

On independent sets of square grids

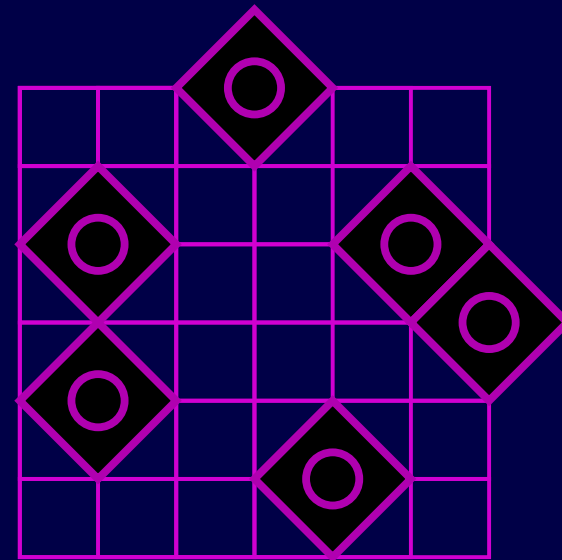
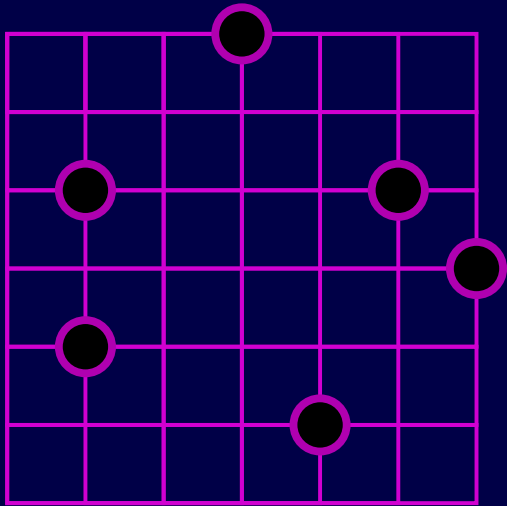
Mireille Bousquet-Mélou, CNRS, Bordeaux, France

Svante Linusson, KTH, Stockholm, Sweden

Eran Nevo, Hebrew University, Jerusalem, Israel

<http://www.labri.fr/~bousquet>

The hard-square model



An independent set of the 7×7 grid, or a hard-square configuration.

Question: find

$$Z_N(t) = \sum_I t^{|I|},$$

where the sum runs over all independent sets of the $N \times N$ grid.

The **partition function** of the hard-square model **at activity** t .

Hard squares are indeed hard

Question: find

$$Z_N(t) = \sum_I t^{|I|},$$

where the sum runs over all independent sets of the $N \times N$ grid.

Unsolved!

- even for $t = 1$,
- even in the thermodynamic limit:

$$\lim_N Z_N(t)^{1/N^2} = ?$$

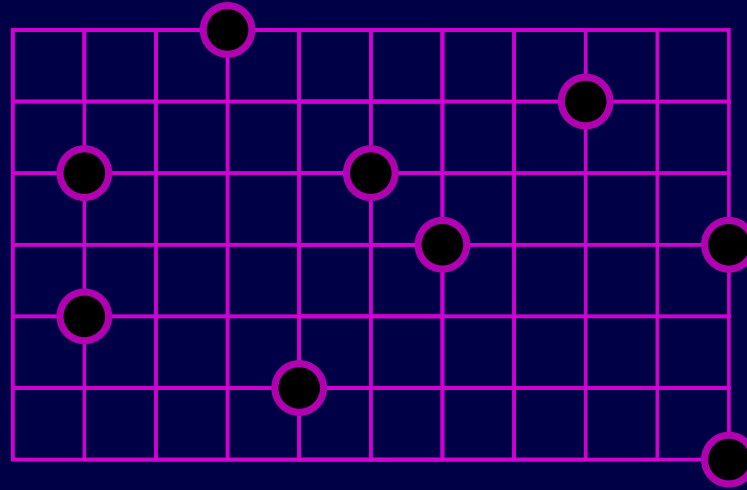
- even for $t = 1$ **and** in the thermodynamic limit:

$$\lim_N Z_N(1)^{1/N^2} \sim 1.503048\dots$$

is the (mysterious) **hard-square constant**.

**Some conjectures of
Fendley, Schoutens and van Eerten (2004)**

Hard squares at activity -1



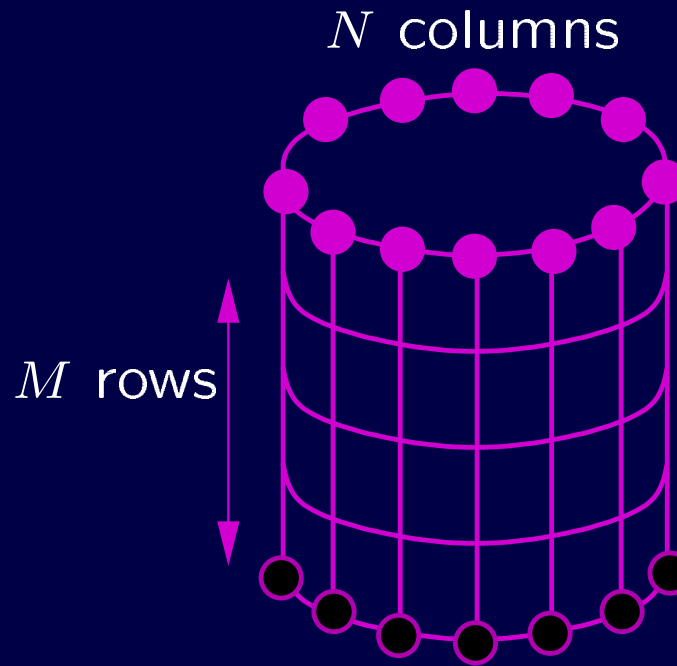
Question: find

$$Z_{M,N} := \sum_I (-1)^{|I|},$$

where the sum runs over all independent sets of the $M \times N$ grid.

The **alternating number** of independent sets.

Independent sets on a torus



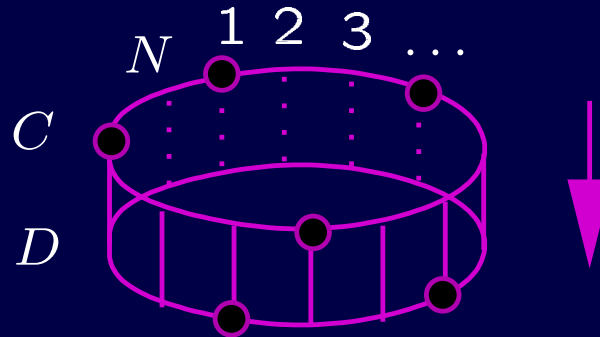
Conjecture [Fendley et al. 04]

For a torus with M rows and N columns, with M and N coprime,

$$Z_t(M, N) := \sum_I (-1)^{|I|} = 1.$$

\implies Not true for a rectangle (with open boundary conditions), nor for a cylinder.

Transfer matrices



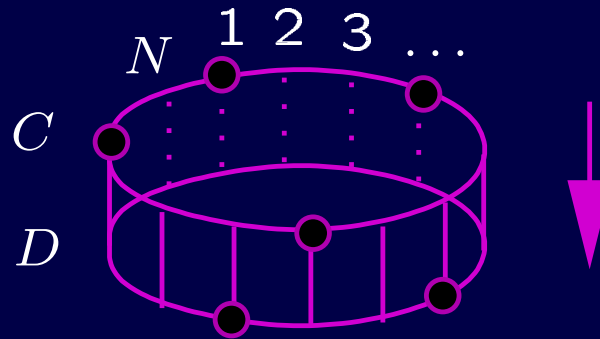
For two independent sets C and D on the N -point circle:

$$\mathbb{T}_N(C, D) = \begin{cases} (-1)^{|D|} & \text{if } C \cap D = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Example: For $N = 4$, the independent sets on the 4-point circle are $\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 3\}, \{2, 4\}$, and

$$\mathbb{T}_4 = \begin{pmatrix} 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 0 & -1 & -1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 & -1 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 0 & 1 \\ 1 & -1 & -1 & -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 & 0 & 1 & 0 \end{pmatrix}$$

Transfer matrices



For two independent sets C and D on the N -point circle:

