Self-avoiding walks
interacting with a surface
on the honeycomb lattice

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joint work with
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A. Introduction
What is $c_n$, the number of $n$-step SAWs?

Hard! $c_n$ is known up to $n = 105$ [Jensen 06]
The growth constant

Clearly,

$$c_{m+n} \leq c_m c_n$$

$$\Rightarrow \lim_{n} c_n^{1/n}$$ exists and

$$\mu := \lim_{n} c_n^{1/n} = \inf_{n} c_n^{1/n}$$

Conjecture [Nienhuis 82]: the growth constant is

$$\mu = \sqrt{2 + \sqrt{2}}$$
The growth constant

Clearly,

\[ c_{m+n} \leq c_m c_n \]

\[ \Rightarrow \lim_{n} c_n^{1/n} \text{ exists and} \]

\[ \mu := \lim_{n} c_n^{1/n} = \inf_n c_n^{1/n} \]

Theorem [Duminil-Copin & Smirnov 10]: the growth constant is

\[ \mu = \sqrt{2 + \sqrt{2}} \]
Growth constants and generating functions

• Let \( C(x) \) be the length generating function of SAWs:

\[
C(x) = \sum_{n \geq 0} c_n x^n.
\]

• The radius of convergence of \( C(x) \) is

\[
\rho = 1/\mu,
\]

where

\[
\mu = \lim_n c_n^{1/n}
\]

is the growth constant.
Walks in a half-plane interacting with a “surface”

• Enumeration by contacts of $n$-step walks:

$$\bar{c}_n(y) = \sum_\omega y^{\text{contacts}(\omega)}$$

In statistical physics, the parameter $y$ is called “fugacity”
Walks in a half-plane interacting with a “surface”

- Enumeration by contacts of $n$-step walks:
  \[ \bar{c}_n(y) = \sum_{\omega} y^{\text{contacts}(\omega)} \]

- Generating function
  \[ \bar{C}(x, y) = \sum_{n \geq 0} \bar{c}_n(y)x^n \]

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- Enumeration by contacts of $n$-step walks:
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- Generating function
  \[
  \bar{C}(x, y) = \sum_{n \geq 0} \bar{c}_n(y) x^n
  \]

- Radius and growth constant ($y > 0$ fixed):
  \[
  \rho(y) = \frac{1}{\mu(y)} = \lim_n \bar{c}_n(y)^{-1/n}
  \]

[Hammersley, Torrie and Whittington 82]

In statistical physics, the parameter $y$ is called “fugacity”
The critical fugacity

- Radius and growth constant: for \( y > 0 \),

\[
\rho(y) = \frac{1}{\mu(y)} = \lim_{n} \bar{c}_n(y)^{-1/n}
\]

Proposition: \( \rho(y) \) is a continuous, weakly decreasing function of \( y \in (0, \infty) \). There exists \( y_c > 1 \) such that

\[
\rho(y) \begin{cases} 
= 1/\mu & \text{if } y \leq y_c, \\
< 1/\mu & \text{if } y > y_c,
\end{cases}
\]

where \( \mu \) is the growth constant of (unrestricted) SAWs.
[Whittington 75, Hammersley, Torrie and Whittington 82]
The critical fugacity: probabilistic meaning

Take half-space SAWs of length $n$ under the Boltzmann distribution

$$P_n(\omega) = \frac{y^{\text{contacts}(\omega)}}{\bar{c}_n(y)}.$$

Then for $y < y_c$, the walk escapes from the surface. For $y > y_c$, a positive fraction of its vertices lie on the surface.
The critical fugacity: probabilistic meaning

Take half-space SAWs of length \( n \) under the Boltzmann distribution

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P_n(\omega) = \frac{y^{\text{contacts}(\omega)}}{c_n(y)}.
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Theorem [B-BM-dG-DC-G 12]: this phase transition occurs at

\[
y_c = 1 + \sqrt{2}
\]

(conjectured by Batchelor and Yung in 1995)
B. Three ingredients
1. Duminil-Copin and Smirnov’s “global” identity

Consider the following finite domain $D_{h,\ell}$.

