The Potts model on planar maps

Olivier Bernardi, Brandeis University, Boston
Mireille Bousquet-Mélou, CNRS, LaBRI, Bordeaux

http://www.labri.fr/~bousquet
Outline

I. Planar maps and the Potts model

II. Main result

III. Where does it come from?

IV. Some special cases
Planar maps

There is a finite number of maps with $n$ edges.
Planar maps

There is a finite number of maps with \( n \) edges.
There is a finite number of maps with $n$ edges.
There is a finite number of maps with $n$ edges.
Planar maps: rooted version

There is a finite number of maps with $n$ edges.
With degree constraints: rooted triangulations
With degree constraints: rooted triangulations

A near-triangulation with outer-degree 5
The partition function of the $q$-state Potts model on a planar map $M$:

$$Z_M(q, \nu) = \sum_{c: V(M) \rightarrow \{1,2,...,q\}} \nu^{m(c)}$$

where $m(c)$ is the number of monochromatic edges in the colouring $c$. In fact, $Z_M(q, \nu)$ is a polynomial in $q$ (and $\nu$), divisible by $q$.

Example: When $M$ has one edge and two vertices,

$$Z_M(q, \nu) = q\nu + q(q - 1)$$
The Potts model on planar maps

- Generating function:

$$M_1(q, \nu, w, t) = \frac{1}{q} \sum_M Z_M(q, \nu) w^{v(M)} t^{e(M)} = \frac{1}{q} \sum_{M, c} w^{v(M)} t^{e(M)} \nu^{m(c)}$$

$$= w + (\nu w + \nu w^2 + (q - 1)w^2) t + O(t^2)$$

"The Potts generating function of planar maps"

⇒ Enumeration of $q$-coloured planar maps, counted by vertices, edges, and monochromatic edges.

\[ w \quad \nu w \quad \nu w^2 \quad (q - 1)w^2 \]
A possible answer [Tutte 68]

- Consider the refined Potts generating function:

\[ M(x, y) \equiv M(q, \nu, w, t; x, y) = \frac{1}{q} \sum_M Z_M(q, \nu) w^{v(M)} t^{e(M)} x^{d_v(M)} y^{d_f(M)}, \]

where \( d_v(M) \) (resp. \( d_f(M) \)) is the degree of the root-vertex (resp. root-face).
A possible answer [Tutte 68]

• Consider the refined Potts generating function:

\[ M(x, y) \equiv M(q, \nu, w, t; x, y) = \frac{1}{q} \sum_M Z_M(q, \nu) w^{v(M)} t^{e(M)} x^{d_v(M)} y^{d_f(M)}, \]

where \( d_v(M) \) (resp. \( d_f(M) \)) is the degree of the root-vertex (resp. root-face).

• By deletion/contraction of the root-edge:

\[
M(x, y) = w + xyt ((\nu - 1)(y - 1) + qy) M(x, y) M(1, y) \\
+ xyt/w (x\nu - 1) M(x, y) M(x, 1) \\
+ xytw(\nu - 1) \left( \frac{x M(x, y) - M(1, y)}{x - 1} \right) \\
+ xyt \frac{y M(x, y) - M(x, 1)}{y - 1}.
\]

A discrete partial differential equation with two catalytic variables
Are we happy?

The refined Potts generating function satisfies:

\[ M(x, y) = w + xyt ((\nu - 1)(y - 1) + qy) M(x, y)M(1, y) \]
\[ + xyt/w (x\nu - 1) M(x, y)M(x, 1) \]
\[ + xyw(t(\nu - 1) \frac{xM(x, y) - M(1, y)}{x - 1} + xyt \frac{yM(x, y) - M(x, 1)}{y - 1}. \]

- compute coefficients efficiently (polynomial time)
- What is \( M(1, 1) \equiv M(q, \nu, w, t; 1, 1) \)?
- asymptotics? phase transitions?
Are we happy?

The refined Potts generating function satisfies:

\[
M(x, y) = w + xyt ((\nu - 1)(y - 1) + qy) M(x, y)M(1, y) \\
+ xyt/w(x\nu - 1)M(x, y)M(x, 1) \\
+ ywt(\nu - 1)\frac{xM(x, y) - M(1, y)}{x - 1} + yt\frac{yM(x, y) - M(x, 1)}{y - 1}.
\]

- compute coefficients efficiently (polynomial time)
- What is \( M(1, 1) \equiv M(q, \nu, w, t; 1, 1) \)?
- asymptotics? phase transitions?
Are we happy?

