbb{E}(X_n) = \mathbb{E}(Y_n) \sim n \qquad \mathbb{E}((X_n - Y_n)^2) \sim n, \qquad \mathbb{E}(D_n) \sim 4.15\ldots$$



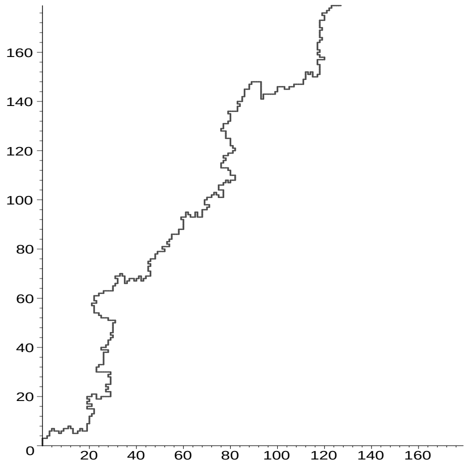
Two-sided walks: random generation (uniform distribution)



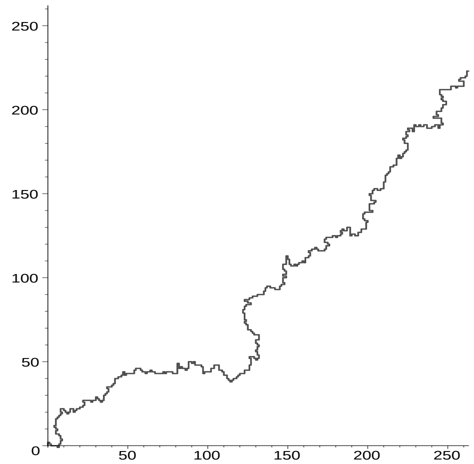
500 steps

• Recursive step-by-step construction à la Wilf $\Rightarrow$ 500 steps (precomputation of $O(n^2)$ large numbers)

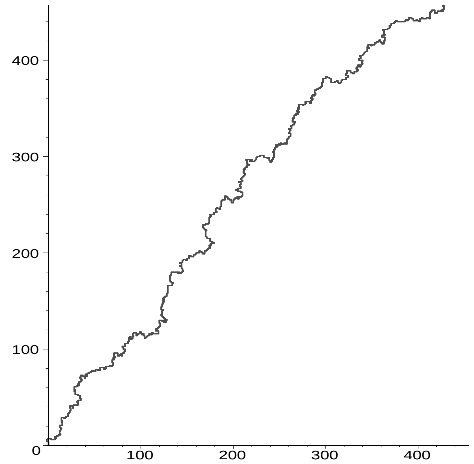
Two-sided walks: random generation (uniform distribution)



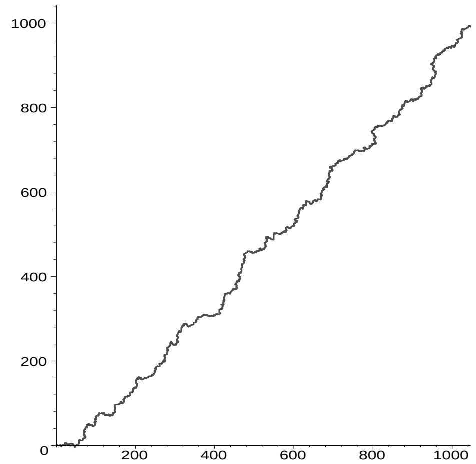
500 steps



780 steps



1354 steps



3148 steps

• Recursive step-by-step construction à la Wilf $\Rightarrow$ 500 steps (precomputation of $O(n^2)$ large numbers)