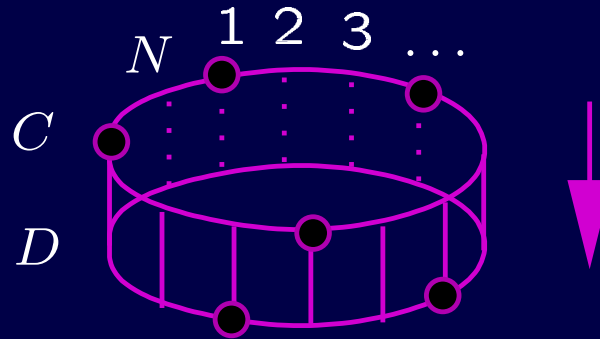
$$\mathbb{T}_N(C, D) = \begin{cases} (-1)^{|D|} & \text{if } C \cap D = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Then:

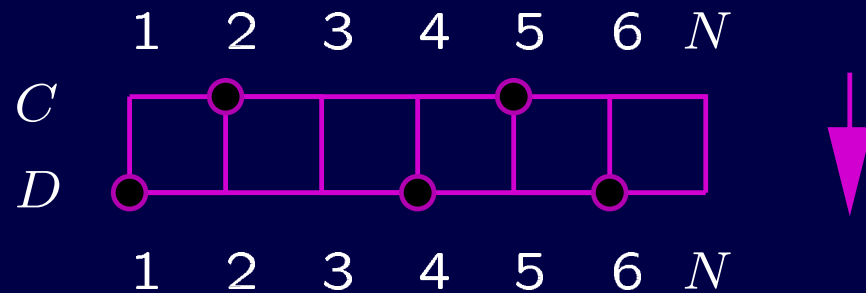
- $(\mathbb{T}_N)^M(C, C)$ is the alternating number of independent sets of the $M \times N$ **torus**, with boundary condition C on row 1
- $\text{tr}(\mathbb{T}_N)^M$ is the alternating number $Z_t(M, N)$ of independent sets of the $M \times N$ **torus**

Conjectures on transfer matrices [Fendley et al. 04]

- For every N , all the eigenvalues of the matrix \mathbb{T}_N are roots of unity



- Same conjecture for the transfer matrix \mathbb{O}_N between two N -point segments



Ex.: The characteristic polynomial of the matrix \mathbb{O}_N

$$P_1(x) = \frac{x^3 + 1}{x + 1},$$

$$P_4(x) = (x - 1)(x^3 + 1)(x^4 - 1),$$

$$P_2(x) = (x - 1)(x^2 + 1),$$

$$P_5(x) = \frac{(x^{10} - 1)(x^4 + 1)}{x + 1},$$

$$P_3(x) = (x - 1)(x^4 + 1),$$

$$P_6(x) = (x - 1)(x^2 + 1)(x^4 - 1)(x^{14} - 1),$$

$$P_7(x) = \frac{(x^3 + 1)(x^4 + 1)(x^{12} - 1)(x^{18} - 1)}{(x + 1)(x^2 + 1)},$$

$$P_8(x) = (x + 1)(x^4 + 1)^2(x^8 + 1)(x^{16} - 1)(x^{22} - 1).$$

Some bad news

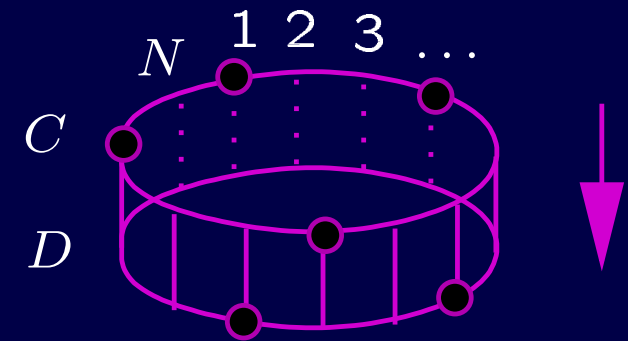
I am not going to prove these conjectures...

... but

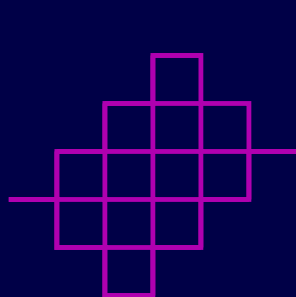
Some good news

(1) **Jacob Jonsson** has proved some of them:

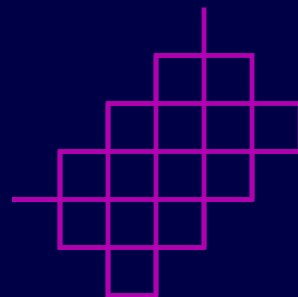
- $Z_t(M, N) = 1$ if M and N are coprime
- All eigenvalues of the transfer matrix from circle to circle are **roots of unity**



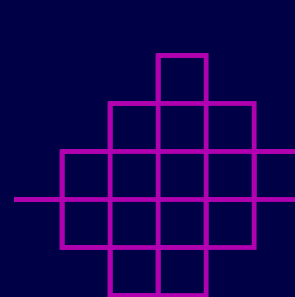
(2) **Similar results hold in greater generality** for other rectangular shapes



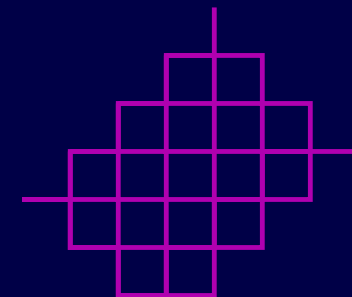
$\mathcal{R}(6, 8)$



$\mathcal{R}(6, 9)$



$\mathcal{R}(7, 8)$



$\mathcal{R}(7, 9)$

Results

Independent sets of tilted rectangles

Let $\mathcal{R}(M, N)$ be the subgraph of \mathbb{Z}^2 induced by the points (x, y) satisfying

$$y \leq x \leq y + M - 1 \quad \text{and} \quad -y \leq x \leq -y + N - 1.$$

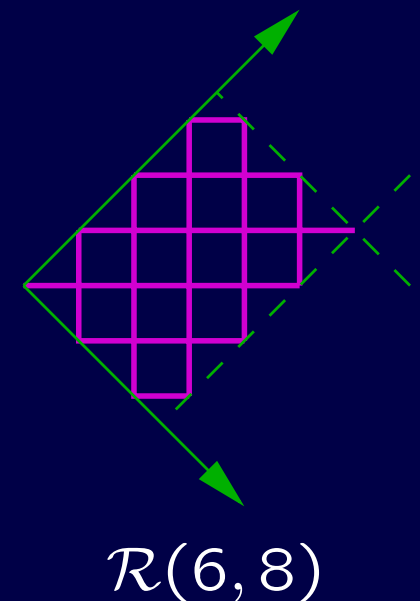
The alternating number of independent sets on $\mathcal{R}(M, N)$ is

$$Z_{\mathcal{R}}(M, N) = \sum_I (-1)^{|I|}.$$

Theorem [MBM-Linusson-Nevo 06]

- If $M \equiv_3 1$ or $N \equiv_3 1$, then $Z_{\mathcal{R}}(M, N) = 0$.
- Otherwise $Z_{\mathcal{R}}(M, N) = (-1)^{mn}$,

with $m = \lceil M/3 \rceil$ and $n = \lceil N/3 \rceil$.



Proof: Involutions on independent sets

Aim: find an involution σ such that

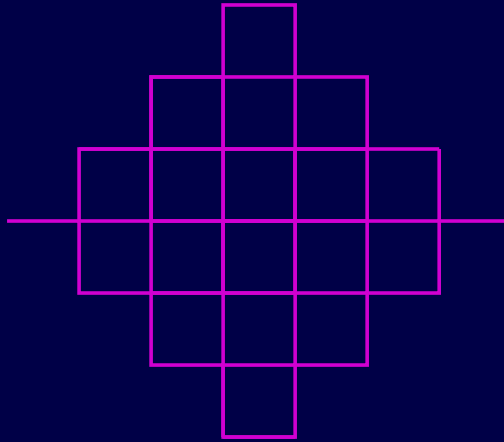
- there are few fixed points (ideally, 0 or 1)
- if $I \neq \sigma(I)$, then the cardinalities of I and $\sigma(I)$ differ by ± 1 . The sets I and $\sigma(I)$ are said to be **matched** together.