The refined Potts generating function satisfies:

\[
M(x, y) = w + xyt ((\nu - 1)(y - 1) + qy) M(x, y)M(1, y) \\
+ xyt/w(x\nu - 1)M(x, y)M(x, 1) \\
+ xyt/\nu(x - 1) \frac{M(x, y) - M(1, y)}{x - 1} + xyt \frac{yM(x, y) - M(x, 1)}{y - 1}.
\]

- compute coefficients efficiently (polynomial time)
- What is \(M(1, 1) \equiv M(q, \nu, w, t; 1, 1)\)?
- asymptotics? phase transitions?
Experience makes us greedier: uncoloured maps ("pure gravity")

\[ G(t; y) \equiv G(y) = \sum_{M} t e(M) y df(M) \]

where \( e(M) \) is the number of edges and \( df(M) \) the degree of the outer face. Then by deletion of the root-edge [Tutte 68]:

\[ G(y) = 1 + ty^2 G(y) + ty y G(y) - G(1) y - 1 \]

A discrete differential equation with one catalytic variable, \( y \).

The solution is an algebraic function!

\[ G(1) = \sum_{M} t e(M) = \left(1 - \frac{1}{2} t \right)^{3/2} - 1 + \frac{1}{8} t - \frac{1}{4} = 0 \]
Experience makes us greedier: uncoloured maps ("pure gravity")

- Let

\[ G(t; y) \equiv G(y) = \sum_{M} t^{e(M)} y^{df(M)} \]

where \( e(M) \) is the number of edges and \( df(M) \) the degree of the outer face. Then by deletion of the root-edge [Tutte 68]:

\[ G(y) = 1 + ty^2 G(y)^2 + ty \frac{yG(y) - G(1)}{y - 1} \]

A discrete differential equation with one catalytic variable, \( y \).
Experience makes us greedier: uncoloured maps (“pure gravity”)

- Let

$$G(t; y) \equiv G(y) = \sum_M t^{e(M)} y^{df(M)}$$

where $e(M)$ is the number of edges and $df(M)$ the degree of the outer face. Then by deletion of the root-edge [Tutte 68]:

$$G(y) = 1 + ty^2 G(y)^2 + ty \frac{yG(y) - G(1)}{y - 1}$$

A discrete differential equation with one catalytic variable, $y$.

- The solution is an algebraic function!

$$G(1) = \sum_M t^{e(M)} = \frac{(1 - 12t)^{3/2} - 1 + 18t}{54t^2}$$

Equivalently,

$$27 t^2 G(1)^2 + (1 - 18t) G(1) + 16t - 1 = 0$$
The refined Potts generating function satisfies:

\[ M(x, y) = w + xyt ((\nu - 1)(y - 1) + qy) M(x, y)M(1, y) + xyt/w(x\nu - 1)M(x, y)M(x, 1) \]
\[ + xyt/w(x\nu - 1)M(x, y)M(x, 1) + xyt \left( xM(x, y) - M(1, y) \right) \frac{x}{x - 1} + xyt \left( yM(x, y) - M(x, 1) \right) \frac{y}{y - 1}. \]

- compute coefficients efficiently (poly. time)
- What is \( M(1, 1) \equiv M(q, \nu, w, t; 1, 1) \)?
- asymptotics? phase transitions?
- Is \( M(1, 1) \) algebraic?
An encouraging sign: the Ising model ($q = 2$)

**Theorem**

Let $\nu = (1 + \nu)/(1 - \nu)$. The Ising generating function of near-triangulations with outer degree 1 is

$$T_1(2, \nu, t) = \frac{(S - \nu)^2 (S - 2 + \nu) (-2 \nu + \nu^2 - Sv - S^2\nu + 3 S^3)}{128t^4 (1 + \nu)^4 S^2},$$

where $S$ is the unique series in $t$ with constant term $\nu$ satisfying:

$$S = \nu + t^3 \frac{64 (1 + \nu)^3 S^2}{(S - 2 + \nu) (2 \nu - \nu^2 + 2 S + S^2 - 4 S^3)}.$$

In particular, it is algebraic.

⇒ Asymptotics, exponents, transition at $\nu = 1 + 1/\sqrt{7}$
[Boulatov & Kazakov 87], [mbm & Schaeffer 02], [Bouttier, Di Francesco & Guitter 04]
Another special case: maps equipped with a spanning tree (Potts in the limit $q \to 0, \nu \to 1$) [Mullin 67]

**Theorem**

The GF of planar maps equipped with a spanning tree is

$$M_1(0, 1, 1, t) = \sum_{n \geq 0} \frac{1}{(n + 1)(n + 2)} \binom{2n}{n} \binom{2n + 2}{n + 1} t^n.$$ 

This series is **transcendental** (= non-algebraic), but **D-finite** (solution of a linear DE with polynomial coefficients).

⇒ Forget about algebraicity in general.