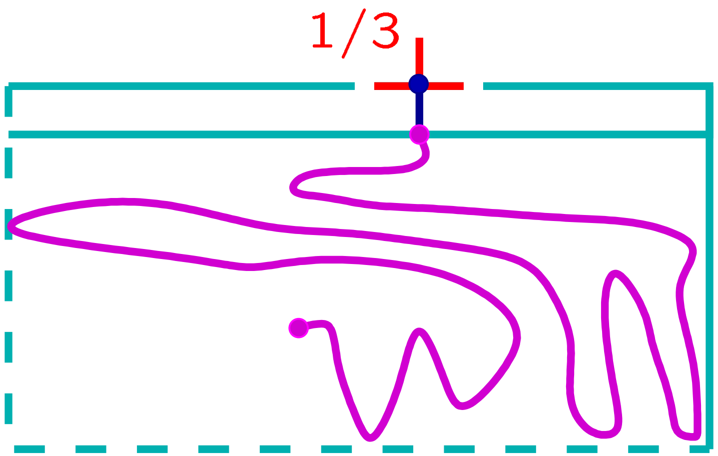
• Boltzmann sampling via the context-free grammar [Duchon-Flajolet-Louchard-Schaeffer 02]

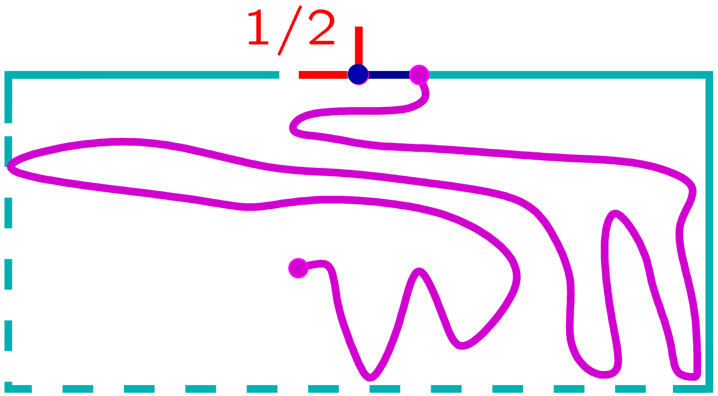
$$\mathbb{E}(X_n) = \mathbb{E}(Y_n) \sim n \qquad \mathbb{E}((X_n - Y_n)^2) \sim n, \qquad \mathbb{E}(D_n) \sim 4.15\ldots$$

Another distribution: Kinetic two-sided walks

At time n, the walk chooses one of the admissible steps with uniform probability.

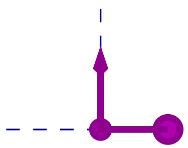
[An admissible step is one that gives a two-sided walk]