Let $A_{h,\ell}(x)$ (resp. $B_{h,\ell}(x)$, $E_{h,\ell}(x)$) be the length generating function of SAWs that start from the origin and end on the bottom (resp. top, right/left) border of the domain $D_{h,\ell}$. These series are polynomials in $x$. 
1. Duminil-Copin and Smirnov’s “global” identity

For \( x = x^* = 1/\sqrt{2 + \sqrt{2}} \) (that is, \( x = 1/\mu \)),

\[
\alpha A_{h,\ell}(x^*) + B_{h,\ell}(x^*) + \varepsilon E_{h,\ell}(x^*) = 1
\]

with \( \alpha = \frac{\sqrt{2-\sqrt{2}}}{2} \) and \( \varepsilon = \frac{1}{\sqrt{2}} \).
1. Duminil-Copin and Smirnov’s “global” identity: refinement with upper contacts

For \( x = x^* = 1/\sqrt{2 + \sqrt{2}} \) (that is, \( x = 1/\mu \)), and for any \( y \),

\[
\alpha A_{h,\ell}(x^*, y) + \frac{y^* - y}{y(y^* - 1)} B_{h,\ell}(x^*, y) + \varepsilon E_{h,\ell}(x^*, y) = 1
\]

with \( \alpha = \frac{\sqrt{2-\sqrt{2}}}{2} \), \( \varepsilon = \frac{1}{\sqrt{2}} \) and \( y^* = 1 + \sqrt{2} \).

\( A_{h,\ell} \) arches,
\( B_{h,\ell} \) bridges,
\( E_{h,\ell} \) ...
2. An alternative description of the critical fugacity

**Proposition:** The radius $\rho(y)$ of $\overline{C}(x, y)$ is a continuous, weakly decreasing function of $y \in (0, \infty)$. There exists $y_c > 1$ such that

$$\rho(y) \begin{cases} 
= \frac{1}{\mu} & \text{if } y \leq y_c, \\
< \frac{1}{\mu} & \text{if } y > y_c,
\end{cases}$$

where $\mu$ is the growth constant of (unrestricted) SAWs.
Our identity

For \( x = x^* = 1/\sqrt{2 + \sqrt{2}} \) (that is, \( x = 1/\mu \)), and for any \( y \),

\[
\alpha A_{h,\ell}(x^*, y) + \frac{y^* - y}{y(y^* - 1)} B_{h,\ell}(x^*, y) + \varepsilon E_{h,\ell}(x^*, y) = 1
\]

with \( \alpha = \frac{\sqrt{2 - \sqrt{2}}}{2} \), \( \varepsilon = \frac{1}{\sqrt{2}} \) and \( y^* = 1 + \sqrt{2} \).

- Walks in a bounded domain
- Contacts on the “wrong” side
- Only valid at \( x = x^* = 1/\mu \)
2. An alternative description of the critical fugacity

Proposition: Let $A_h(x, y)$ be the generating function of arches in a strip of height $h$, counted by length and upper contacts. Let $y_h$ be the radius of convergence of $A_h(x^*, y)$, where $x^* = 1/\mu$ (*).

Then, as $h \to \infty$,

$$y_h \searrow y_c.$$ 

(*) For all $k$, the coefficient of $y^k$ in $A_h(x, y)$ is finite at $x^* = 1/\mu$

(uses [van Rensburg, Orlandini and Whittington 06])
2. An alternative description of the critical fugacity

Proposition: Let $A_h(x, y)$ be the generating function of arches in a strip of height $h$, counted by length and upper contacts. Let $y_h$ be the radius of convergence of $A_h(x^*, y)$, where $x^* = 1/\mu$ (*).