OK, but... is Potts D-finite?
A hierarchy of formal power series

- Rational series
  \[ A(t) = \frac{P(t)}{Q(t)} \]

- Algebraic series
  \[ P(t, A(t)) = 0 \]

- Differentially finite series (D-finite)
  \[ \sum_{i=0}^{d} P_i(t) A^{(i)}(t) = 0 \]

- D-algebraic series
  \[ P(t, A(t), A'(t), \ldots, A^{(d)}(t)) = 0 \]
Another special case: maps equipped with a connected subgraph
(Potts in the limit $q \to 0$)

**Theorem** [mbm & Courtiel 13(a)]

The generating function of triangulations equipped with a connected subgraph, counted by edges and by the size of the subgraph, is not **D-finite**.

But... it is **D-algebraic** (2nd order non-linear DE).

⇒ Forget about D-finiteness in general

OK, but... is Potts D-algebraic?
II. Main result
The Potts generating function is D-algebraic

**Theorem**

The Potts generating function of planar maps:

\[ M_1(q, \nu, w, t) = \frac{1}{q} \sum_M Z_M(q, \nu) w^{v(M)} t^{e(M)}, \]

is

- **algebraic** if \( q = 2 + 2 \cos \frac{j \pi}{m}, \ q \neq 0, 4 \) (includes \( q = 2, 3 \))
- **D-algebraic** (over \( \mathbb{Q}(q, \nu, w, t) \)) when \( q \) is an indeterminate

The same holds for triangulations.

[mbm-Bernardi 09(a)] Counting coloured planar maps: algebraicity results.
[mbm-Bernardi 15(a)] Counting coloured planar maps: differential equations

cf. [Eynard & Bonnet 99]: algebraicity w.r.t. the catalytic variable \( y \) (for near-triangulations)
An explicit differential system for Potts on planar maps

Let \( D(t, u) = (q\nu + \beta^2)u^2 - q(\nu + 1)u + \beta t(q - 4)(wq + \beta) + q, \)
with \( \beta = \nu - 1. \)
An explicit differential system for Potts on planar maps

Let $D(t, u) = (q\nu + \beta^2)u^2 - q(\nu + 1)u + \beta t(q - 4)(wq + \beta) + q$, with $\beta = \nu - 1$.

- There exists a unique 11-tuple $(P_0(t), \ldots, P_4(t), Q_0(t), \ldots, Q_2(t), R_0(t), \ldots, R_2(t))$ of series in $t$ with coefficients in $\mathbb{Q}(q, \nu, w)$ such that
An explicit differential system for Potts on planar maps

Let \( D(t, u) = (q\nu + \beta^2)u^2 - q(\nu + 1)u + \beta t(q - 4)(wq + \beta) + q, \) 
with \( \beta = \nu - 1. \)

- There exists a unique 11-tuple \((P_0(t), \ldots, P_4(t), Q_0(t), \ldots, Q_2(t), R_0(t), \ldots, R_2(t))\) of series in \( t \) with coefficients in \( \mathbb{Q}(q, \nu, w) \) such that
  \[
  P_4(t) = Q_2(t) = 1, \quad R_2(t) = \nu + 1 - w(q + 2\beta),
  \]
  and
Let $D(t, u) = (q\nu + \beta^2)u^2 - q(\nu + 1)u + \beta t(q - 4)(wq + \beta) + q$, with $\beta = \nu - 1$.

- There exists a unique 11-tuple $(P_0(t), \ldots, P_4(t), Q_0(t), \ldots, Q_2(t), R_0(t), \ldots, R_2(t))$ of series in $t$ with coefficients in $\mathbb{Q}(q, \nu, w)$ such that

$$P_4(t) = Q_2(t) = 1, \quad R_2(t) = \nu + 1 - w(q + 2\beta),$$

and

$$\frac{1}{Q} \frac{\partial}{\partial t} \left( \frac{Q^2}{PD^2} \right) = \frac{1}{R} \frac{\partial}{\partial u} \left( \frac{R^2}{PD^2} \right),$$

with $P \equiv P(t, u) = P_0(t) + P_1(t)u + \cdots + P_4(t)u^4$ and so on,
An explicit differential system for Potts on planar maps

Let \( D(t, u) = (q\nu + \beta^2)u^2 - q(\nu + 1)u + \beta t(q - 4)(wq + \beta) + q, \) with \( \beta = \nu - 1. \)

- There exists a unique 11-tuple \((P_0(t), \ldots, P_4(t), Q_0(t), \ldots, Q_2(t), R_0(t), \ldots, R_2(t))\) of series in \( t \) with coefficients in \( \mathbb{Q}(q, \nu, w) \) such that

\[
P_4(t) = Q_2(t) = 1, \quad R_2(t) = \nu + 1 - w(q + 2\beta),
\]

and

\[
\frac{1}{Q} \frac{\partial}{\partial t} \left( \frac{Q^2}{PD^2} \right) = \frac{1}{R} \frac{\partial}{\partial u} \left( \frac{R^2}{PD^2} \right),
\]

with \( P \equiv P(t, u) = P_0(t) + P_1(t)u + \cdots + P_4(t)u^4 \) and so on, and the initial conditions (at \( t = 0 \)):

\[
P(0, u) = u^2(u - 1)^2 \quad \text{and} \quad Q(0, u) = u(u - 1).
\]
An explicit differential system for Potts on planar maps

Let \( D(t, u) = (q \nu + \beta^2)u^2 - q(\nu + 1)u + \beta t(q - 4)(wq + \beta) + q \),
with \( \beta = \nu - 1 \).

