

**Self-avoiding walks
interacting with a surface
on the honeycomb lattice**

Mireille Bousquet-Mélou, CNRS, LaBRI, Bordeaux, France

joint work with

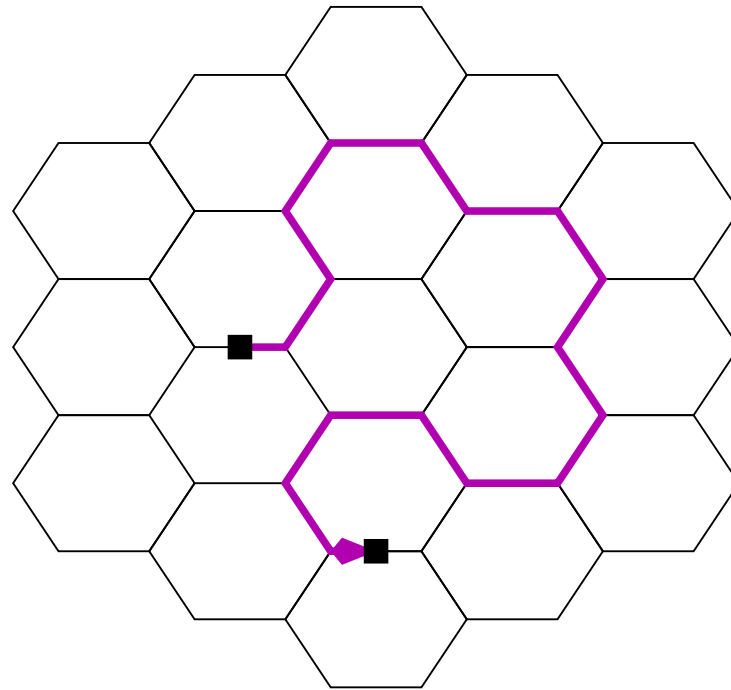
Nick Beaton, Hugo Duminil-Copin, Jan de Gier and

Tony Guttmann

ArXiv 1109.0358 (v3 in June 2012)

A. Introduction

Self-avoiding walks (SAW)



What is c_n , the number of n -step SAWs?

Hard! c_n is known up to $n = 105$ [Jensen 06]

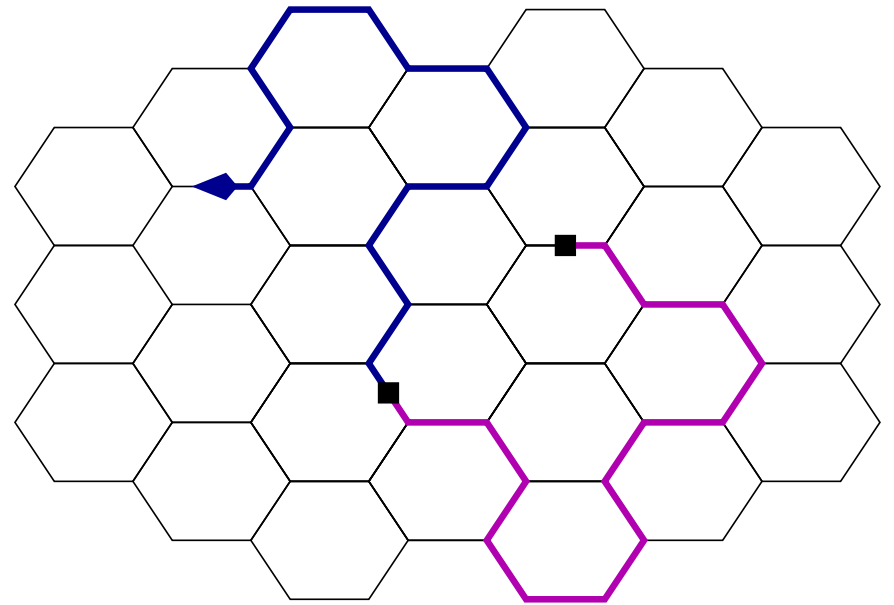
The growth constant

Clearly,

$$c_{m+n} \leq c_m c_n$$

$\Rightarrow \lim_n c_n^{1/n}$ exists and

$$\mu := \lim_n c_n^{1/n} = \inf_n c_n^{1/n}$$



Conjecture [Nienhuis 82]: the growth constant is

$$\mu = \sqrt{2 + \sqrt{2}}$$

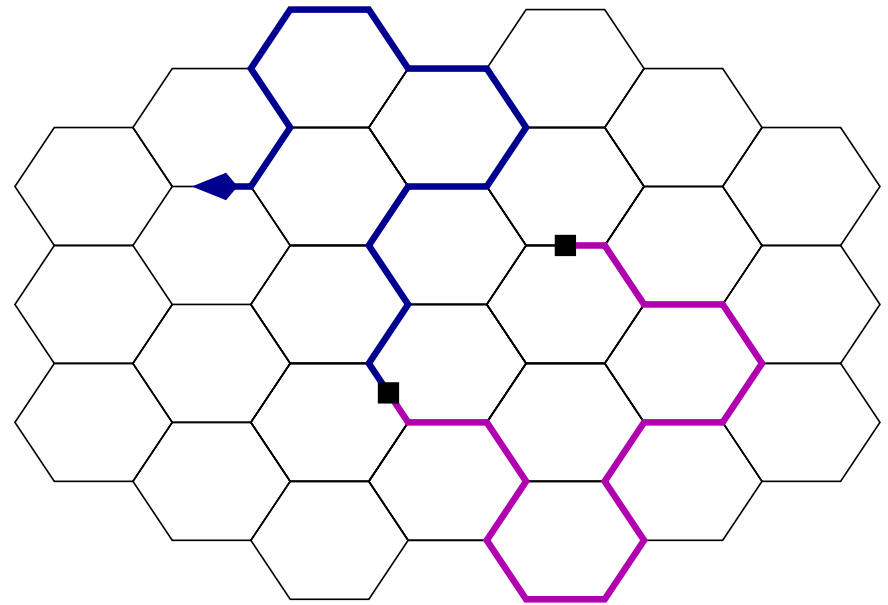
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Theorem [Duminil-Copin & Smirnov 10]: the growth constant is

$$\mu = \sqrt{2 + \sqrt{2}}$$

Growth constants and generating functions

- Let $C(x)$ be the length generating function of SAWs:

$$C(x) = \sum_{n \geq 0} c_n x^n.$$

- The radius of convergence of $C(x)$ is

$$\rho = 1/\mu,$$

where

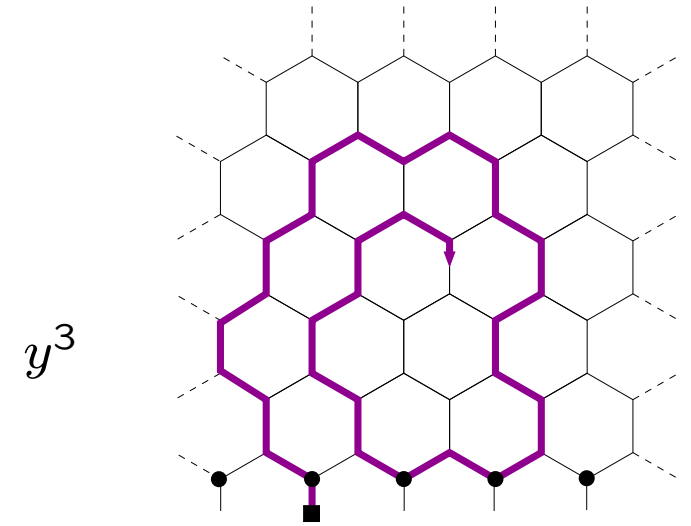
$$\mu = \lim_n c_n^{1/n}$$

is the growth constant.