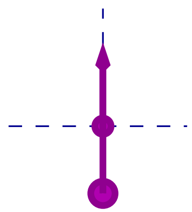




Remark: Walks of length n are no longer uniform

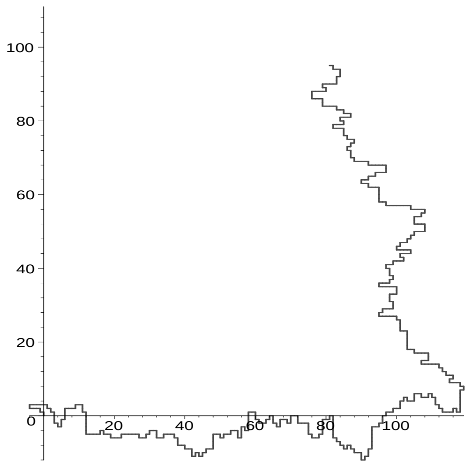
$\frac{1}{4} \cdot \frac{1}{2}$



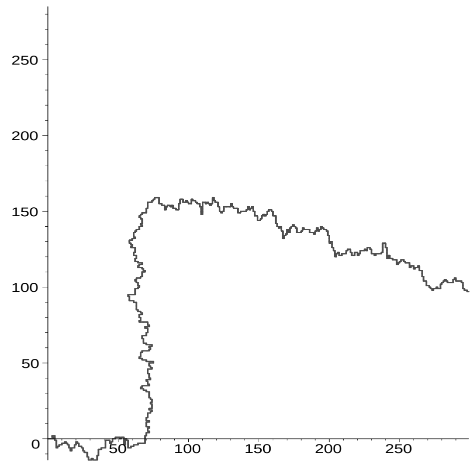


$\frac{1}{4} \cdot \frac{1}{3}$

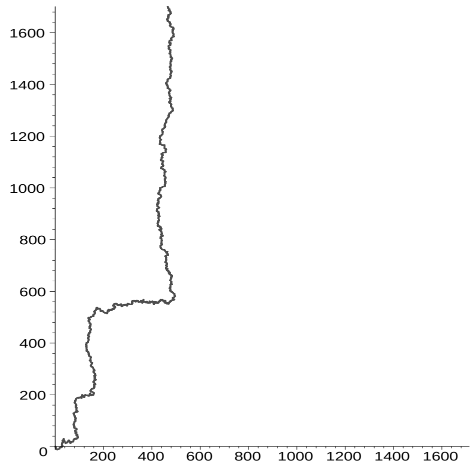
Another distribution: Kinetic two-sided walks



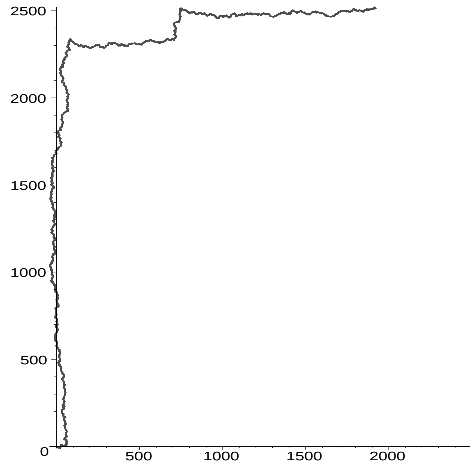
500 steps



1000 steps



5000 steps



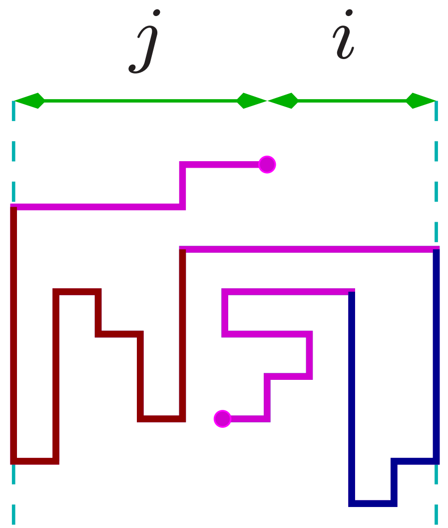
10000 steps

- Random generation: Recursive step-by-step construction à la Wilf (no precomputation)

- Asymptotic properties (from exact enumeration)

$$\mathbb{E}(X_n) = \mathbb{E}(Y_n) \sim n \qquad \mathbb{E}((X_n - Y_n)^2) \sim n^2, \qquad \mathbb{E}(D_n) \sim \sqrt{n}$$

III. Three-sided prudent walks



Three-sided walks: two catalytic variables

$$\left(1 - \frac{uvt(1-t^2)}{(u-tv)(v-tu)}\right) T(t;u,v) = 1 + \cdots - \frac{t^2 v}{u-tv} T(t;tv,v) - \frac{t^2 u}{v-tu} T(t;u,tu)$$

- Cancel the kernel by an appropriate choice of $v \equiv v(t;u)$
- This kernel is homogeneous in u and v

...

Three-sided prudent walks: exact enumeration

- The length generating function of three-sided prudent walks is:

$$P(t) = \frac{1}{1-2t-t^2}\left(\frac{1+3t+tq(1-3t-2t^2)}{1-tq} + 2t^2q\; T(t;1,t)\right)$$

where

$$T(t;1,t) = \sum_{k\geq 0} (-1)^k \frac{\prod_{i=0}^{k-1}\left(\frac{t}{1-tq} - U(q^{i+1})\right)}{\prod_{i=0}^{k}\left(\frac{tq}{q-t} - U(q^i)\right)} \left(1 + \frac{U(q^k)-t}{t(1-tU(q^k))} + \frac{U(q^{k+1})-t}{t(1-tU(q^{k+1}))}\right)$$

with

$$U(w) = \frac{1-tw+t^2+t^3w-\sqrt{(1-t^2)(1+t-tw+t^2w)(1-t-tw-t^2w)}}{2t},$$

and

$$q = U(1) = \frac{1-t+t^2+t^3-\sqrt{(1-t^4)(1-2t-t^2)}}{2t}.$$

Three-sided prudent walks: asymptotic enumeration and singularities

- The length generating function of three-sided prudent walks is:

$$P(t) = \frac{1}{1-2t-t^2}\left(\frac{1+3t+tq(1-3t-2t^2)}{1-tq} + 2t^2 q\, T(t;1,t)\right)$$