$\Rightarrow \sigma$ is sign-reversing on the matched sets.

$$Z = \sum_I (-1)^{|I|} = \sum_{I \text{ unmatched}} (-1)^{|I|}$$

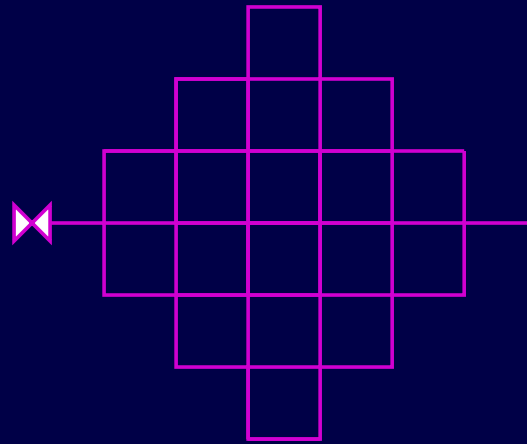
How to match independent sets

$\mathcal{R}(8, 8)$



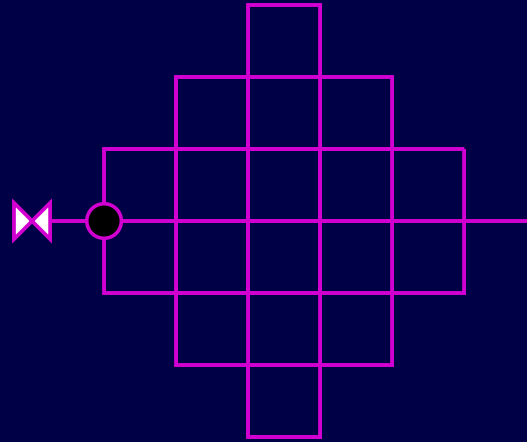
How to match independent sets

$\mathcal{R}(8, 8)$



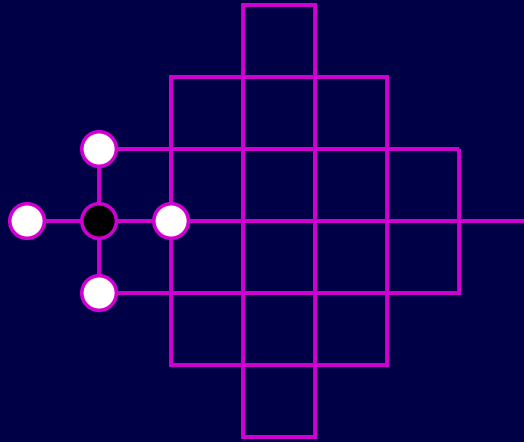
How to match independent sets

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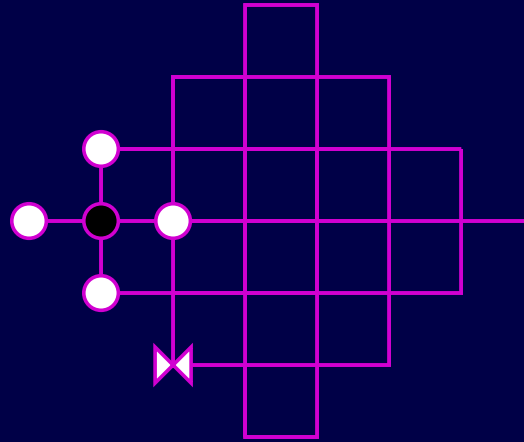
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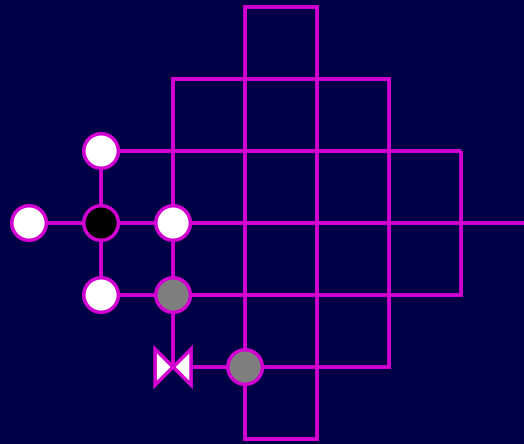
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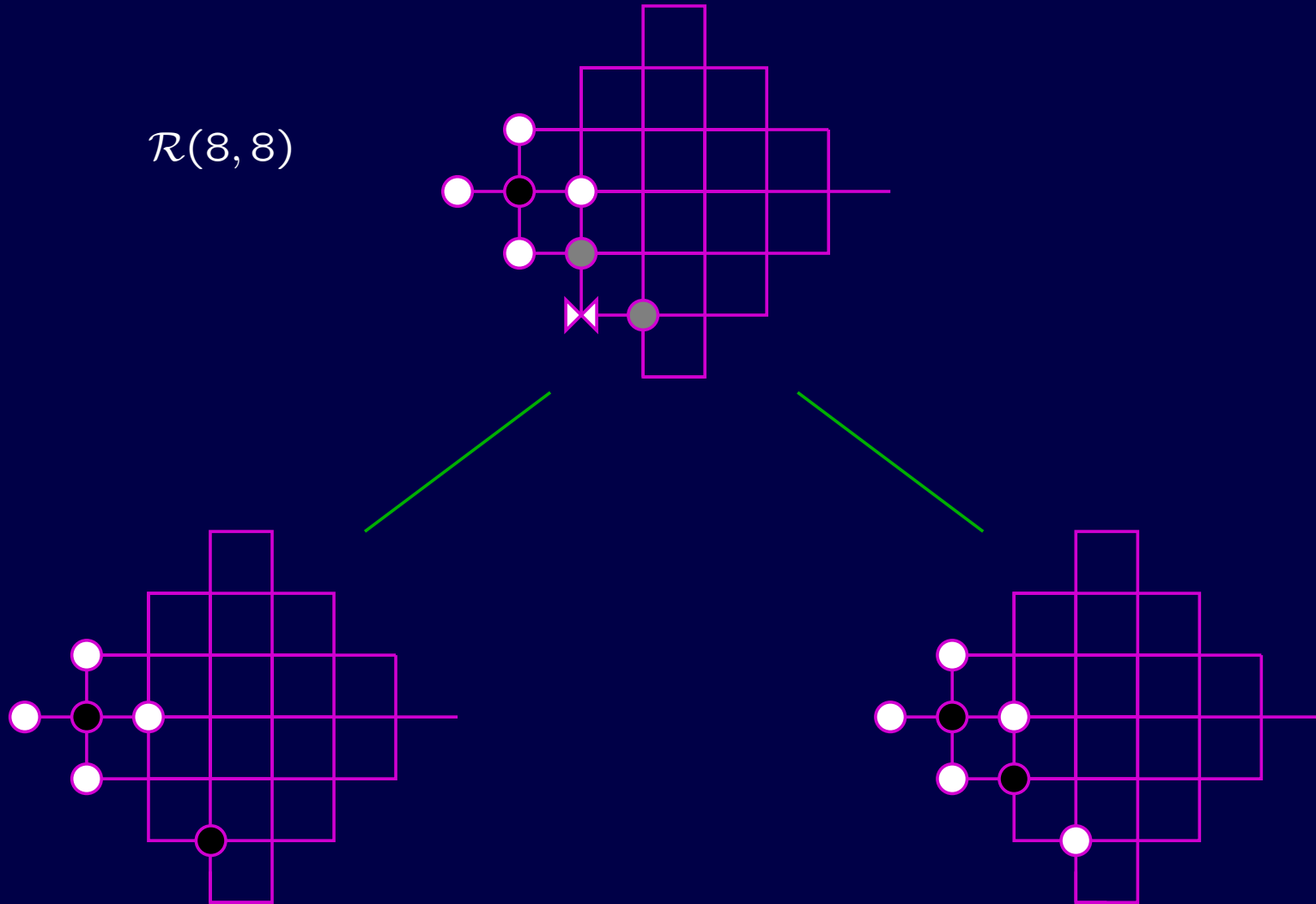
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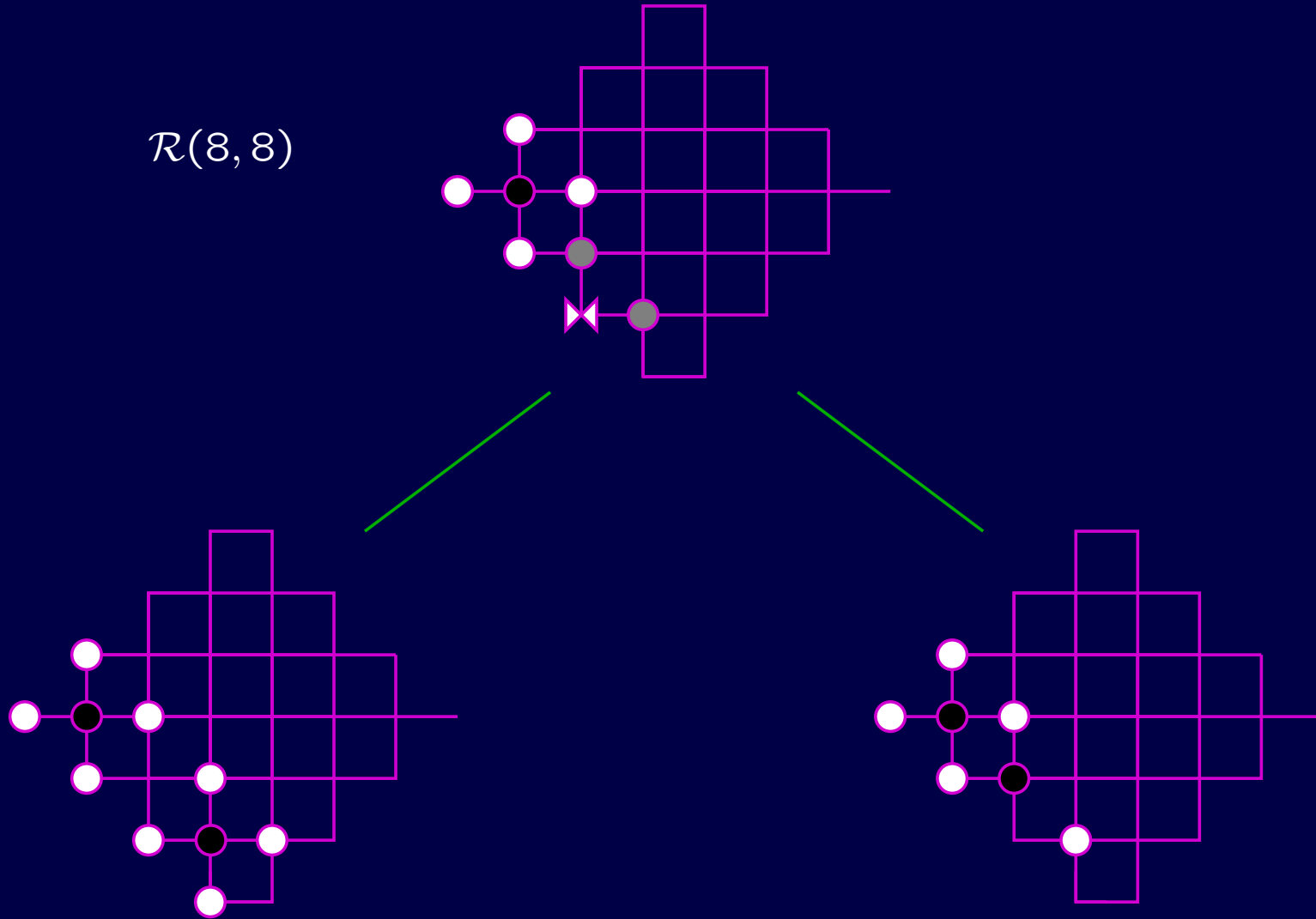
How to match independent sets