Walks in a half-plane interacting with a “surface”

- Enumeration by contacts of n -step walks:

$$\bar{c}_n(y) = \sum_{\omega} y^{\text{contacts}(\omega)}$$



In statistical physics, the parameter y is called “fugacity”

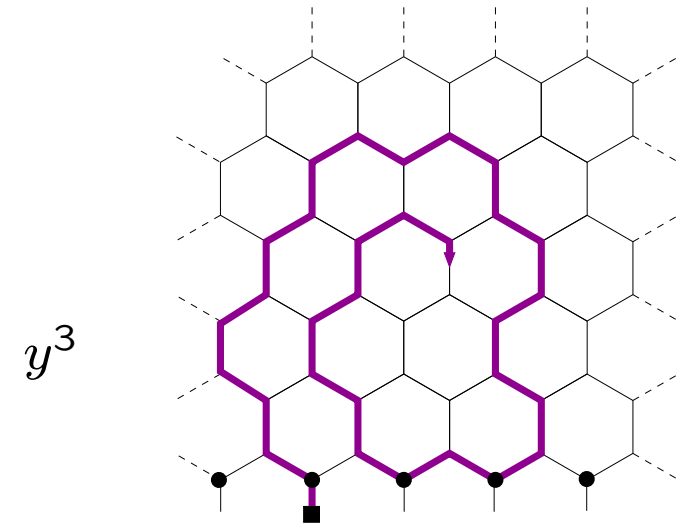
Walks in a half-plane interacting with a “surface”

- Enumeration by contacts of n -step walks:

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- Generating function

$$\bar{C}(x, y) = \sum_{n \geq 0} \bar{c}_n(y) x^n$$



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Walks in a half-plane interacting with a “surface”

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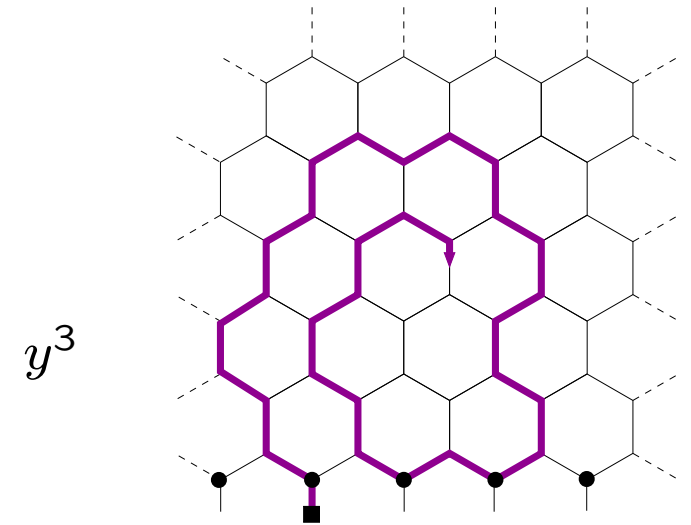
$$\bar{C}(x, y) = \sum_{n \geq 0} \bar{c}_n(y) x^n$$

- Radius and growth constant ($y > 0$ fixed):

$$\rho(y) = \frac{1}{\mu(y)} = \lim_n \bar{c}_n(y)^{-1/n}$$

[Hammersley, Torrie and Whittington 82]

In statistical physics, the parameter y is called “fugacity”



The critical fugacity

- Radius and growth constant: for $y > 0$,

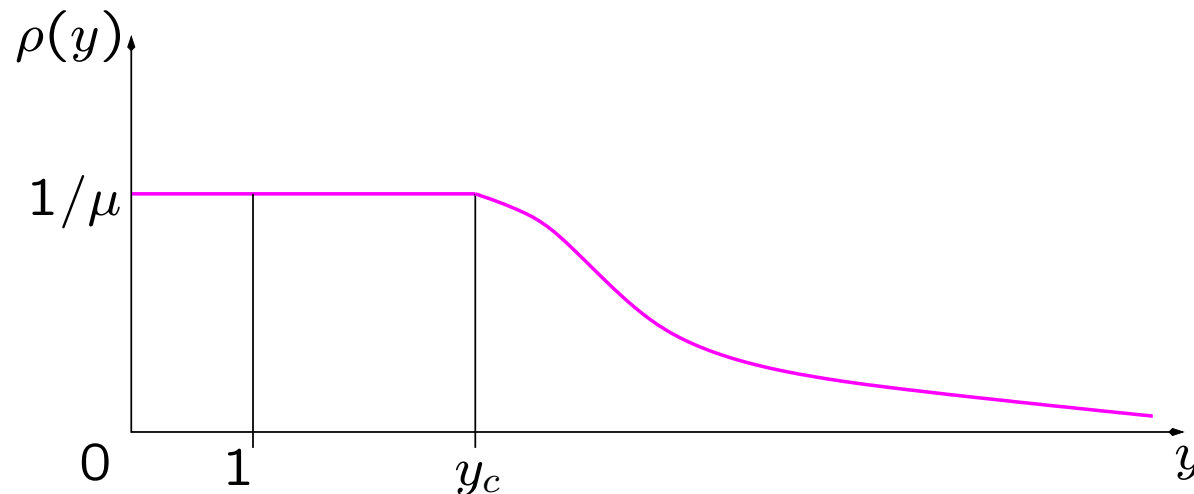
$$\rho(y) = \frac{1}{\mu(y)} = \lim_n \bar{c}_n(y)^{-1/n}$$

Proposition: $\rho(y)$ is a continuous, weakly decreasing function of $y \in (0, \infty)$. There exists $y_c > 1$ such that

$$\rho(y) \begin{cases} = 1/\mu & \text{if } y \leq y_c, \\ < 1/\mu & \text{if } y > y_c, \end{cases}$$

where μ is the growth constant of (unrestricted) SAWs.

[Whittington 75, Hammersley, Torrie and Whittington 82]

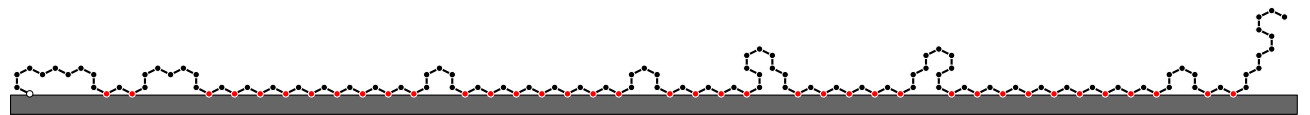
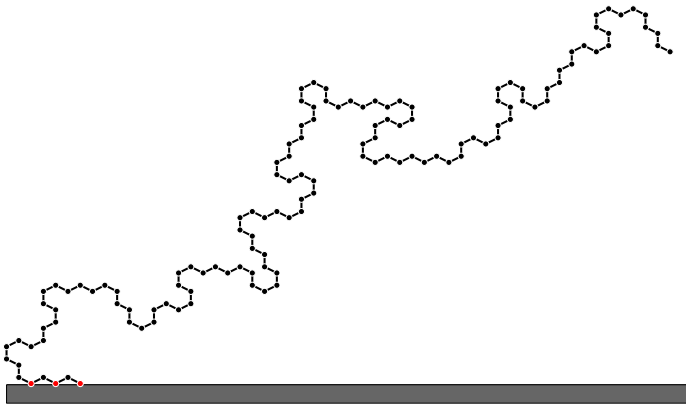


The critical fugacity: probabilistic meaning

Take half-space SAWs of length n under the Boltzmann distribution

$$\mathbb{P}_n(\omega) = \frac{y^{\text{contacts}(\omega)}}{\bar{c}_n(y)}.$$

Then for $y < y_c$, the walk escapes from the surface. For $y > y_c$, a positive fraction of its vertices lie on the surface.