Then, as $h \to \infty$,

$$y_h \searrow y_c.$$

The same holds for the generating function $B_h(x, y)$ of bridges, and for the generating function $C_h(x, y)$ of all SAWs in the $h$-strip, with the same value of $y_h$.

(*) For all $k$, the coefficient of $y^k$ in $A_h(x, y)$ is finite at $x^* = 1/\mu$

(uses [van Rensburg, Orlandini and Whittington 06])
For $y > 0$ fixed, let $\rho_h(y)$ be the radius of $A_h(x,y)$ (or $B_h(x,y)$, or $C_h(x,y)$). Then $\rho_h(y)$ decreases to $\rho(y)$ as $h \to \infty$. 

\[ x^* = \frac{1}{\mu} \]
3. Bridges of height $h$

**Proposition:** The length generating function $B_h(x, 1)$ of bridges of height $h$, taken at $x^* = 1/\mu$, satisfies

$$B_h(x^*, 1) \to 0 \quad \text{as } h \to \infty.$$  

**Remark:** the global identity implies that $B_h(x^*, 1)$ converges.

**Conjecture** (from SLE):

$$B_h(x^*, 1) \simeq h^{-1/4}$$

Inspired by [Duminil-Copin & Hammond], “The self-avoiding walk is sub-ballistic”, ArXiv 2012
C. Putting everything together
1. A lower bound on $y_c$

- For $x = x^* = 1/\sqrt{2 + \sqrt{2}}$ (that is, $x = 1/\mu$), and for any $y$,

\[ \alpha A_{h,\ell}(x^*, y) + \frac{y^* - y}{y(y^* - 1)} B_{h,\ell}(x^*, y) + \varepsilon E_{h,\ell}(x^*, y) = 1 \]

with $\alpha = \frac{\sqrt{2} - \sqrt{2}}{2}$, $\varepsilon = \frac{1}{\sqrt{2}}$ and $y^* = 1 + \sqrt{2}$.

- Let $y = y^*$. Then $A_{h,\ell}(x^*, y^*)$ increases with $\ell$ but remains bounded: its limit is $A_h(x^*, y^*)$ (arches in an $h$-strip), and is finite.

Thus

\[ y^* \leq y_h, \]

and by taking the limit on $h$,

\[ y^* \leq y_c. \]
2. A limit identity

We have

\[ \alpha A_{h, \ell}(x^*, y) + \frac{y^* - y}{y(y^* - 1)} B_{h, \ell}(x^*, y) + \varepsilon E_{h, \ell}(x^*, y) = 1. \]

with \( \alpha = \frac{\sqrt{2} - \sqrt{2}}{2} \), \( \varepsilon = \frac{1}{\sqrt{2}} \) and \( y^* = 1 + \sqrt{2} \).

Let \( \ell \to \infty \). For \( y < y_h \),

\[ \alpha A_h(x^*, y) + \frac{y^* - y}{y(y^* - 1)} B_h(x^*, y) = 1 \]
2. A limit identity

We have
\[ \alpha A_{h,\ell}(x^*, y) + \frac{y^* - y}{y(y^* - 1)} B_{h,\ell}(x^*, y) + \varepsilon E_{h,\ell}(x^*, y) = 1 \]
with \( \alpha = \frac{\sqrt{2} - \sqrt{2}}{2} \), \( \varepsilon = \frac{1}{\sqrt{2}} \) and \( y^* = 1 + \sqrt{2} \).

Let \( h \to \infty \). For \( y < y_h \),
\[ \alpha A_h(x^*, y) + \frac{y^* - y}{y(y^* - 1)} B_h(x^*, y) = 1 \]

Proof. \( A_{h,\ell}(x^*, y) \) and \( B_{h,\ell}(x^*, y) \) converge, and \( E_{h,\ell}(x^*, y) \) tends to 0 (because it counts walks of length at least \( \ell \), and \( C_h(x^*, y) \), the generating function of all SAWs in the \( h \)-strip, converges for \( y < y_h \)).