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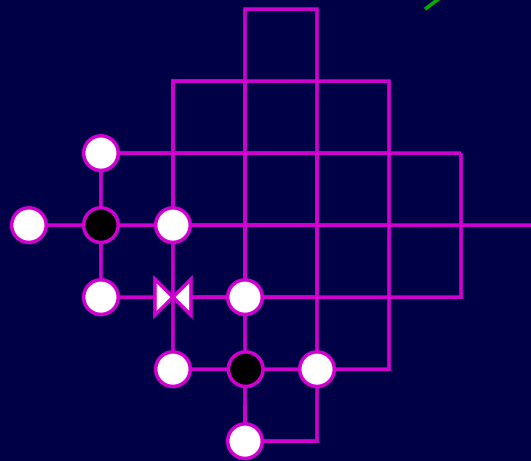
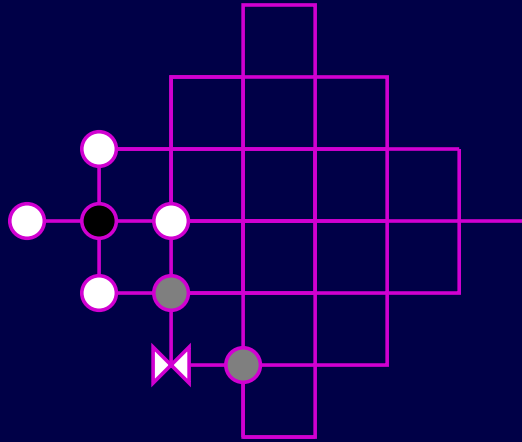
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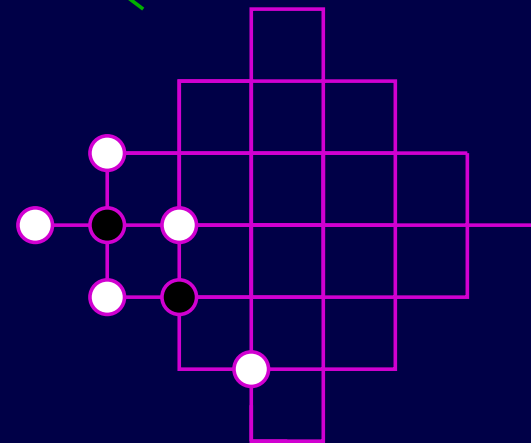


How to match independent sets

$\mathcal{R}(8,8)$

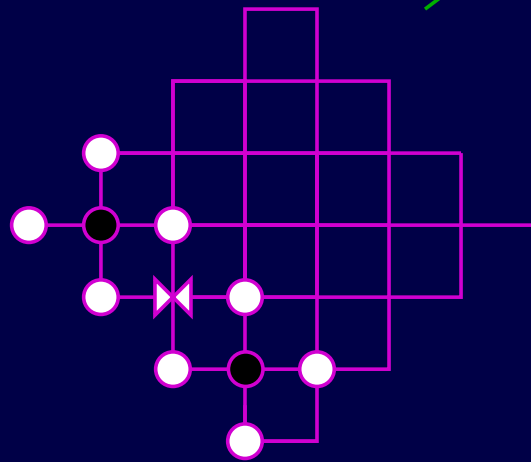
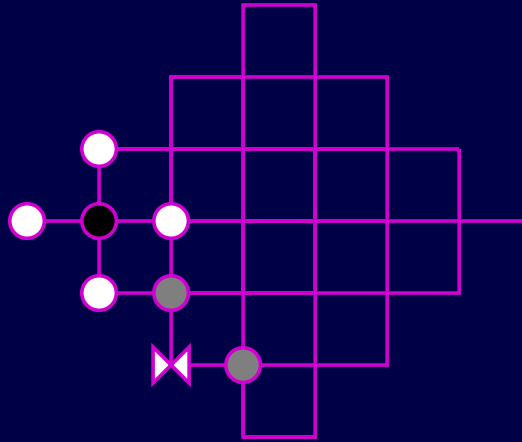


Perfect matching

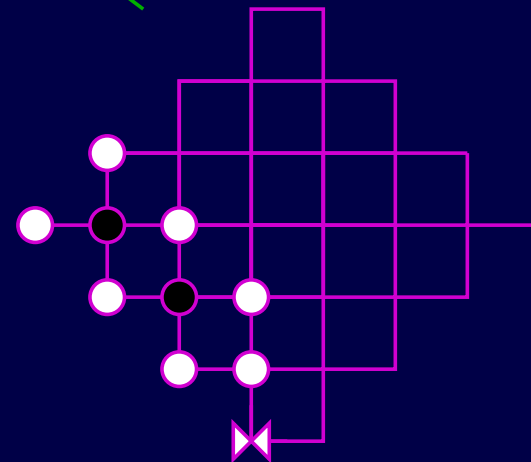


How to match independent sets

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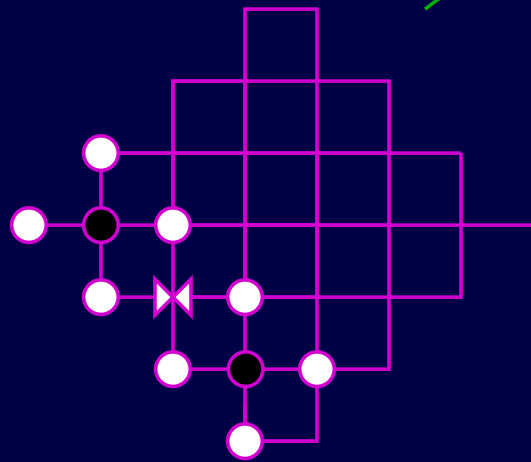
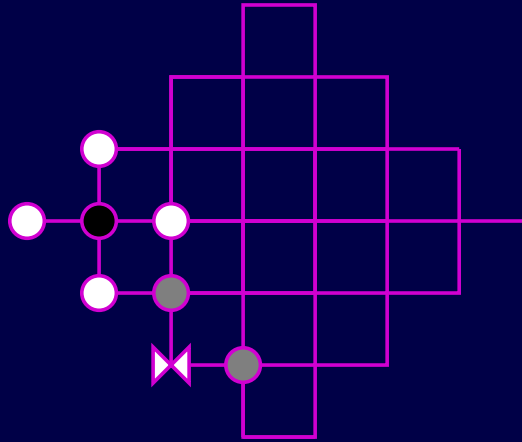