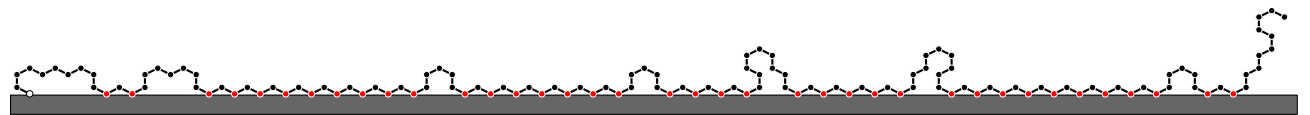
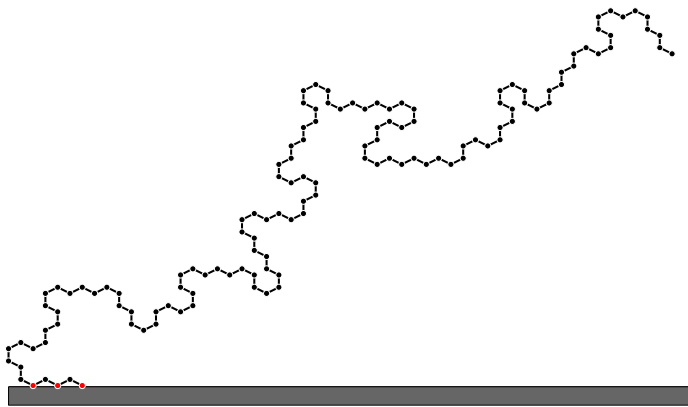
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Take half-space SAWs of length n under the Boltzmann distribution

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Theorem [B-BM-dG-DC-G 12]: this phase transition occurs at

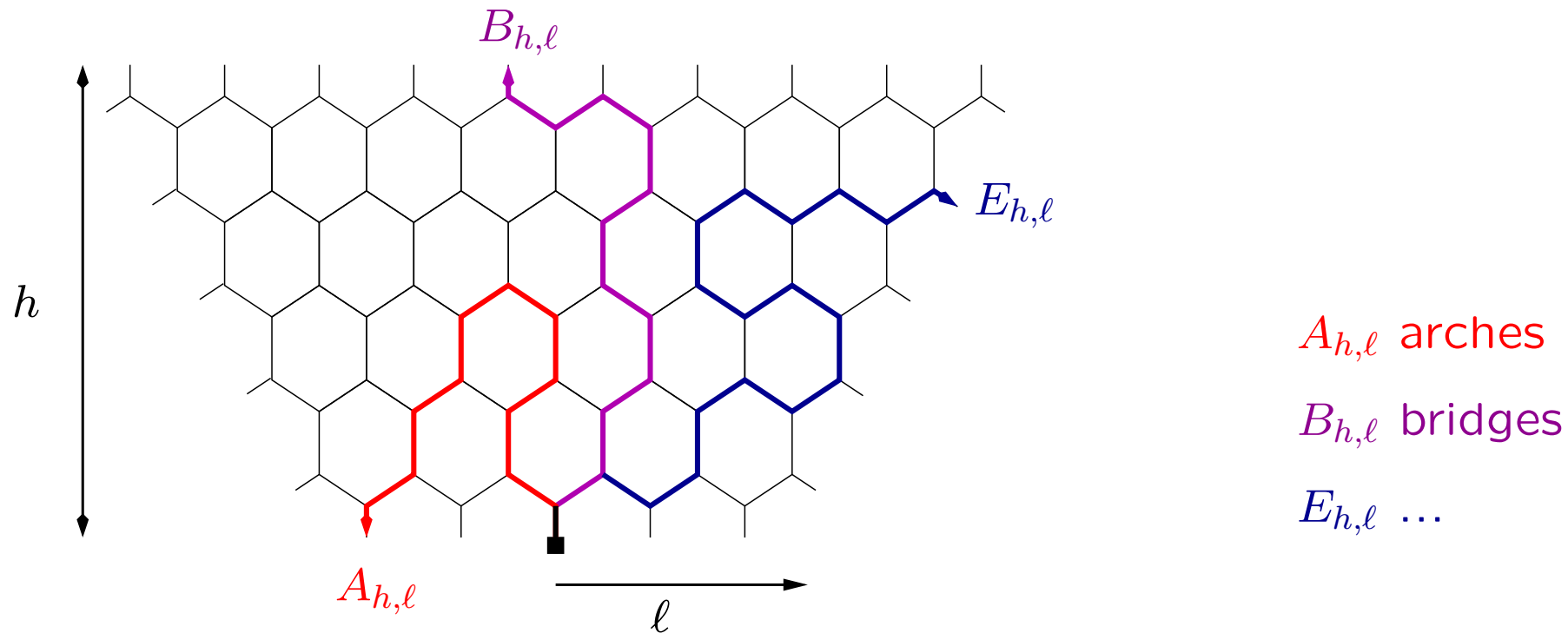
$$y_c = 1 + \sqrt{2}$$

(conjectured by Batchelor and Yung in 1995)

B. Three ingredients

1. Duminil-Copin and Smirnov's "global" identity

Consider the following finite domain $D_{h,\ell}$.



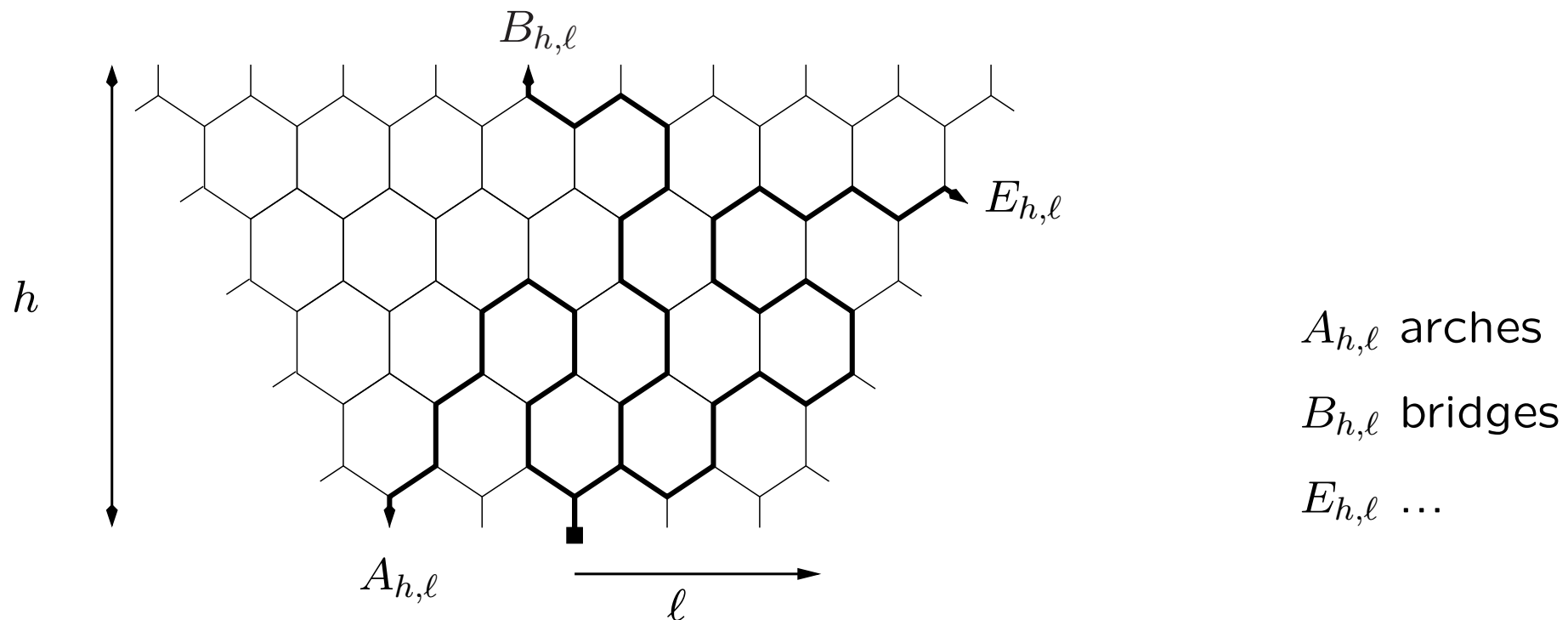
Let $A_{h,\ell}(x)$ (resp. $B_{h,\ell}(x)$, $E_{h,\ell}(x)$) be the length generating function of SAWs that start from the origin and end on the bottom (resp. top, right/left) border of the domain $D_{h,\ell}$. These series are **polynomials** in x .