• There exists a unique 11-tuple \((P_0(t), \ldots, P_4(t), Q_0(t), \ldots, Q_2(t), R_0(t), \ldots, R_2(t))\) of series in \( t \) with coefficients in \( \mathbb{Q}(q, \nu, w) \) such that

\[
P_4(t) = Q_2(t) = 1, \quad R_2(t) = \nu + 1 - w(q + 2\beta),
\]

and

\[
\frac{1}{Q} \frac{\partial}{\partial t} \left( \frac{Q^2}{PD^2} \right) = \frac{1}{R} \frac{\partial}{\partial u} \left( \frac{R^2}{PD^2} \right),
\]

with \( P \equiv P(t, u) = P_0(t) + P_1(t)u + \cdots + P_4(t)u^4 \) and so on, and the initial conditions (at \( t = 0 \)):

\[
P(0, u) = u^2(u - 1)^2 \quad \text{and} \quad Q(0, u) = u(u - 1).
\]

• The Potts GF of planar maps \( M_1 \) is an explicit polynomial in the \( P_i \)'s and \( Q_i \)'s.
Is this a differential system?

... or a partial differential equation?

\[
\frac{1}{Q} \frac{\partial}{\partial t} \left( \frac{Q^2}{PD^2} \right) = \frac{1}{R} \frac{\partial}{\partial u} \left( \frac{R^2}{PD^2} \right),
\]

with \( P \equiv P(t, u) = P_0(t) + P_1(t)u + \cdots + P_4(t)u^4 \) and so on.
Is this a differential system?

... or a partial differential equation?

\[ \frac{1}{Q} \frac{\partial}{\partial t} \left( \frac{Q^2}{PD^2} \right) = \frac{1}{R} \frac{\partial}{\partial u} \left( \frac{R^2}{PD^2} \right), \]

with \( P \equiv P(t, u) = P_0(t) + P_1(t)u + \cdots + P_4(t)u^4 \) and so on.

**Both:** The equation reads

\[ 2Q_t PD - QP_t D - 2QPD_t = 2R_u PD - RP_u D - 2RPD_u. \]

Extracting the coefficient of \( u^0, \ldots, u^7 \) gives a system of 8 DEs (in \( t \)) between the 8 unknowns series. For \( u^7 \) for instance, one finds:

\[ P'_3(t) - 2Q'_1(t) + 4(1 + \nu) - 4w(2\beta + q) = 0 \]
Is this a differential system?

... or a partial differential equation?

\[
\frac{1}{Q} \frac{\partial}{\partial t} \left( \frac{Q^2}{PD^2} \right) = \frac{1}{R} \frac{\partial}{\partial u} \left( \frac{R^2}{PD^2} \right),
\]

with \( P \equiv P(t, u) = P_0(t) + P_1(t)u + \cdots + P_4(t)u^4 \) and so on.

Both: The equation reads

\[
2Q_tPD - QP_tD - 2QPD_t = 2R_uPD - RP_uD - 2RPD_u.
\]

Extracting the coefficient of \( u^0, \ldots, u^7 \) gives a system of 8 DEs (in \( t \)) between the 8 unknowns series. For \( u^7 \) for instance, one finds:

\[
P_3'(t) - 2Q_1'(t) + 4(1 + \nu) - 4w(2\beta + q) = 0
\]

Elimination \( \Rightarrow \) existence of a DE of order 5 for \( M_1 \).
An explicit differential system for Potts on planar maps

Let \( D(t, u) = (q\nu + \beta^2)u^2 - q(\nu + 1)u + \beta t(q - 4)(wq + \beta) + q, \) with \( \beta = \nu - 1. \)

• There exists a unique 11-tuple \((P_0(t), \ldots, P_4(t), Q_0(t), \ldots, Q_2(t), R_0(t), \ldots, R_2(t))\) of series in \( t \) with coefficients in \( \mathbb{Q}(q, \nu, w) \) such that

\[
P_4(t) = Q_2(t) = 1, \quad R_2(t) = \nu + 1 - w(q + 2\beta),
\]

and

\[
\frac{1}{Q} \frac{\partial}{\partial t} \left( \frac{Q^2}{PD^2} \right) = \frac{1}{R} \frac{\partial}{\partial u} \left( \frac{R^2}{PD^2} \right),
\]

with \( P \equiv P(t, u) = P_0(t) + P_1(t)u + \cdots + P_4(t)u^4 \) and so on, and the initial conditions (at \( t = 0 \)):

\[
P(0, u) = u^2(u - 1)^2 \quad \text{and} \quad Q(0, u) = u(u - 1).
\]