Perfect matching

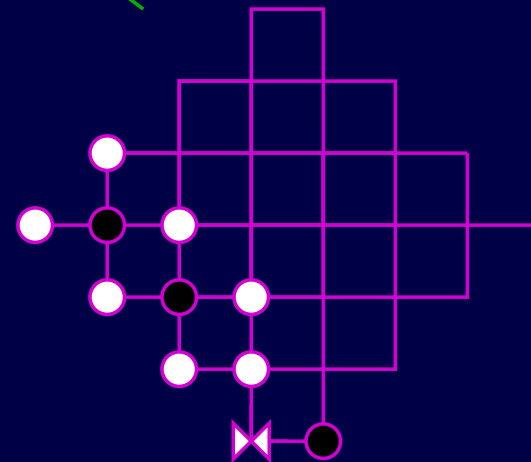


How to match independent sets

$\mathcal{R}(8,8)$

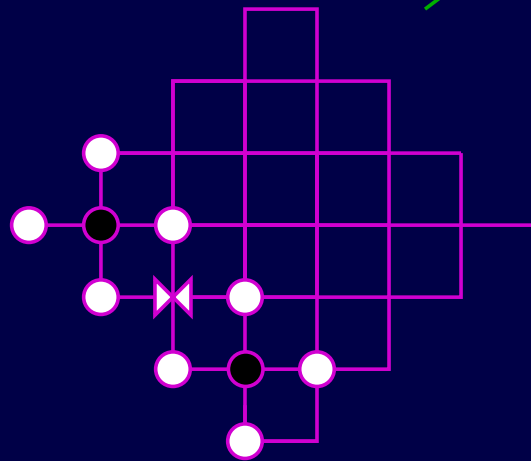
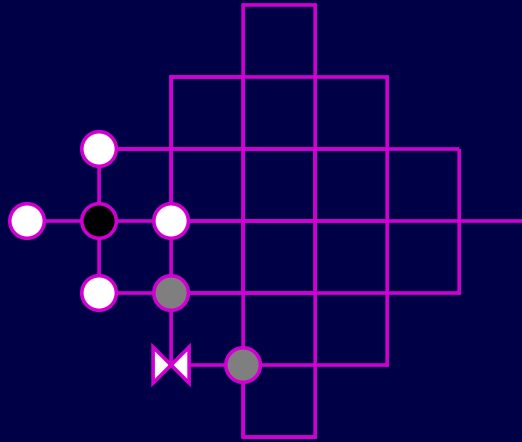


Perfect matching

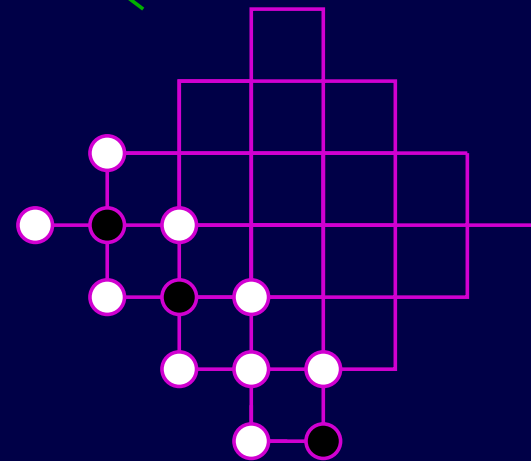


How to match independent sets

$\mathcal{R}(8, 8)$



Perfect matching



$\mathcal{R}(8, 5) \Rightarrow$ Recursion

Independent sets of tilted rectangles

Let $\mathcal{R}(M, N)$ be the subgraph of \mathbb{Z}^2 induced by the points (x, y) satisfying

$$y \leq x \leq y + M - 1 \quad \text{and} \quad -y \leq x \leq -y + N - 1.$$

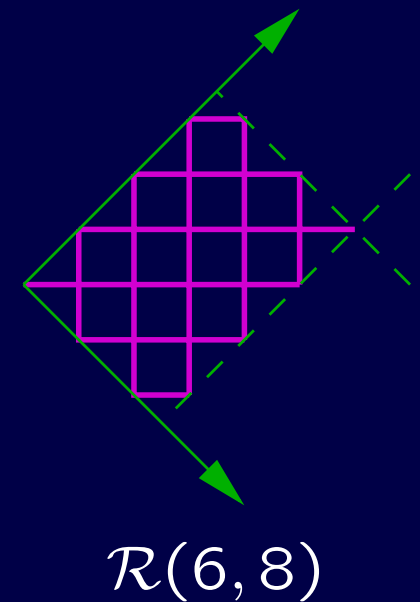
The alternating number of independent sets on $\mathcal{R}(M, N)$ is

$$Z_{\mathcal{R}}(M, N) = \sum_I (-1)^{|I|}.$$

Theorem [BM-L-N 06]

- If $M \equiv_3 1$ or $N \equiv_3 1$, then $Z_{\mathcal{R}}(M, N) = 0$.
- Otherwise $Z_{\mathcal{R}}(M, N) = (-1)^{mn}$,

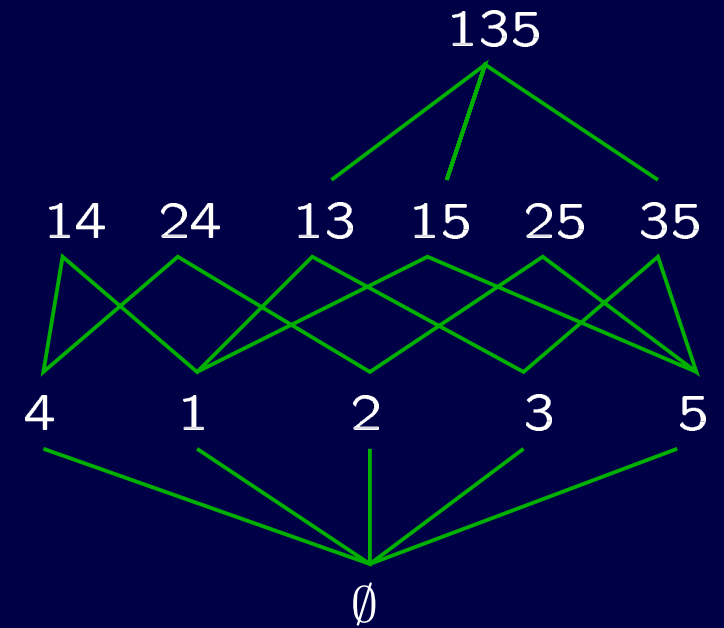
with $m = \lceil M/3 \rceil$ and $n = \lceil N/3 \rceil$.