Remark. Implies that \( B_h(x^*, 1) \) converges as \( h \to \infty \).
3. Upper bound on $y_c$

- Arches that hit the top boundary:

$$A_{h+1}(x^*, y) - A_h(x^*, 1) \leq x^* B_{h+1}(x^*, y) B_h(x^*, 1)$$
3. Upper bound on $y_c$

- Arches that hit the top boundary:

$$A_{h+1}(x^*, y) - A_h(x^*, 1) \leq x^*B_{h+1}(x^*, y)B_h(x^*, 1)$$

- Combine this with

$$\alpha A_{h+1}(x^*, y) + \frac{y^* - y}{y(y^* - 1)}B_{h+1}(x^*, y) = 1 = \alpha A_h(x^*, 1) + B_h(x^*, 1).$$

- This gives, for $0 < y < y_{h+1}$ (and in particular for $y = y_c$),

$$\frac{1}{B_{h+1}(x^*, y)} \leq \alpha x^* + \frac{1}{B_h(x^*, 1)} \frac{y^* - y}{y(y^* - 1)}.$$
3. Upper bound on $y_c$ (end)

We have

$$\frac{1}{B_{h+1}(x^*, y_c)} \leq \alpha x^* + \frac{1}{B_h(x^*, 1)} \frac{y^* - y_c}{y_c(y^* - 1)}.$$ 

In particular,

$$0 \leq \alpha x^* + \frac{1}{B_h(x^*, 1)} \frac{y^* - y_c}{y_c(y^* - 1)}.$$ 

Given that $\lim_{h} B_h(x^*, 1) = 0$, this implies that

$$y_c \leq y^*.$$
Concluding remarks

- A similar result has been proved by Nick Beaton for SAWs confined to the half-plane \( \{x \geq 0\} \) (rather than \( \{y \geq 0\} \) here):

\[
y_c = \sqrt[3]{\frac{2 + \sqrt{2}}{1 + \sqrt{2} - \sqrt{2 + \sqrt{2}}}}.
\]

(conjectured by [Batchelor, Bennett-Wood and Owczarek 98])
B1. The first ingredient: Duminil-Copin and Smirnov’s global identity
Duminil-Copin and Smirnov’s global identity

\[
\frac{\sqrt{2 - \sqrt{2}}}{2} A_{h,\ell}(x^*) + B_{h,\ell}(x^*) + \frac{1}{\sqrt{2}} E_{h,\ell}(x^*) = 1
\]

Where does it come from?

From a local identity that is re-summed over all vertices of the domain.
Let $D \equiv D_{h,\ell}$ be our domain, $a$ the origin of the walks, and $p$ a point in the domain. Let

$$F(p) \equiv F(x, \alpha; p) = \sum_{\omega:a \leadsto p} x^{|\omega|} e^{i \alpha W(\omega)},$$

where $|\omega|$ is the length of $\omega$, and $W(\omega)$ its **winding number**:

$$W(\omega) = \frac{\pi}{3} (\text{left turns} - \text{right turns}).$$
A local identity

Let

\[ F(p) \equiv F(x, \alpha; p) = \sum_{\omega:a \sim p \text{ in } D} x|\omega|e^{i\alpha W(\omega)}, \]

If \( p, q \) and \( r \) are the 3 mid-edges around a vertex \( v \) of the honeycomb lattice, then, for \( x = x^* \) and \( \alpha = -5/8 \),

\[ (p - v)F(p) + (q - v)F(q) + (r - v)F(r) = 0. \]

Rem: \((p - v)\) is here a complex number!
A local identity

Proof: Group walks three by three or two by two as follows

- Walks that visit $p$, $q$ and $r$:

- Walks that only visit one or two of $p$, $q$ and $r$:

The contribution of all walks in a group is zero.
A local identity

• Walks that visit $p$, $q$ and $r$:

\[ e^{4i\alpha \pi/3} + j^2 e^{-4i\alpha \pi/3} = 2/j \cos \left( \frac{\pi}{3} + 4\alpha \frac{\pi}{3} \right) = 0 \]

• Walks that only visit one or two of $p$, $q$ and $r$:

\[ 1 + jx^* e^{-i\alpha \pi/3} + j^2 x^* e^{i\alpha \pi/3} = 0 \]
Proof of the global identity

Sum the local identity

\[(p - v)F(p) + (q - v)F(q) + (r - v)F(r) = 0\]

over all vertices \(v\) of the domain \(D_{h,\ell}\).