1. Duminil-Copin and Smirnov's "global" identity

For $x = x^* = 1/\sqrt{2 + \sqrt{2}}$ (that is, $x = 1/\mu$),

$$\alpha A_{h,\ell}(x^*) + B_{h,\ell}(x^*) + \varepsilon E_{h,\ell}(x^*) = 1$$

with $\alpha = \frac{\sqrt{2-\sqrt{2}}}{2}$ and $\varepsilon = \frac{1}{\sqrt{2}}$.

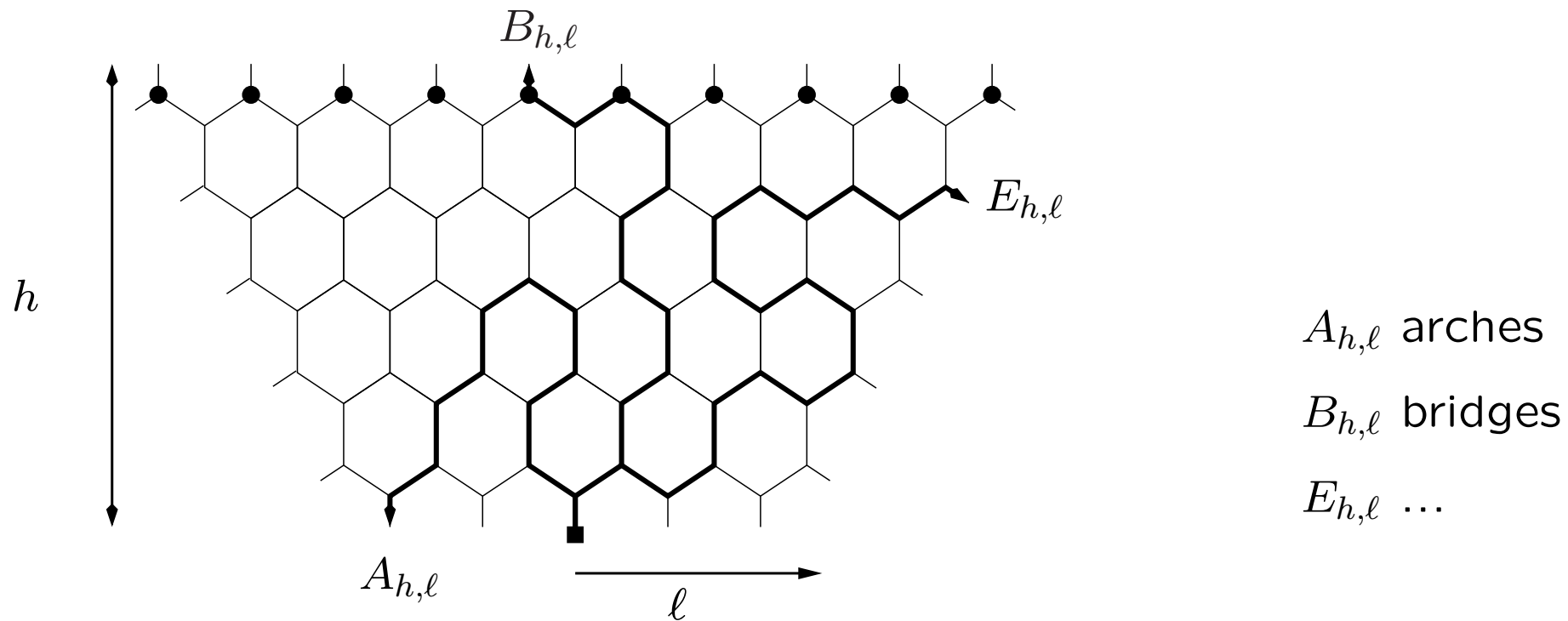


1. Duminil-Copin and Smirnov's "global" identity: refinement with *upper* contacts

For $x = x^* = 1/\sqrt{2 + \sqrt{2}}$ (that is, $x = 1/\mu$), and for **any** y ,

$$\alpha A_{h,\ell}(x^*, y) + \frac{y^* - y}{y(y^* - 1)} B_{h,\ell}(x^*, y) + \varepsilon E_{h,\ell}(x^*, y) = 1$$

with $\alpha = \frac{\sqrt{2-\sqrt{2}}}{2}$, $\varepsilon = \frac{1}{\sqrt{2}}$ and $y^* = 1 + \sqrt{2}$.

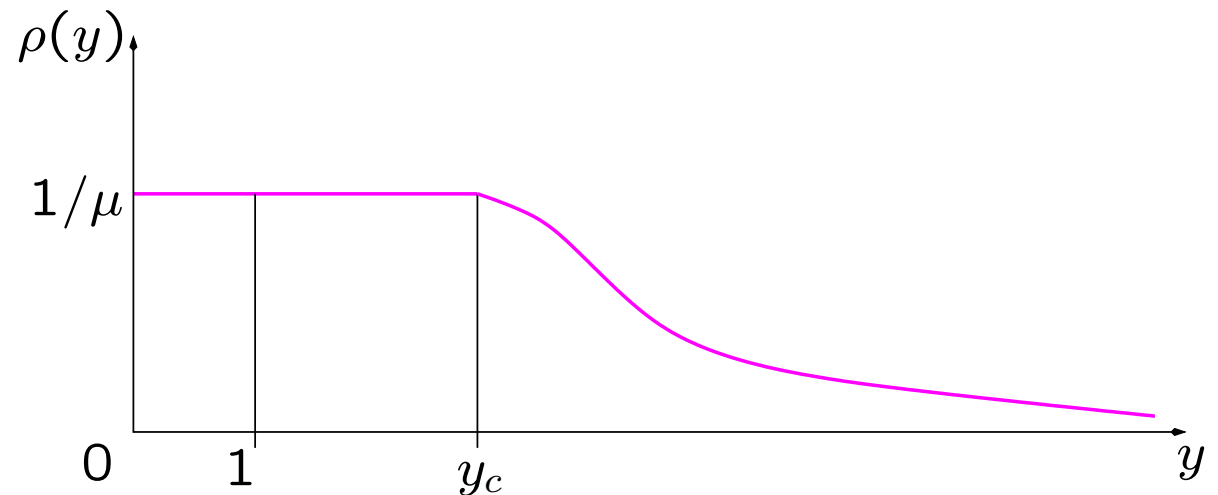
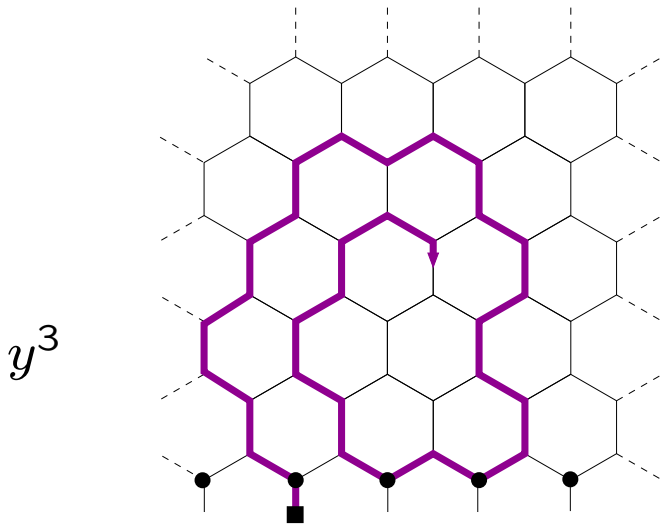


2. An alternative description of the critical fugacity

Proposition: The radius $\rho(y)$ of $\bar{C}(x, y)$ is a continuous, weakly decreasing function of $y \in (0, \infty)$. There exists $y_c > 1$ such that

$$\rho(y) \begin{cases} = 1/\mu & \text{if } y \leq y_c, \\ < 1/\mu & \text{if } y > y_c, \end{cases}$$

where μ is the growth constant of (unrestricted) SAWs.