**Bonus: The topology of the independence
complex**

The topology of the independence complex

- Graph G



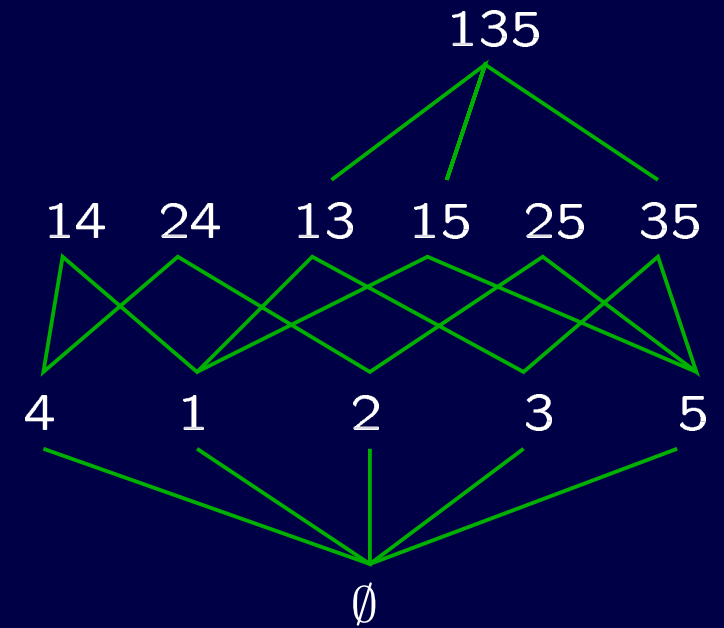
Independence complex $\Sigma(G)$

The topology of the independence complex

- Graph G



$$Z(G) = 1 - 5 + 6 - 1$$



Independence complex $\Sigma(G)$

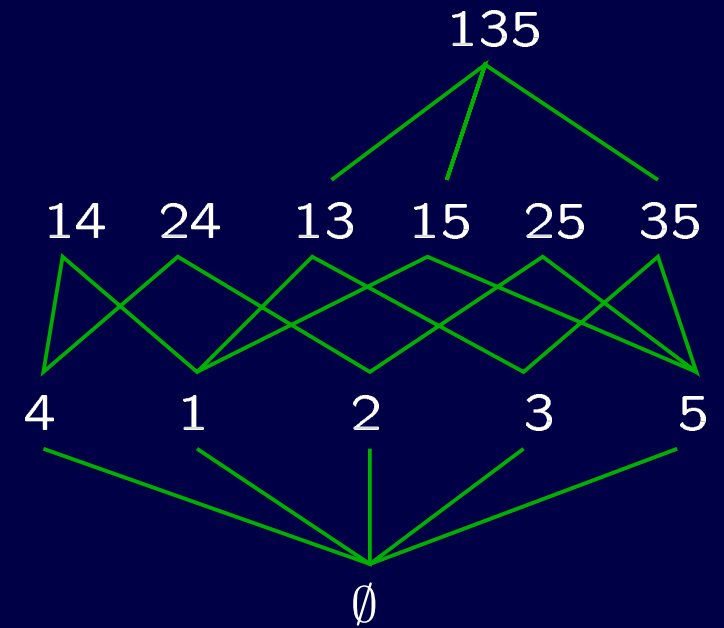
The topology of the independence complex

- Graph G



$$\begin{aligned} Z(G) &= 1 - 5 + 6 - 1 \\ &= -\chi(\Sigma(G)) = 1 \end{aligned}$$

The Euler characteristic
of the complex



Independence complex $\Sigma(G)$

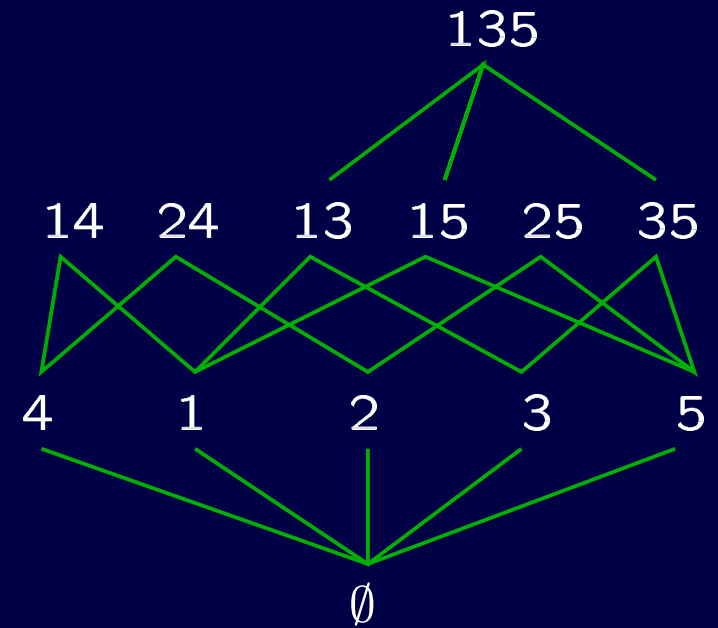
The topology of the independence complex

- Graph G



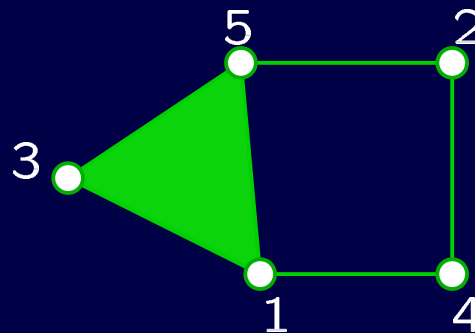
$$\begin{aligned} Z(G) &= 1 - 5 + 6 - 1 \\ &= -\chi(\Sigma(G)) = 1 \end{aligned}$$

The Euler characteristic of the complex



Independence complex $\Sigma(G)$

- Topologic realization



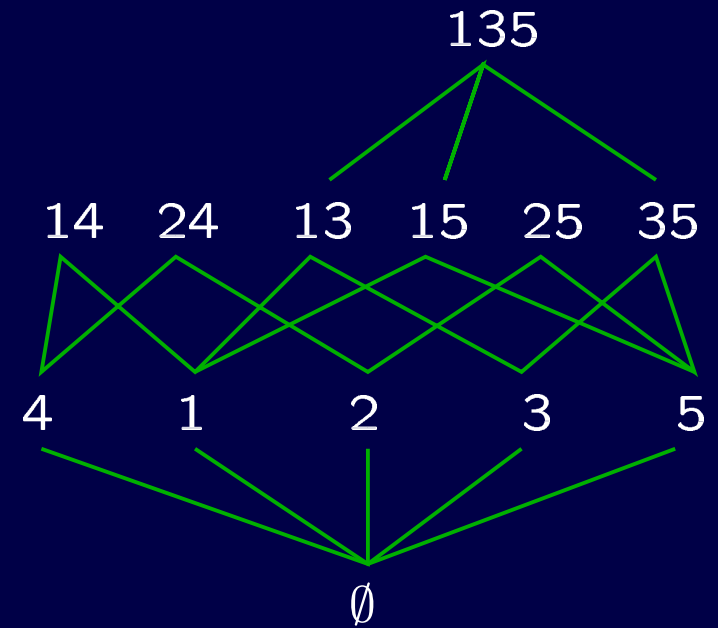
The topology of the independence complex

- Graph G



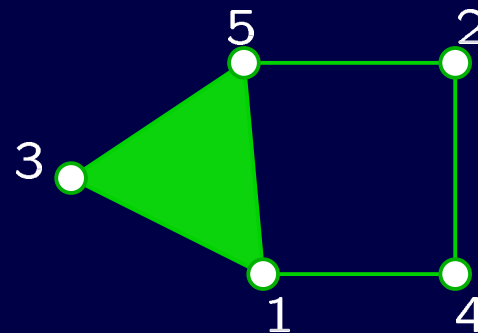
$$\begin{aligned} Z(G) &= 1 - 5 + 6 - 1 \\ &= -\chi(\Sigma(G)) = 1 \end{aligned}$$

The Euler characteristic of the complex



Independence complex $\Sigma(G)$

- Topologic realization



\simeq a circle

$$\chi(\Sigma(G)) = -1$$

The independence complex of tilted rectangles

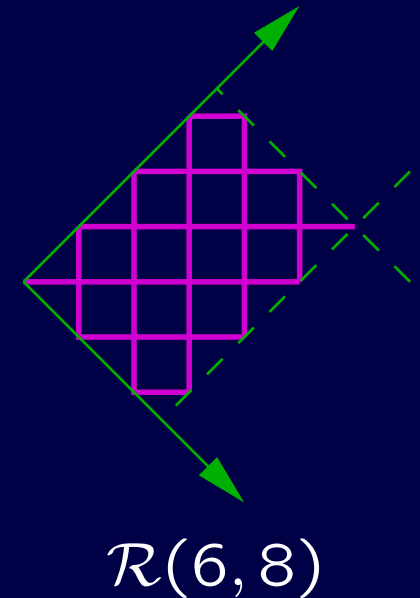
Let $\mathcal{R}(M, N)$ be the subgraph of \mathbb{Z}^2 induced by the points (x, y) satisfying

$$y \leq x \leq y + M - 1 \quad \text{and} \quad -y \leq x \leq -y + N - 1.$$

Theorem

- If $M \equiv_3 1$ or $N \equiv_3 1$, then $\Sigma(\mathcal{R}(M, N))$ is **contractible** and $Z_{\mathcal{R}}(M, N) = 0$.
- Otherwise, $\Sigma(\mathcal{R}(M, N))$ is homotopy equivalent to a **sphere of dimension $mn - 1$** , and $Z_{\mathcal{R}}(M, N) = (-1)^{mn}$,

with $m = \lceil M/3 \rceil$ and $n = \lceil N/3 \rceil$.