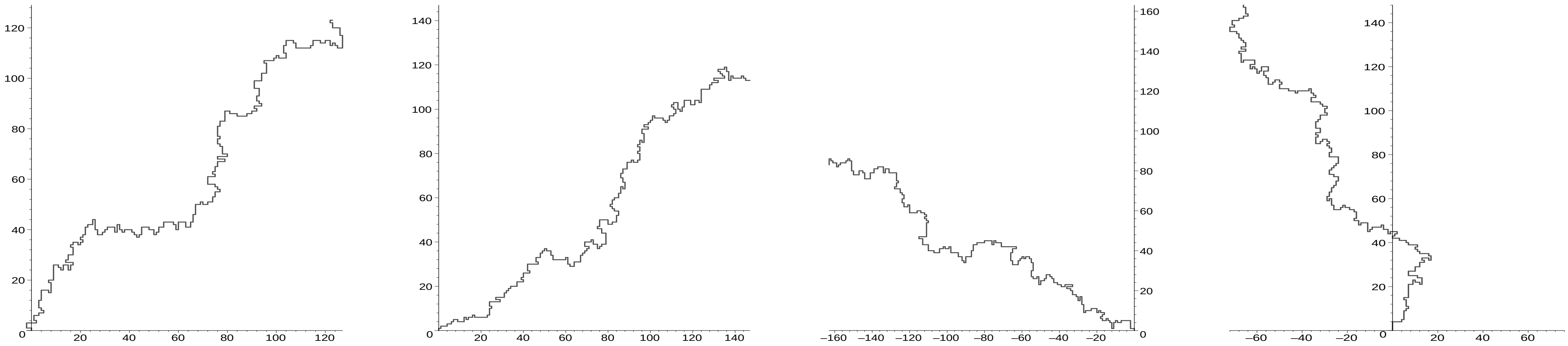
$$T(t;1,t) = \sum_{k\ge 0}(-1)^k \frac{\prod_{i=0}^{k-1}\left(\frac{t}{1-tq} - U(q^{i+1})\right)}{\prod_{i=0}^{k}\left(\frac{tq}{q-t} - U(q^i)\right)}\left(1 + \frac{U(q^k)-t}{t(1-tU(q^k))} + \frac{U(q^{k+1})-t}{t(1-tU(q^{k+1}))}\right)$$

- Asymptotic enumeration: The dominant singularity is (again) a simple pole for $1-2t-2t^2+2t^3=0$. Asymptotically,

$$p(n) \sim \kappa\, (2.48...)^n$$

- Singularity analysis: The series $P(t)$ has infinitely many poles, satisfying $\frac{tq}{q-t} = U(q^i)$ for some $i \ge 0$. Hence it is neither algebraic, nor even D-finite.

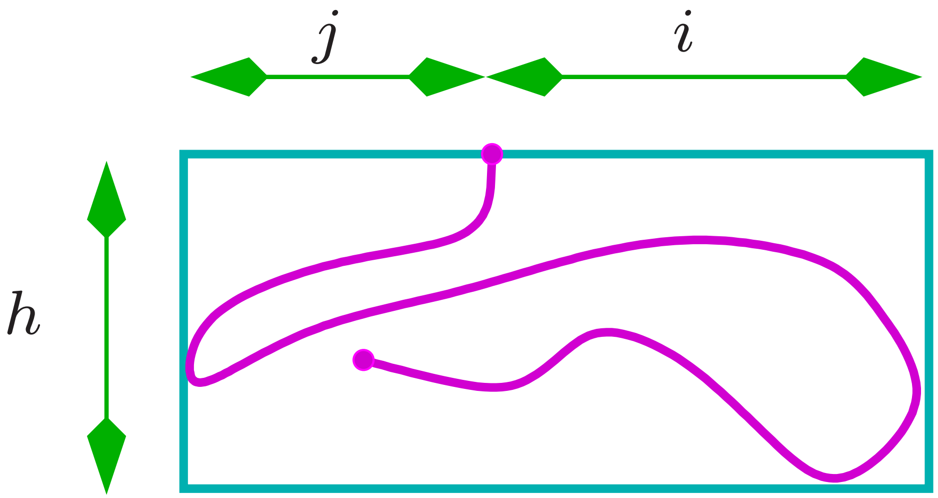
Three-sided prudent walks:
random generation and asymptotic properties