• The Potts GF of planar maps \( M_1 \) is an explicit polynomial in the \( P_i \)'s and \( Q_i \)'s \( \Rightarrow \) DE of order 5
Let \( D(t, u) = q\nu^2 u^2 + \beta(4\beta + q)u + q\beta\nu(q - 4)t + \beta^2 \), with \( \beta = \nu - 1 \).

- There exists a unique 9-tuple \((P_0(t), \ldots, P_3(t), Q_0(t), \ldots, Q_2(t), R_0(t), R_1(t))\) of series in \( t \) with coefficients in \( \mathbb{Q}(q, \nu) \) such that
  \[ P_3(t) = 1, \quad Q_2(t) = 2\nu, \]

and

\[
\frac{1}{Q} \frac{\partial}{\partial t} \left( \frac{Q^2}{PD^2} \right) = \frac{1}{R} \frac{\partial}{\partial u} \left( \frac{R^2}{PD^2} \right),
\]

with \( P \equiv P(t, u) = P_0(t) + P_1(t)u + \cdots + P_3(t)u^3 \) and so on, and the initial conditions (at \( t = 0 \)):

\[ P(0, u) = u^2(u + 1/4) \quad \text{and} \quad Q(0, u) = u(2\nu u + 1). \]

- The Potts GF \( T_1 \) of near-triangulations (outer degree 1) \( T_1 \) is an explicit polynomial in the \( P_i \)'s and \( Q_i \)'s ⇒ DE of order 4
III. Where does it come from?

\[
\frac{1}{Q} \frac{\partial}{\partial t} \left( \frac{Q^2}{PD^2} \right) = \frac{1}{R} \frac{\partial}{\partial u} \left( \frac{R^2}{PD^2} \right)
\]

A (vague) idea of the proof
In the footsteps of W. Tutte

- For the GF $T(x, y)$ of properly $q$-coloured triangulations:

$$T(x, y) = x(q - 1) + xyzT(x, y)T(1, y) + xz \frac{T(x, y) - T(x, 0)}{y} - x^2 yz \frac{T(x, y) - T(1, y)}{x - 1}$$

[Tutte 73] Chromatic sums for rooted planar triangulations: the cases $\lambda = 1$ and $\lambda = 2$

[Tutte 73] Chromatic sums for rooted planar triangulations, II: the case $\lambda = \tau + 1$

[Tutte 73] Chromatic sums for rooted planar triangulations, III: the case $\lambda = 3$

[Tutte 73] Chromatic sums for rooted planar triangulations, IV: the case $\lambda = \infty$

[Tutte 74] Chromatic sums for rooted planar triangulations, V: special equations

[Tutte 78] On a pair of functional equations of combinatorial interest

[Tutte 82] Chromatic solutions

[Tutte 82] Chromatic solutions II

[Tutte 84] Map-colourings and differential equations

[Tutte 95]: Chromatic sums revisited
1. Tutte’s equation with two catalytic variables:

\[ M(x, y) = w + xyt ((\nu - 1)(y - 1) + qy) M(x, y)M(1, y) \]
\[ + xyt/w(x\nu - 1)M(x, y)M(x, 1) \]
\[ + xywt(\nu - 1) \frac{xM(x, y) - M(1, y)}{x - 1} + xyt \frac{yM(x, y) - M(x, 1)}{y - 1} \]
1. Tutte’s equation with two catalytic variables:

\[ \text{Pol}(M(x, y), M(x, 1), M(1, y), x, y) = 0 \]
Structure of the proof

1. Tutte's equation with two catalytic variables:
   \[ \text{Pol}(M(x, y), M(x, 1), M(1, y), x, y) = 0 \]

2. For \( q = 2 + 2 \cos(j \pi/m) \), there also exists an equation with only one catalytic variable defining \( M(1, y) \):
   \[ \text{Pol}_{j,m}(M(1, y), M(1, 1), M_y(1, 1), M_{y,y}(1, 1), \ldots, y) = 0 \]
1. Tutte’s equation with two catalytic variables:

\[
\text{Pol}(M(x, y), M(x, 1), M(1, y), x, y) = 0
\]

2. For \( q = 2 + 2\cos(j\pi/m) \), there also exists an equation with only one catalytic variable defining \( M(1, y) \):

\[
\text{Pol}_{j,m}(M(1, y), M(1, 1), M_y(1, 1), M_{y,y}(1, 1), \ldots, y) = 0
\]