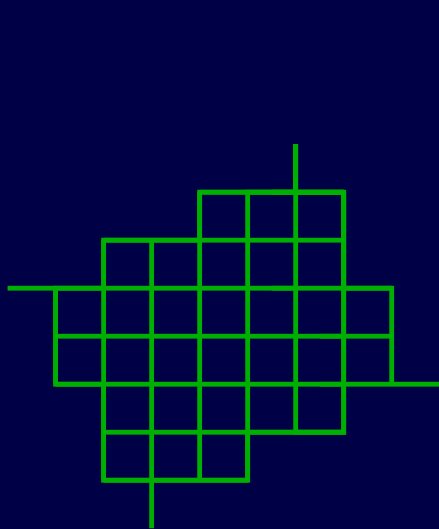


Proof: Our involution defines a **Morse matching** of the complex $\Sigma(\mathcal{R}(M, N))$.
Discrete Morse theory [Forman 95]

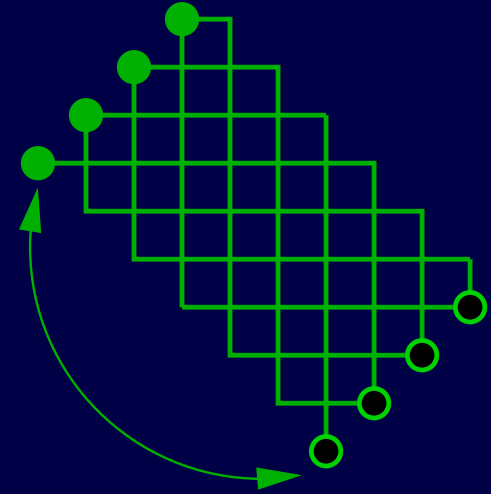
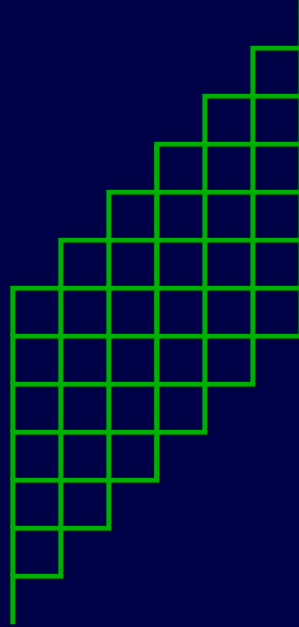
Extension to other shapes

Other shapes

- Shapes with lots of vertices of degree 2

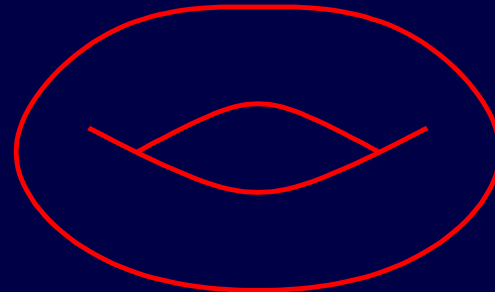
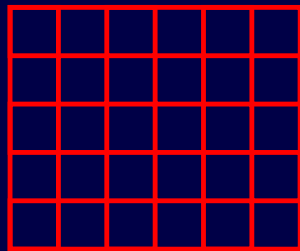


Open boundary conditions



Cylindric boundary conditions

- Does **not** work for:



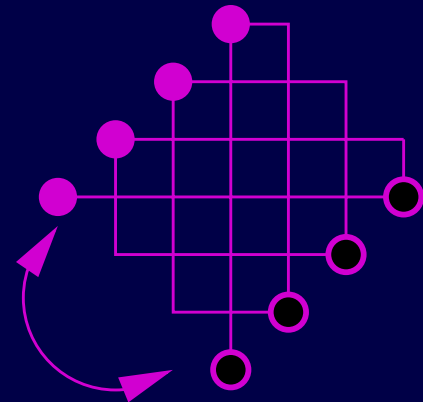
Tilted rectangles with cylindric boundary conditions

For M even, let $\mathcal{R}^c(M, N)$ be obtained by wrapping $\mathcal{R}(M + 1, N)$ on a cylinder (identifying the vertices (i, i) and $(M/2 + i, -M/2 + i)$)

Theorem

- If $N \equiv_3 1$, then $\Sigma(\mathcal{R}^c(M, N))$ is **contractible** and $Z_{\mathcal{R}}^c(M, N) = 0$.
- Otherwise,
 - If $M \equiv_3 0$, then $\Sigma(\mathcal{R}^c(M, N))$ is homotopy equivalent to a **wedge of 2^n spheres of dimension $mn - 1$** , and $Z_{\mathcal{R}}^c(M, N) = 2^n$.
 - If $M \equiv_3 1$ or 2 then $\Sigma(\mathcal{R}^c(M, N))$ is homotopy equivalent to a **single sphere of dimension $mn - 1$** , and $Z_{\mathcal{R}}^c(M, N) = (-1)^n$.

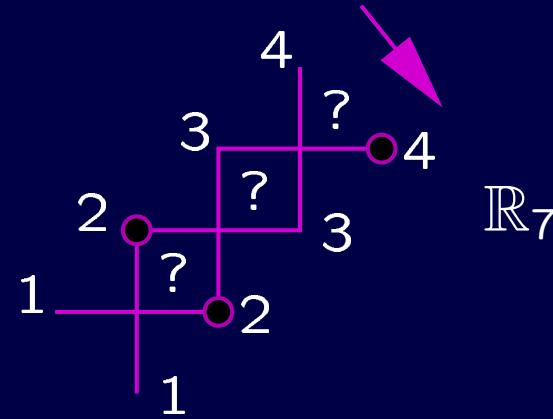
with $m = \lfloor \frac{M+1}{3} \rfloor$ and $n = \lceil N/3 \rceil$.



$\mathcal{R}^c(6, 8)$

Diagonal transfer matrices

$$\text{tr}(\mathbb{R}_N)^M = Z_{\mathcal{R}}^c(M, N)$$



$$\mathbb{R}_N(C, D) = \begin{cases} (-1)^{|D|} & \text{if } {}^c C \cap {}^c D \cap {}^c (C-1) \cap {}^c (D-1) = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem [BM-L-N]

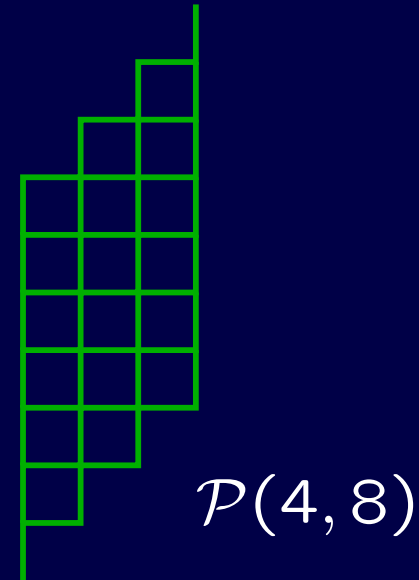
- If $N \equiv_3 1$, then \mathbb{R}_N is **nilpotent** (all its eigenvalues are 0).
- Otherwise, \mathbb{R}_N has eigenvalues:
 - **0** with multiplicity $d_N - 2^n$,
 - **1** with multiplicity $(2^n + 2(-1)^n)/3$,
 - **j** and **j^2** with multiplicity $(2^n - (-1)^n)/3$,

where $d_N := 2^{\lceil N/2 \rceil}$, $n = \lceil N/3 \rceil$, and $j = e^{2i\pi/3}$.