• Random generation: Recursive method à la Wilf $\Rightarrow$ 400 steps (pre-computation of $O(n^3)$ numbers)

• Asymptotic properties: The average width of the box is $\sim n$

IV. Four-sided (i.e. general) prudent walks



General prudent walks: three catalytic variables

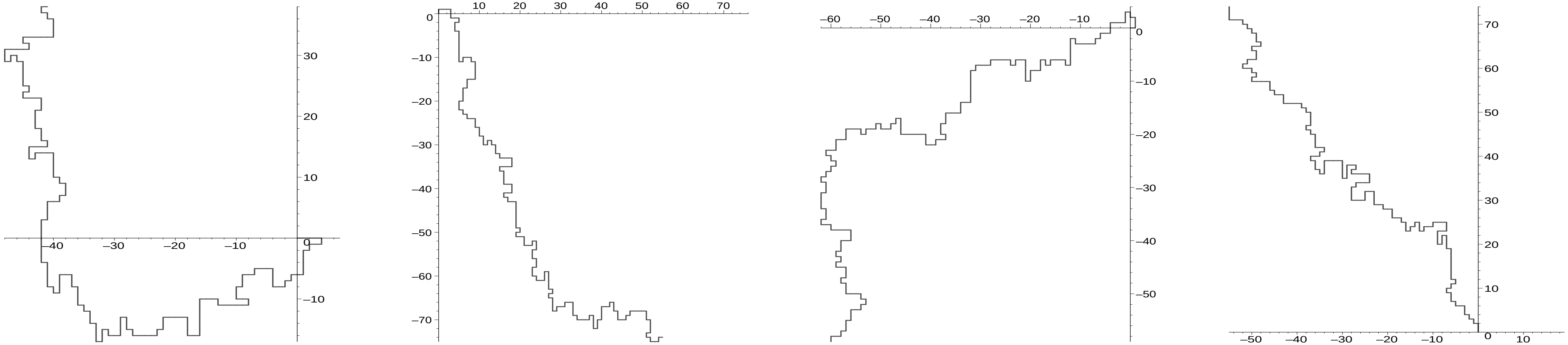
$$\left(1 - \frac{uvwt(1-t^2)}{(u-tv)(v-tu)}\right) T(u,v,w) = 1 + \mathcal{T}(w,u) + \mathcal{T}(w,v) - tv\frac{\mathcal{T}(v,w)}{u-tv} - tu\frac{\mathcal{T}(u,w)}{v-tu}$$

with $\mathcal{T}(u,v) = tvT(u,tu,v)$.