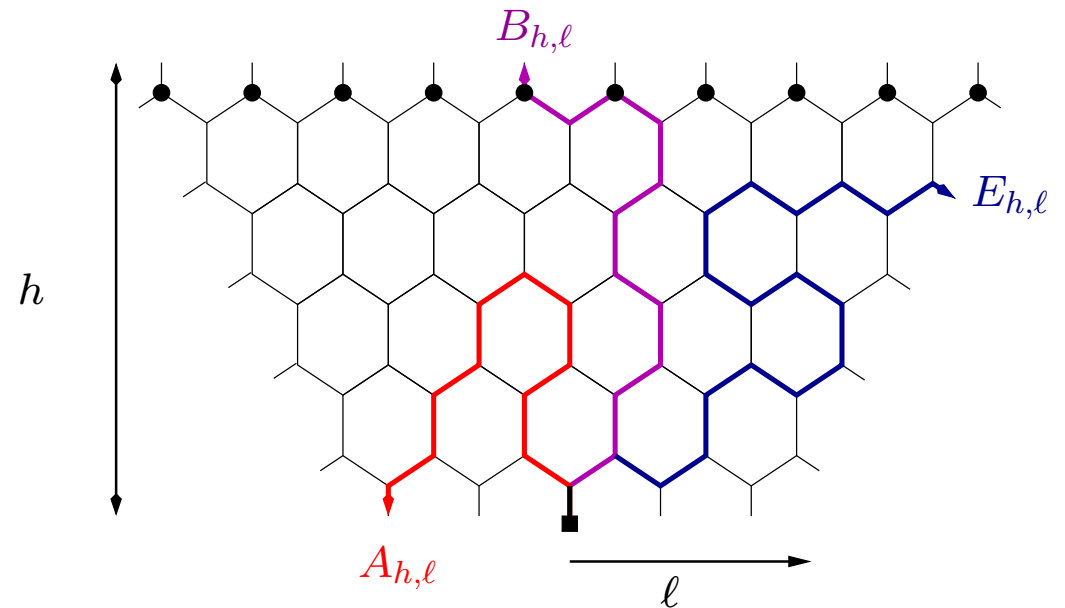
Our identity

For $x = x^* = 1/\sqrt{2 + \sqrt{2}}$ (that is, $x = 1/\mu$), and for any y ,

$$\alpha A_{h,\ell}(x^*, y) + \frac{y^* - y}{y(y^* - 1)} B_{h,\ell}(x^*, y) + \varepsilon E_{h,\ell}(x^*, y) = 1$$

with $\alpha = \frac{\sqrt{2-\sqrt{2}}}{2}$, $\varepsilon = \frac{1}{\sqrt{2}}$ and $y^* = 1 + \sqrt{2}$.

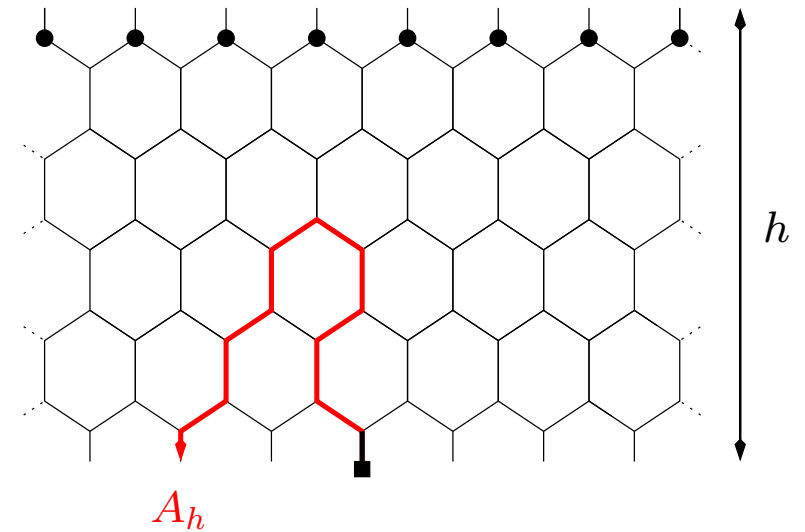
- Walks in a bounded domain
- Contacts on the “wrong” side
- Only valid at $x = x^* = 1/\mu$



2. An alternative description of the critical fugacity

Proposition: Let $A_h(x, y)$ be the generating function of arches in a strip of height h , counted by length and upper contacts. Let y_h be the radius of convergence of $A_h(x^*, y)$, where $x^* = 1/\mu$ (*). Then, as $h \rightarrow \infty$,

$$y_h \searrow y_c.$$



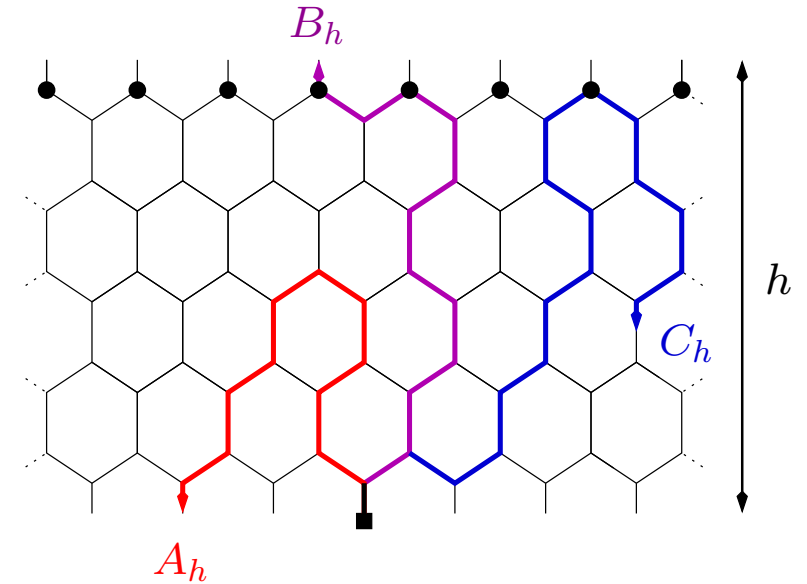
(*) For all k , the coefficient of y^k in $A_h(x, y)$ is finite at $x^* = 1/\mu$

(uses [van Rensburg, Orlandini and Whittington 06])