Independent sets of parallelograms $\mathcal{P}(K, N)$

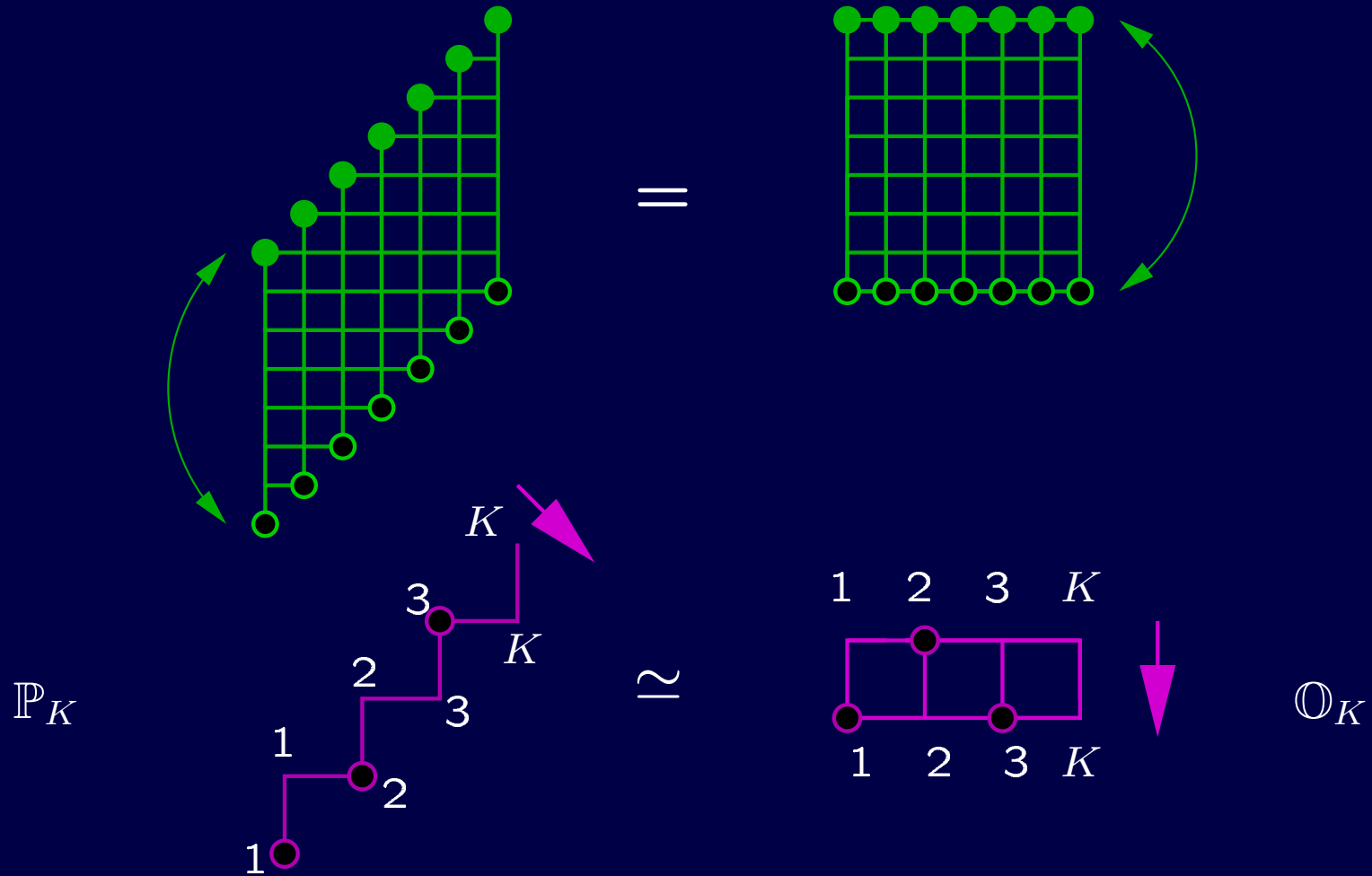
Theorem

- If $K \equiv_3 1$, then
 - if $N \equiv_3 1$ then $Z_{\mathcal{P}}(K, N) = 0$,
 - otherwise, $Z_{\mathcal{P}}(K, N) = (-1)^n$ with $n = \lceil N/3 \rceil$.
- If $K \equiv_3 2$, write $N = 2qK + r$, with $0 \leq r \leq 2K - 1$.
 - If $r \equiv_3 1, 2$, then $Z_{\mathcal{P}}(K, N) = 0$,
 - otherwise $Z_{\mathcal{P}}(K, N) = 1$.
- If $K \equiv_3 0$, write $N = 2q(K+1) + r$ with $0 \leq r \leq 2K + 1$.
 - If $r \equiv_3 0$ with $r \geq 1$, or $r \equiv_3 1$ with $r \leq 2K$, then $Z_{\mathcal{P}}(K, N) = 0$,
 - Otherwise, $Z_{\mathcal{P}}(K, N) = 1$.



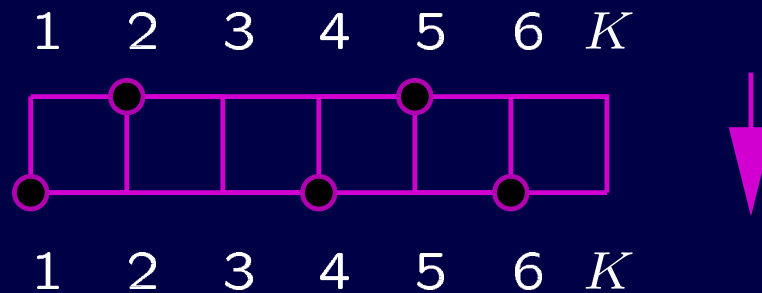
$\mathcal{P}(4, 8)$

From parallelograms to rectangles



The matrices \mathbb{P}_K and \mathbb{O}_K have **the same spectrum**, except from the multiplicity of the null eigenvalue

Some eigenvalues of the ordinary transfer matrix \mathbb{O}_K



Theorem [BM-L-N]

- If $K \equiv_3 1$, then $e^{i\pi/3}$ and $e^{-i\pi/3}$ are eigenvalues of \mathbb{O}_K .
- If $K \equiv_3 2$, then **all the $2K$ th roots of unity**, except maybe -1 , are eigenvalues of \mathbb{O}_K .
- If $K \equiv_3 0$, then **all the $(2K + 2)$ th roots of unity**, except maybe -1 and, if K is odd, $\pm i$, are eigenvalues of \mathbb{O}_K .

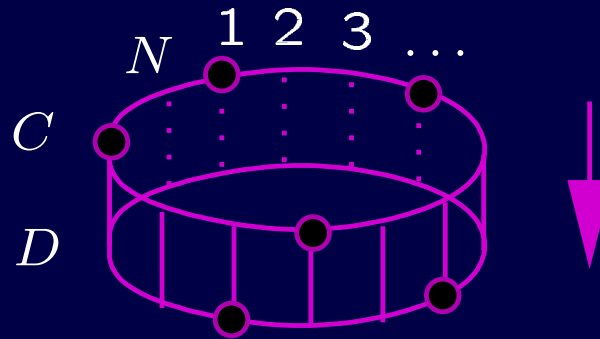
Perspectives

(1) Range of application

- Other shapes?
- Fixed boundary conditions \Rightarrow more eigenvalues of the transfer matrix \mathbb{O}_K ?

(2) Comparizon with J. Jonsson's methods

Transfer matrices



For two independent sets C and D on the N -point circle:

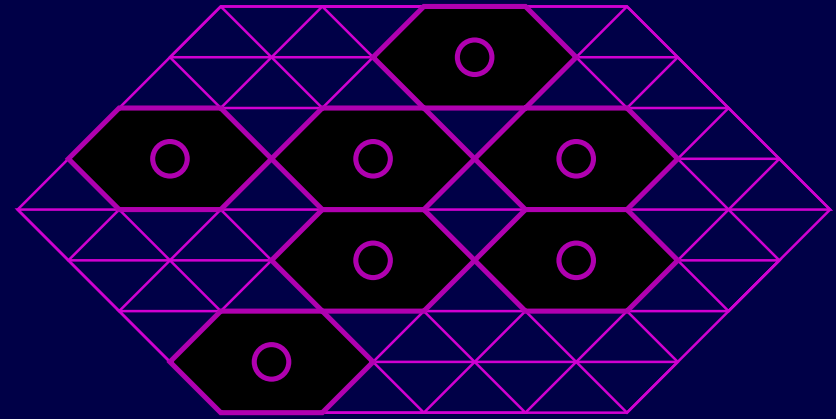
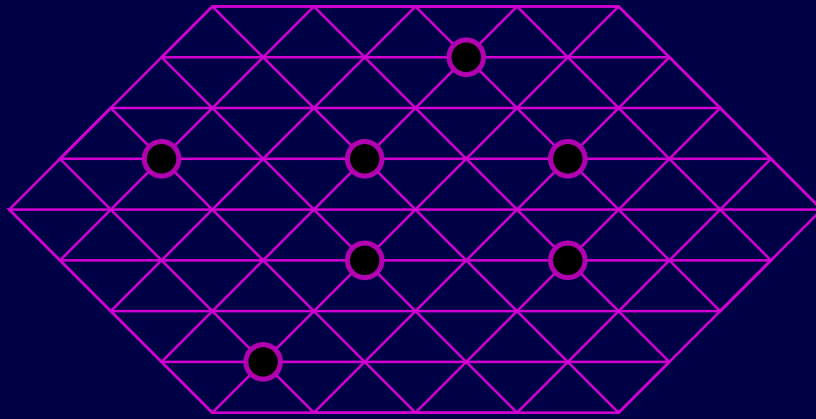
$$\mathbb{T}_N(C, D) = \begin{cases} (-1)^{|D|} & \text{if } C \cap D = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Example: For $N = 3$, the independent sets on the 3-point line are $\emptyset, \{1\}, \{2\}, \{3\}, \{1, 3\}$, and

$$\mathbb{O}_3 = \begin{pmatrix} 1 & -1 & -1 & -1 & 1 \\ 1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & -1 & 1 \\ 1 & -1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \end{pmatrix}$$



Hard hexagons are slightly less hard



Question: find

$$Z_N(t) = \sum_I t^{|I|},$$

where the sum runs over all independent sets of the $N \times N \times N$ hexagon.

Solved by Baxter in the thermodynamic limit:

$$\lim_N Z_N(t)^{1/N^2} = \dots$$

[Baxter, 1980]