3. Derive from it a polynomial equation for the Potts generating function \( M_1 = M(1, 1) \).
1. Tutte’s equation with two catalytic variables:

\[ \text{Pol}(M(x, y), M(x, 1), M(1, y), x, y) = 0 \]

2. For \( q = 2 + 2\cos(j\pi/m) \), there also exists an equation with only one catalytic variable defining \( M(1, y) \):

\[ \text{Pol}_{j,m}(M(1, y), M(1, 1), M_y(1, 1), M_{y,y}(1, 1), \ldots, y) = 0 \]  

[B-mbm 09(a)]

3. Derive from it a differential system for the Potts generating function \( M_1 = M(1, 1) \).
1. Tutte's equation with two catalytic variables:
   \[
   \text{Pol}(M(x, y), M(x, 1), M(1, y), x, y) = 0
   \]

2. For \( q = 2 + 2 \cos(j\pi/m) \), there also exists an equation with only one catalytic variable defining \( M(1, y) \):
   \[
   \text{Pol}_{j,m}(M(1, y), M(1, 1), M_y(1, 1), M_{y,y}(1, 1), \ldots, y) = 0
   \]

3. Derive from it a differential system for the Potts generating function \( M_1 = M(1, 1) \).

4. It depends polynomially on \( q \), and is thus valid for any \( q \).
1. Tutte's equation with two catalytic variables:

\[ \text{Pol}(M(x, y), M(x, 1), M(1, y), x, y) = 0 \]

2. For \( q = 2 + 2\cos(j\pi/m) \), there also exists an equation with only one catalytic variable defining \( M(1, y) \):

\[ \text{Pol}_{j,m}(M(1, y), M(1, 1), M_y(1, 1), M_{xy}(1, 1), \ldots, y) = 0 \]  

[B-mbm 09(a)]

3. Derive from it a polynomial equation for the Potts generating function \( M_1 = M(1, 1) \).

4. It depends polynomially on \( q \), and is thus valid for any \( q \).
1. Tutte’s equation with two catalytic variables:
\[ \text{Pol}(M(x, y), M(x, 1), M(1, y), x, y) = 0 \]

2. For \( q = 2 + 2 \cos(j \pi / m) \), there also exists an equation with only one catalytic variable defining \( M(1, y) \):
\[ \text{Pol}_{j,m}(M(1, y), M(1, 1), M_y(1, 1), M_{y,y}(1, 1), \ldots, y) = 0 \] [B-mbm 09(a)]

3. Derive from it a differential system for the Potts generating function \( M_1 = M(1, 1) \).

4. It depends polynomially on \( q \), and is thus valid for any \( q \).
Planar maps: algebraic solution

- The functional equation

\[ G(y) = 1 + ty^2 G(y)^2 + ty \frac{y G(y) - G(1)}{y - 1} \]
Planar maps: algebraic solution

- The functional equation, written with a square:

\[ (2ty^2(y - 1)G(y) + ty^2 - y + 1)^2 \]

\[ = (y - 1 - y^2t)^2 - 4ty^2(y - 1)^2 + 4t^2y^3(y - 1)G_1 := \Delta(y) \]

(a polynomial in $y$)
Planar maps: algebraic solution

- The functional equation, written with a square:
  \[(2ty^2(y - 1)G(y) + ty^2 - y + 1)^2\]
  \[= (y - 1 - y^2t)^2 - 4ty^2(y - 1)^2 + 4t^2y^3(y - 1)G_1 \equiv \Delta(y)\]
  (a polynomial in \(y\))

- There exists a (unique) series \(Y \equiv Y(t)\) that cancels the LHS:
  \[Y = 1 + tY^2 + 2tY^2(Y - 1)G(Y).\]

  \(\Rightarrow\) characterizes inductively the coefficient of \(t^n\)
Planar maps: algebraic solution

- The functional equation, written with a square:

\[
(2ty^2(y - 1)G(y) + ty^2 - y + 1)^2 = (y - 1 - y^2 t)^2 - 4ty^2(y - 1)^2 + 4t^2y^3(y - 1)G_1 := \Delta(y)
\]

(a polynomial in \(y\))

- There exists a (unique) series \(Y \equiv Y(t)\) that cancels the LHS:

\[
Y = 1 + tY^2 + 2tY^2(Y - 1)G(Y).
\]

\(\Rightarrow\) characterizes inductively the coefficient of \(t^n\)

- This series \(Y\) must be a root of \(\Delta(y)\), and in fact a \textbf{double} root.
Planar maps: algebraic solution

- The functional equation, written with a square:

\[(2ty^2(y - 1)G(y) + ty^2 - y + 1)^2 = (y - 1 - y^2t)^2 - 4ty^2(y - 1)^2 + 4t^2y^3(y - 1)G_1 := \Delta(y)\]

(a polynomial in \(y\))

- There exists a (unique) series \(Y \equiv Y(t)\) that cancels the LHS:

\[Y = 1 + tY^2 + 2tY^2(Y - 1)G(Y)\]

\(\Rightarrow\) characterizes inductively the coefficient of \(t^n\)

- This series \(Y\) must be a root of \(\Delta(y)\), and in fact a double root.