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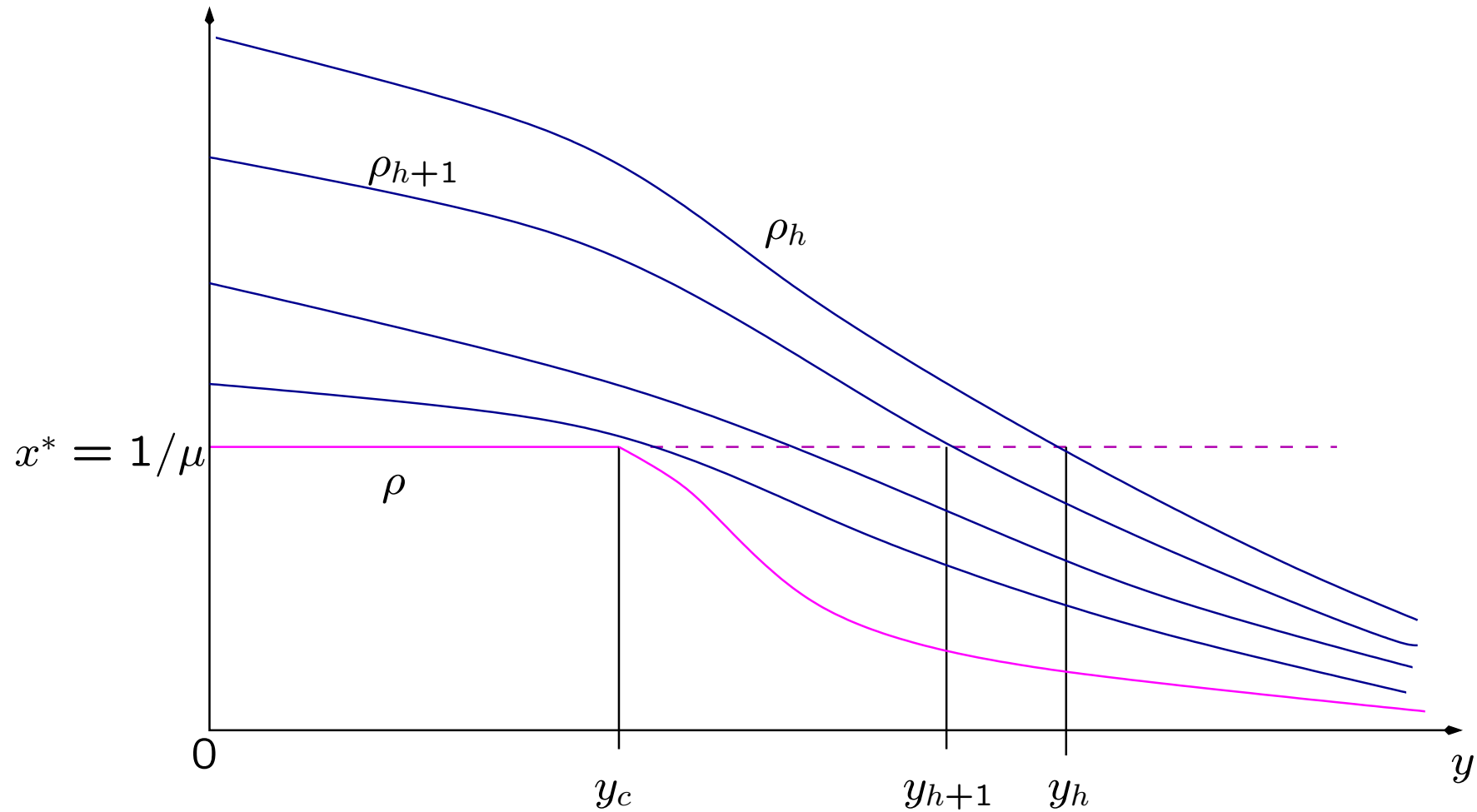
The same holds for the generating function $B_h(x, y)$ of bridges, and for the generating function $C_h(x, y)$ of all SAWs in the h -strip, with the *same value* of y_h .

(*) For all k , the coefficient of y^k in $A_h(x, y)$ is finite at $x^* = 1/\mu$

(uses [van Rensburg, Orlandini and Whittington 06])

The complete picture

For $y > 0$ fixed, let $\rho_h(y)$ be the radius of $A_h(x, y)$ (or $B_h(x, y)$, or $C_h(x, y)$). Then $\rho_h(y)$ decreases to $\rho(y)$ as $h \rightarrow \infty$.



3. Bridges of height h

Proposition: The length generating function $B_h(x, 1)$ of bridges of height h , taken at $x^* = 1/\mu$, satisfies

$$B_h(x^*, 1) \rightarrow 0 \quad \text{as } h \rightarrow \infty.$$

Remark: the global identity implies that $B_h(x^*, 1)$ converges.

Conjecture (from SLE):

$$B_h(x^*, 1) \simeq h^{-1/4}$$

Inspired by [Duminil-Copin & Hammond], “The self-avoiding walk is sub-ballistic”,
ArXiv 2012

C. Putting everything together

1. A lower bound on y_c

- For $x = x^* = 1/\sqrt{2 + \sqrt{2}}$ (that is, $x = 1/\mu$), and for any y ,

$$\alpha A_{h,\ell}(x^*, y) + \frac{y^* - y}{y(y^* - 1)} B_{h,\ell}(x^*, y) + \varepsilon E_{h,\ell}(x^*, y) = 1$$

with $\alpha = \frac{\sqrt{2-\sqrt{2}}}{2}$, $\varepsilon = \frac{1}{\sqrt{2}}$ and $y^* = 1 + \sqrt{2}$.

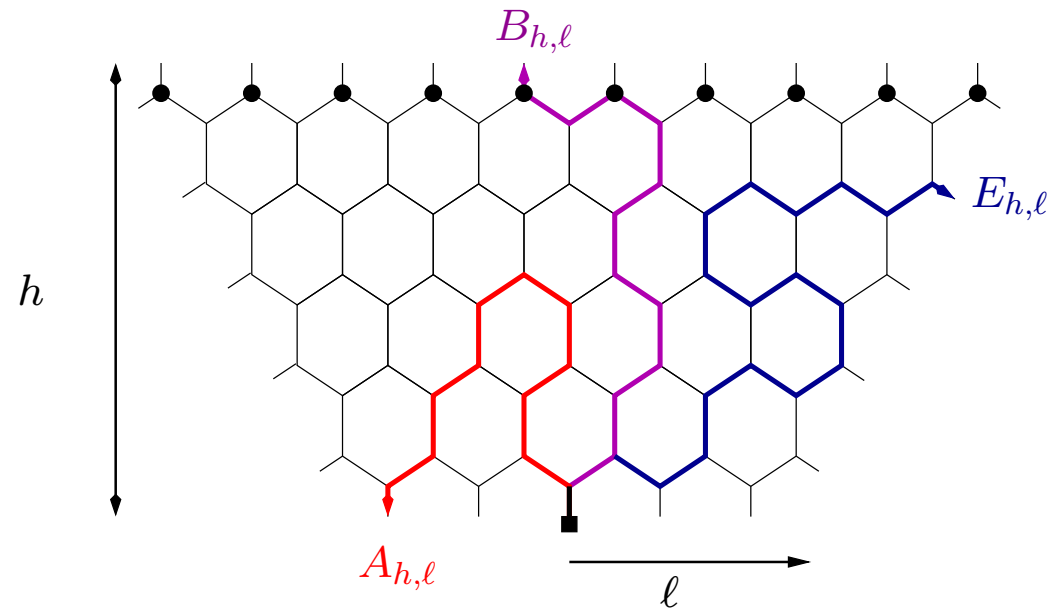
- Let $y = y^*$. Then $A_{h,\ell}(x^*, y^*)$ increases with ℓ but remains bounded: its limit is $A_h(x^*, y^*)$ (arches in an h -strip), and is finite.