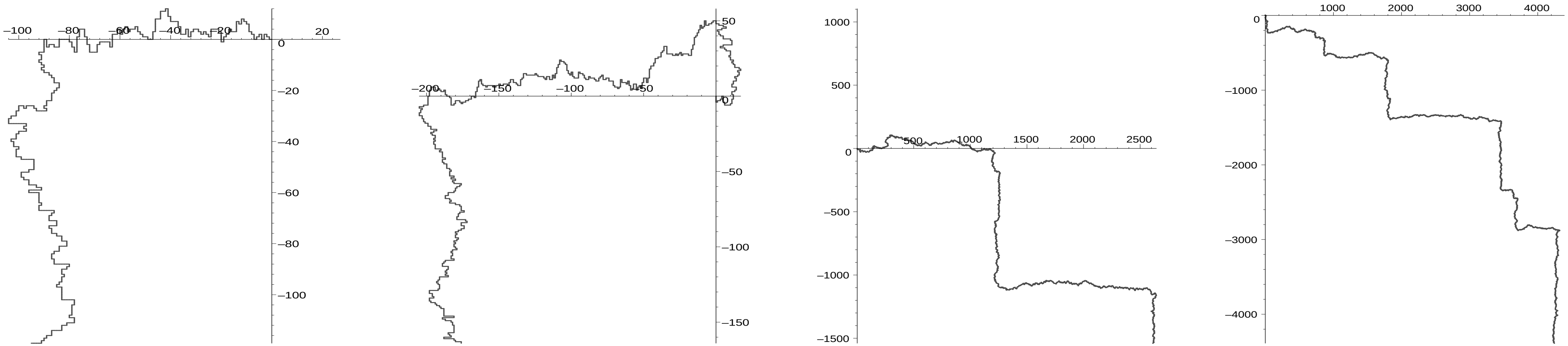
?

Random prudent walks

- Uniform model: recursive generation, 195 steps (sic! $O(n^4)$ numbers)



- Kinetic model: recursive generation with no precomputation



500 steps   1000 steps   10000 steps   20000 steps

Conjectures, and summary of the results

	Nature of the g.f.	Asympt. growth	End-to-end distance
1-sided (part. dir)	Rat.	$(2.41...)^n$	n
2-sided	Alg. [Duchi 05]	$(2.48...)^n$	n
3-sided	not D-finite	$(2.48...)^n$	n
4-sided (general)	not D-finite	$(2.48...)^n$	n
square lattice SAW	?	$(2.63...)^n \, n^{11/32}$	$n^{3/4}$

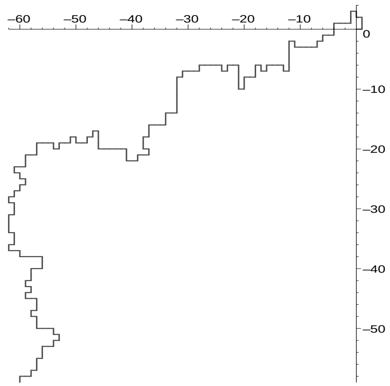
Conjectures: [Dethridge, Guttmann, Jensen 07]

What's next?

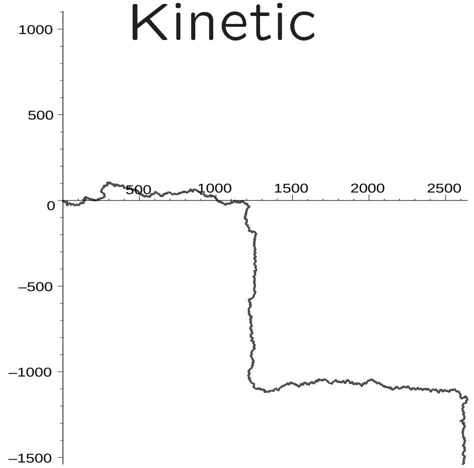
• Exact enumeration: General prudent walks on the square lattice – Growth constant?

• Uniform random generation: better algorithms (maximal length 200 for general prudent walks...)

Uniform



Kinetic



• Kinetic models

• Limit processes?

• More general walks (with A. Bacher), with growth constant 2.54...

Triangular prudent walks

The length generating function of triangular prudent walks is

$$P(t;1) = \frac{6t(1+t)}{1-3t-2t^2}\Big(1 + t\,(1+2t)\,R(t;1,t)\Big)$$

with

$$R(t;1,t) = (1+Y)(1+tY) \sum_{k\geq 0} \frac{t^{\binom{k+1}{2}} \left(Y(1-2t^2)\right)^k}{(Y(1-2t^2);t)_{k+1}} \left(\frac{Yt^2}{1-2t^2};t\right)_k$$

and

$$Y = \frac{1-2t-t^2-\sqrt{(1-t)(1-3t-t^2-t^3)}}{2t^2}$$

Notation:

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}).$$

• The series $P(t;1)$ is neither algebraic, nor even D-finite (infinitely many poles at $Yt^k(1-2t^2)=0$)