- The inner mid-edges do not contribute.
- The winding number of walks ending on the boundary is known.
- The domain has a right-left symmetry.
Proof of the global identity

Sum the local identity

\[(p - v)F(p) + (q - v)F(q) + (r - v)F(r) = 0\]

over all vertices \(v\) of the domain \(D_{h,\ell}\).

- The inner mid-edges do not contribute.
- The winding number of walks ending on the boundary is known.
- The domain has a right-left symmetry.

This gives:

\[
\frac{\sqrt{2} - \sqrt{2}}{2} A_{h,\ell}(x^*) + B_{h,\ell}(x^*) + \frac{1}{\sqrt{2}} E_{h,\ell}(x^*) = 1.
\]
B3. The third ingredient: Bridges of height $h$
Proposition: The length generating function $B_h(x, 1)$ of bridges of height $h$, taken at $x^* = 1/\mu$, satisfies

$$B_h(x^*, 1) \to 0 \quad \text{as} \quad h \to \infty.$$  

Remark: the global identity implies that $B_h(x^*, 1)$ converges.

Conjecture (from SLE):

$$B_h(x^*, 1) \approx h^{-1/4}$$

Inspired by [Duminil-Copin & Hammond], “The self-avoiding walk is sub-ballistic”, ArXiv 2012
Some tools and steps in the proof

- Irreducible bridges

Reducible

Irreducible
Some tools and steps in the proof

- Irreducible bridges

Proposition [Kesten 63]:

\[ \sum_{\omega \in iB} x_c|\omega| = 1. \]

(Related to \( B(x) = 1/(1 - IB(x)) \))

One can thus define random irreducible bridges by \( P_{iB}(\omega) = x_c|\omega| \).
• **Lemma:** (renewal theory) As $h \to \infty$,

\[ B_h(x_c, 1) \to \frac{1}{\mathbb{E}_{iB}(H(\omega))} \]

where $H(\omega)$ is the height of $\omega$. 
• **Lemma:** (renewal theory) As $h \to \infty$,

\[ B_h(x_c, 1) \to \frac{1}{\mathbb{E}_{iB}(H(\omega))} \]

where $H(\omega)$ is the height of $\omega$.

⇒ Prove that $\mathbb{E}_{iB}(H(\omega)) = \infty$!

We assume this is not the case.
• **Lemma:** (renewal theory) As \( h \to \infty \),

\[
B_h(x_c, 1) \to \frac{1}{\mathbb{E}_{iB}(H(\omega))}
\]

where \( H(\omega) \) is the height of \( \omega \).

\[ \Rightarrow \text{Prove that } \mathbb{E}_{iB}(H(\omega)) = \infty! \]

We assume this is not the case.

• **Proposition:** If \( \mathbb{E}_{iB}(H(\omega)) < \infty \), then \( \mathbb{E}_{iB}(W(\omega)) < \infty \).

The proof uses a global identity on a rectangle.

• **Corollary:** If \( \mathbb{E}_{iB}(H(\omega)) < \infty \), a random infinite bridge is tall and skinny.

Proof: law of large numbers
Proposition: If $E_{iB}(H(\omega)) < \infty$, a positive fraction of the renewal points of a random infinite bridge are **diamond points**.
Stickbreaking in bridges
Contradicts the fact that \( E_{iB}(H(\omega)) < \infty \), and thus

\[
B_h(x_c, 1) \to \frac{1}{E_{iB}(H(\omega))} = 0.
\]