Thus

$$y^* \leq y_h,$$

and by taking the limit on h ,

$$y^* \leq y_c.$$

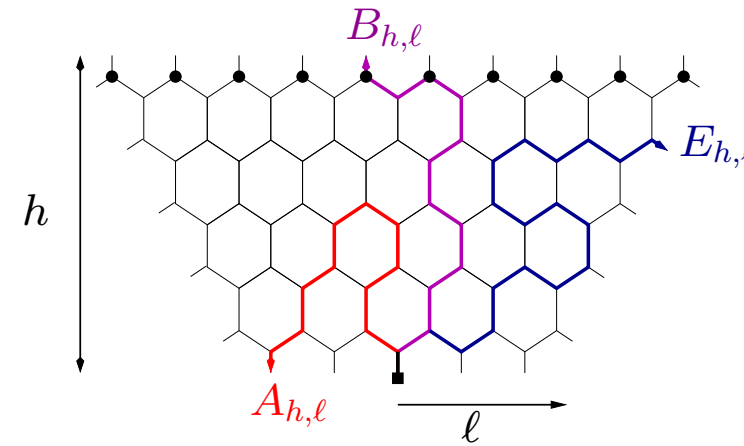


2. A limit identity

We have

$$\alpha A_{h,\ell}(x^*, y) + \frac{y^* - y}{y(y^* - 1)} B_{h,\ell}(x^*, y) + \varepsilon E_{h,\ell}(x^*, y) = 1.$$

with $\alpha = \frac{\sqrt{2-\sqrt{2}}}{2}$, $\varepsilon = \frac{1}{\sqrt{2}}$ and $y^* = 1 + \sqrt{2}$.



Let $\ell \rightarrow \infty$. For $y < y_h$,

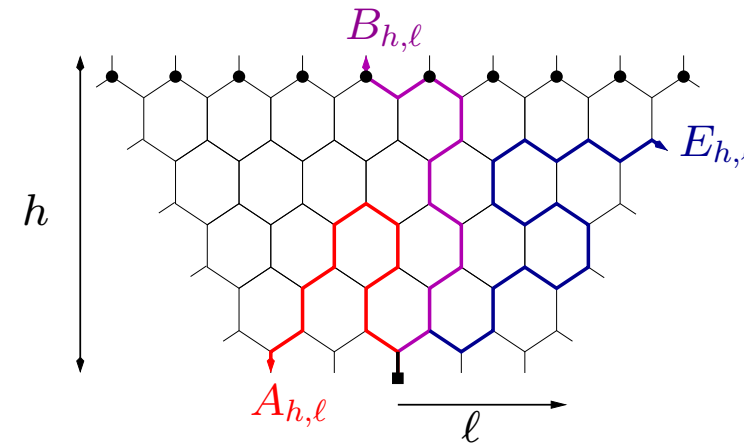
$$\alpha A_h(x^*, y) + \frac{y^* - y}{y(y^* - 1)} B_h(x^*, y) = 1$$

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Let $h \rightarrow \infty$. For $y < y_h$,

$$\alpha A_h(x^*, y) + \frac{y^* - y}{y(y^* - 1)} B_h(x^*, y) = 1$$

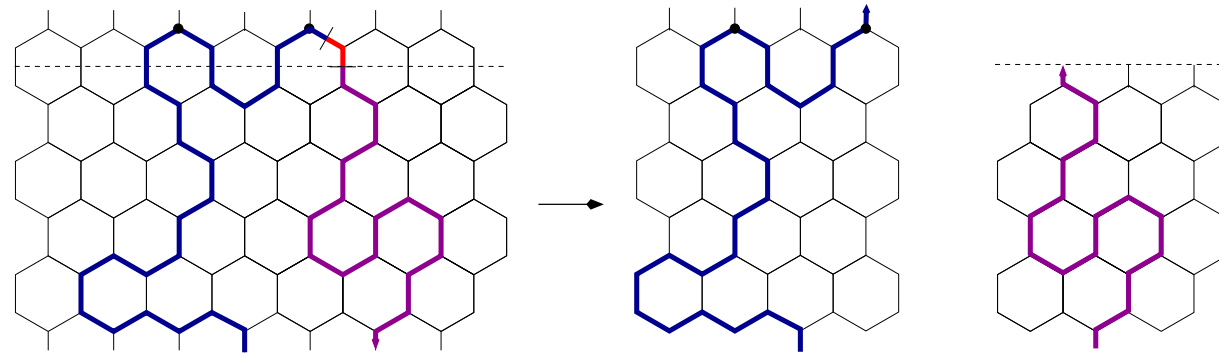
Proof. $A_{h,\ell}(x^*, y)$ and $B_{h,\ell}(x^*, y)$ converge, and $E_{h,\ell}(x^*, y)$ tends to 0 (because it counts walks of length at least ℓ , and $C_h(x^*, y)$, the generating function of all SAWs in the h -strip, converges for $y < y_h$).

Remark. Implies that $B_h(x^*, 1)$ converges as $h \rightarrow \infty$.

3. Upper bound on y_c

- Arches that hit the top boundary:

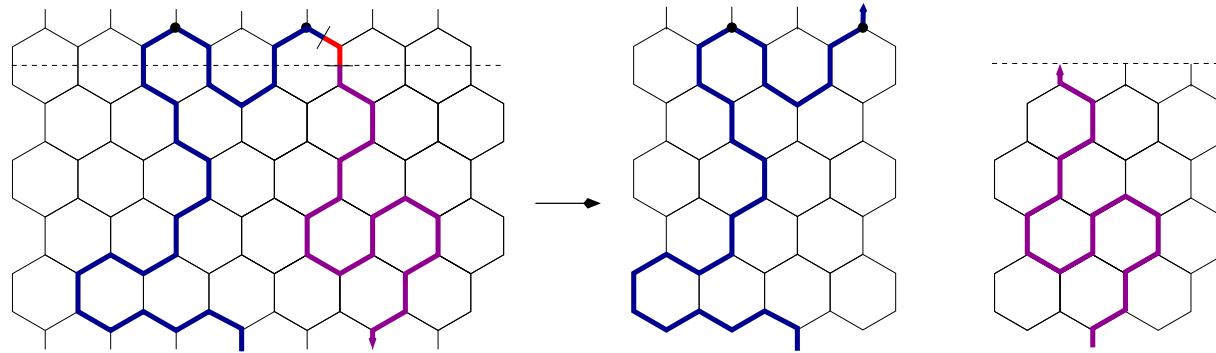
$$A_{h+1}(x^*, y) - A_h(x^*, 1) \leq x^* B_{h+1}(x^*, y) B_h(x^*, 1)$$



3. Upper bound on y_c

- Arches that hit the top boundary:

$$A_{h+1}(x^*, y) - A_h(x^*, 1) \leq x^* B_{h+1}(x^*, y) B_h(x^*, 1)$$



- Combine this with

$$\alpha A_{h+1}(x^*, y) + \frac{y^* - y}{y(y^* - 1)} B_{h+1}(x^*, y) = 1 = \alpha A_h(x^*, 1) + B_h(x^*, 1).$$

- This gives, for $0 < y < y_{h+1}$ (and in particular for $y = y_c$),

$$\frac{1}{B_{h+1}(x^*, y)} \leq \alpha x^* + \frac{1}{B_h(x^*, 1)} \frac{y^* - y}{y(y^* - 1)}.$$

3. Upper bound on y_c (end)

We have

$$\frac{1}{B_{h+1}(x^*, y_c)} \leq \alpha x^* + \frac{1}{B_h(x^*, 1)} \frac{y^* - y_c}{y_c(y^* - 1)}.$$