- Algebraic consequence: The discriminant of \(\Delta(y)\) w.r.t. \(y\) is zero:

\[27t^2G_1^2 + (1 - 18t)G_1 + 16t - 1 = 0\]
Planar maps: differential solution

- The polynomial

\[ \Delta(y) = (y - 1 - y^2 t)^2 - 4ty^2(y - 1)^2 + 4t^2y^3(y - 1)G_1 \]

has degree 4 in \( y \), and admits a double root \( Y(t) \):

\[ \Delta(t; y) = P(t; y)(y - Y(t))^2, \quad P \text{ of degree 2 in } y \]
Planar maps: differential solution

- The polynomial
  \[ \Delta(y) = (y - 1 - y^2 t)^2 - 4ty^2(y - 1)^2 + 4t^2y^3(y - 1)G_1 \]
  has degree 4 in \( y \), and admits a double root \( Y(t) \):

  \[ \Delta(t; y) = P(t; y)(y - Y(t))^2, \quad P \text{ of degree 2 in } y \]
  \[ \Delta_y(t; y) = Q(t; y)(y - Y(t)), \quad Q \text{ of degree 2 in } y \]
Planar maps: differential solution

- The polynomial
  \[ \Delta(y) = (y - 1 - y^2t)^2 - 4ty^2(y - 1)^2 + 4t^2y^3(y - 1)G_1 \]
  has degree 4 in \( y \), and admits a double root \( Y(t) \):

  \[
  \Delta(t; y) = P(t; y)(y - Y(t))^2, \quad P \text{ of degree 2 in } y \\
  \Delta_y(t; y) = Q(t; y)(y - Y(t)), \quad Q \text{ of degree 2 in } y \\
  \Delta_t(t; y) = R(t; y)(y - Y(t)), \quad R \text{ of degree 3 in } y
  \]
Planar maps: differential solution

- The polynomial

\[
\Delta(y) = (y - 1 - y^2t)^2 - 4ty^2(y - 1)^2 + 4t^2y^3(y - 1)G_1
\]

has degree 4 in \(y\), and admits a double root \(Y(t)\):

\[
\begin{align*}
\Delta(t; y) &= P(t; y)(y - Y(t))^2, & P \text{ of degree 2 in } y \\
\Delta_y(t; y) &= Q(t; y)(y - Y(t)), & Q \text{ of degree 2 in } y \\
\Delta_t(t; y) &= R(t; y)(y - Y(t)), & R \text{ of degree 3 in } y
\end{align*}
\]

- Elimination of \(Y\) and \(\Delta\):

\[
\frac{1}{Q} \frac{\partial}{\partial t} \left( \frac{Q^2}{P} \right) = \frac{1}{R} \frac{\partial}{\partial y} \left( \frac{R^2}{P} \right).
\]
Planar maps: differential solution

• The polynomial

\[ \Delta(y) = (y - 1 - y^2 t)^2 - 4ty^2(y - 1)^2 + 4t^2y^3(y - 1)G_1 \]

has degree 4 in \( y \), and admits a double root \( Y(t) \):

\[
\begin{align*}
\Delta(t; y) &= P(t; y)(y - Y(t))^2, & P \text{ of degree 2 in } y \\
\Delta_y(t; y) &= Q(t; y)(y - Y(t)), & Q \text{ of degree 2 in } y \\
\Delta_t(t; y) &= R(t; y)(y - Y(t)), & R \text{ of degree 3 in } y \\
\end{align*}
\]

• Elimination of \( Y \) and \( \Delta \):

\[
\frac{1}{Q} \frac{\partial}{\partial t} \left( \frac{Q^2}{P} \right) = \frac{1}{R} \frac{\partial}{\partial y} \left( \frac{R^2}{P} \right).
\]

• Writing \( P(t; y) = P_0(t) + yP_1(t) + y^2P_2(t) \) and so on for \( Q \) and \( R \), this gives a system of differential equations in \( t \) relating the series \( P_i, Q_i \) and \( R_i \).
Planar maps: differential solution

• The polynomial

$$\Delta(y) = (y - 1 - y^2 t)^2 - 4ty^2(y - 1)^2 + 4t^2y^3(y - 1)G_1$$

has degree 4 in y, and admits a double root \( Y(t) \):

$$\Delta(t; y) = P(t; y)(y - Y(t))^2, \quad P \text{ of degree 2 in } y$$

$$\Delta_y(t; y) = Q(t; y)(y - Y(t)), \quad Q \text{ of degree 2 in } y$$

$$\Delta_t(t; y) = R(t; y)(y - Y(t)), \quad R \text{ of degree 3 in } y$$

• Elimination of \( Y \) and \( \Delta \):

$$\frac{1}{Q} \frac{\partial}{\partial t} \left( \frac{Q^2}{P} \right) = \frac{1}{R} \frac{\partial}{\partial y} \left( \frac{R^2}{P} \right).$$

• Writing \( P(t; y) = P_0(t) + yP_1(t) + y^2P_2(t) \) and so on for \( Q \) and \( R \), this gives a system of differential equations in \( t \) relating the series \( P_i, Q_i \) and \( R_i \).

• Finally, the GF \( G_1 \) is a polynomial in the series \( P_i, Q_i, R_i \).
IV. Special values of $q$ and $\nu$
<table>
<thead>
<tr>
<th>Specialization</th>
<th>general maps</th>
<th>triangulations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q = 2, 3$</td>
<td>algebraic</td>
<td>algebraic</td>
</tr>
<tr>
<td>Proper colourings ($\nu = 0$)</td>
<td>$D = ((\nu + 1)u - 2)^2$ order 3</td>
<td>$D = (4 - q)u + 1$ order 2</td>
</tr>
<tr>
<td>Four colours ($q = 4$)</td>
<td>$D = (2\nu u + \beta)^2$ order 2</td>
<td></td>
</tr>
<tr>
<td>Connected subgraphs (spanning forests) ($q = 0$)</td>
<td>$D = \beta^2(1 + 4u)$ order 2</td>
<td></td>
</tr>
<tr>
<td>Self-dual model ($q = \beta^2, w = 1/\beta$)</td>
<td>$R_2 = 0$</td>
<td>order 3</td>
</tr>
</tbody>
</table>
### Specializations: explicit DE for the Potts generating function

<table>
<thead>
<tr>
<th>Specialization</th>
<th>general maps</th>
<th>triangulations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q = 2, 3$</td>
<td>algebraic</td>
<td>algebraic</td>
</tr>
<tr>
<td><strong>Proper colourings</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(\nu = 0)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Four colours</strong></td>
<td>$D = ((\nu + 1)u - 2)^2$</td>
<td>$D = (2\nu u + \beta)^2$</td>
</tr>
<tr>
<td>$(q = 4)$</td>
<td>order 3</td>
<td>order 2</td>
</tr>
<tr>
<td><strong>Connected subgraphs</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(spanning forests)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(q = 0)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Self-dual model</strong></td>
<td>$R_2 = 0$</td>
<td></td>
</tr>
<tr>
<td>$(q = \beta^2, w = 1/\beta)$</td>
<td>order 3</td>
<td></td>
</tr>
</tbody>
</table>
Two combinatorial problems:

1. Properly 3-coloured planar maps [Bernardi-mbm 09(a)]

Let $A$ be the quartic series in $t$ defined by

$$A = t \frac{(1 + 2A)^3}{1 - 2A^3}.$$ 

Then the generating function of properly 3-coloured planar maps is

$$M_1(3, 0, 1, t) = \frac{(1 + 2A)(1 - 2A^2 - 4A^3 - 4A^4)}{(1 - 2A^3)^2}.$$ 

The proof is at the moment horrible ...
Two combinatorial problems:

2. Properly $q$-coloured triangulations [Tutte 84]

The number $c(n)$ of properly $q$-coloured triangulations having $n$ vertices satisfies:

$$(n + 1)(n + 2)c(n + 2) = (q - 4)(3n - 1)(3n - 2)c(n + 1)$$

$$+ 2 \sum_{i=1}^{n} i(i + 1)(3n - 3i + 1)c(i + 1)c(n + 2 - i),$$

with the initial condition $c(2) = q - 1$.

The proof is at the moment horrible...
More questions are left...

A. The differential system
Simpler, and/or more combinatorial derivation? Done for

- the Ising model ($q = 2$)
  [mbm & Schaeffer 02], [Bouttier, Di Francesco & Guitter 04]
- spanning forests/connected subgraphs ($q \to 0$)
  [Bouttier et al. 07], [mbm-Courtiel 13(a)]

B. Asymptotics, phase transitions

- Asymptotic number of properly $q$-coloured triangulations when $q \in (28/11, 4] \cup [5, \infty)$ [Odlyzko-Richmond 83])
- Critical point of the Potts model on planar maps when $q \in (0, 4)$ (for the outer degree) [Borot et al. 12])
- Critical exponents for near-triangulations (for the outer degree) [Eynard & Bonnet 99]