In particular,

$$0 \leq \alpha x^* + \frac{1}{B_h(x^*, 1)} \frac{y^* - y_c}{y_c(y^* - 1)}.$$

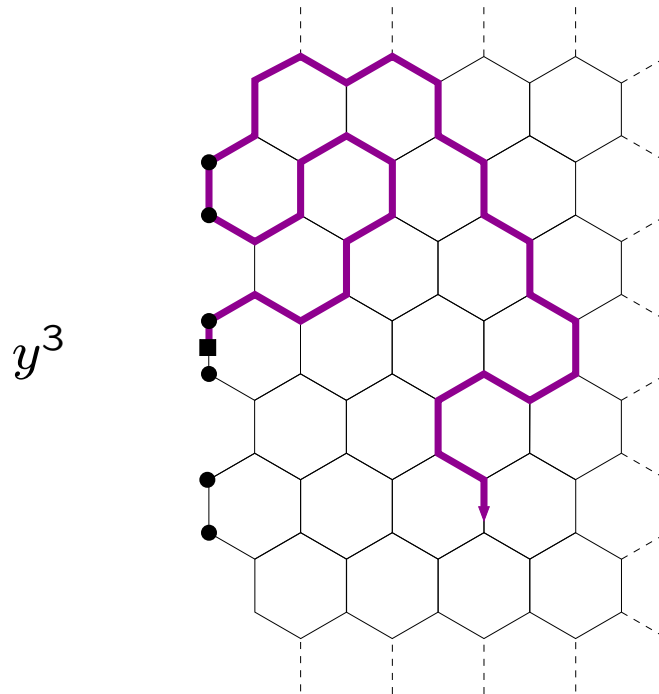
Given that $\lim_h B_h(x^*, 1) = 0$, this implies that

$$y_c \leq y^*.$$

Concluding remarks

- A similar result has been proved by Nick Beaton for SAWs confined to the half-plane $\{x \geq 0\}$ (rather than $\{y \geq 0\}$ here):

$$y_c = \sqrt{\frac{2 + \sqrt{2}}{1 + \sqrt{2} - \sqrt{2 + \sqrt{2}}}}$$



(conjectured by [Batchelor, Bennett-Wood and Owczarek 98])

B1. The first ingredient: Duminił-Copin and Smirnov's global identity

Duminil-Copin and Smirnov's global identity

$$\frac{\sqrt{2 - \sqrt{2}}}{2} A_{h,\ell}(x^*) + B_{h,\ell}(x^*) + \frac{1}{\sqrt{2}} E_{h,\ell}(x^*) = 1$$

Where does it come from?

From a local identity that is re-summed over all vertices of the domain.

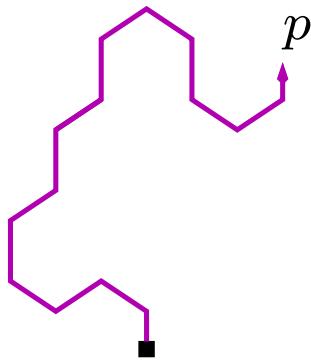
A local identity

Let $D \equiv D_{h,\ell}$ be our domain, a the origin of the walks, and p a point in the domain. Let

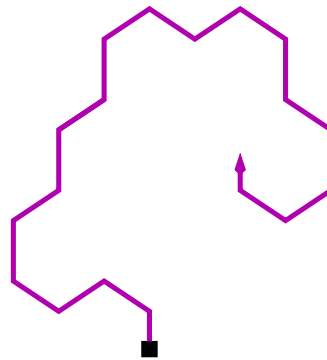
$$F(p) \equiv F(x, \alpha; p) = \sum_{\omega: a \rightsquigarrow p} x^{|\omega|} e^{i\alpha W(\omega)},$$

where $|\omega|$ is the length of ω , and $W(\omega)$ its **winding number**:

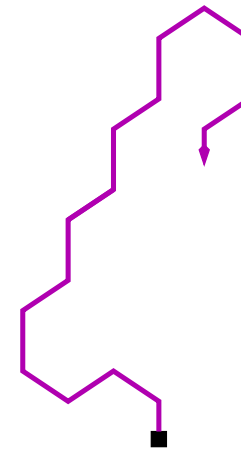
$$W(\omega) = \frac{\pi}{3} (\text{left turns} - \text{right turns}).$$



$$W(\omega) = 0$$



$$W(\omega) = -2\pi$$



$$W(\omega) = -\pi$$

A local identity

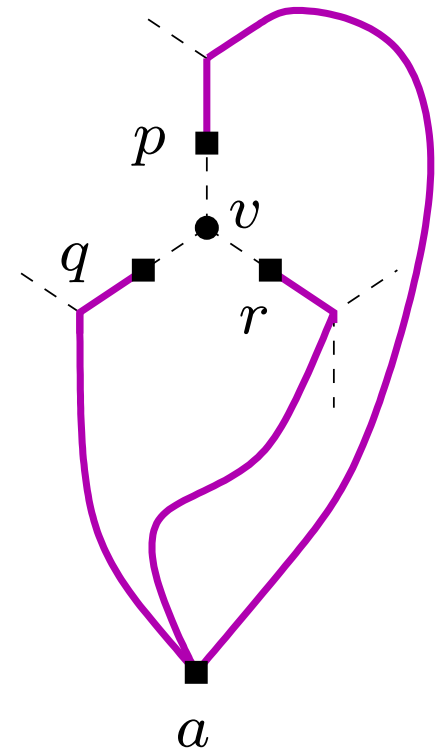
Let

$$F(p) \equiv F(x, \alpha; p) = \sum_{\omega: a \rightsquigarrow p \text{ in } D} x^{|\omega|} e^{i\alpha W(\omega)},$$

If p , q and r are the 3 mid-edges around a vertex v of the honeycomb lattice, then, for $x = x^*$ and $\alpha = -5/8$,

$$(p - v)F(p) + (q - v)F(q) + (r - v)F(r) = 0.$$

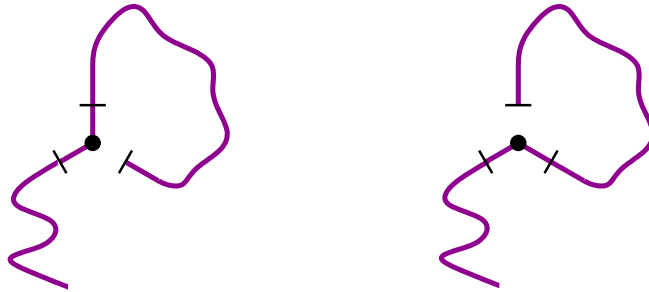
Rem: $(p - v)$ is here a complex number!



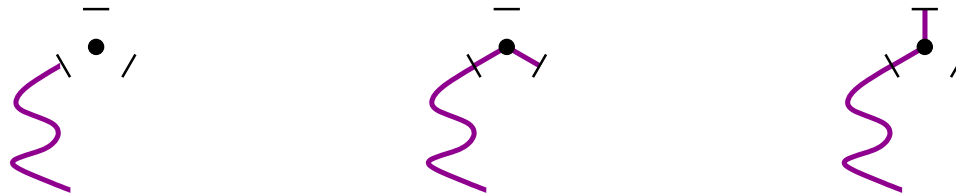
A local identity

Proof: Group walks three by three or two by two as follows

- Walks that visit p , q and r :



- Walks that only visit one or two of p , q and r :



The contribution of all walks in a group is zero.

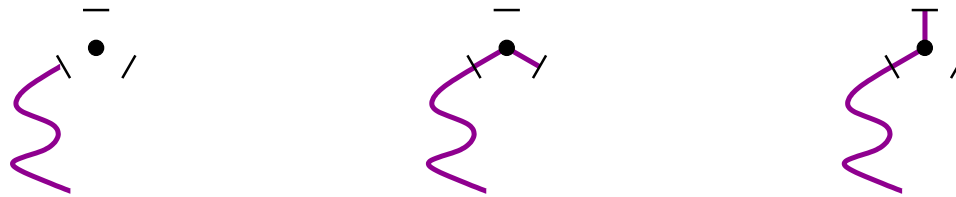
A local identity

- Walks that visit p , q and r :



$$e^{4i\alpha\pi/3} + j^2 e^{-4i\alpha\pi/3} = 2/j \cos\left(\frac{\pi}{3} + 4\alpha\frac{\pi}{3}\right) = 0$$

- Walks that only visit one or two of p , q and r :



$$1 + jx^* e^{-i\alpha\pi/3} + j^2 x^* e^{i\alpha\pi/3} = 0$$

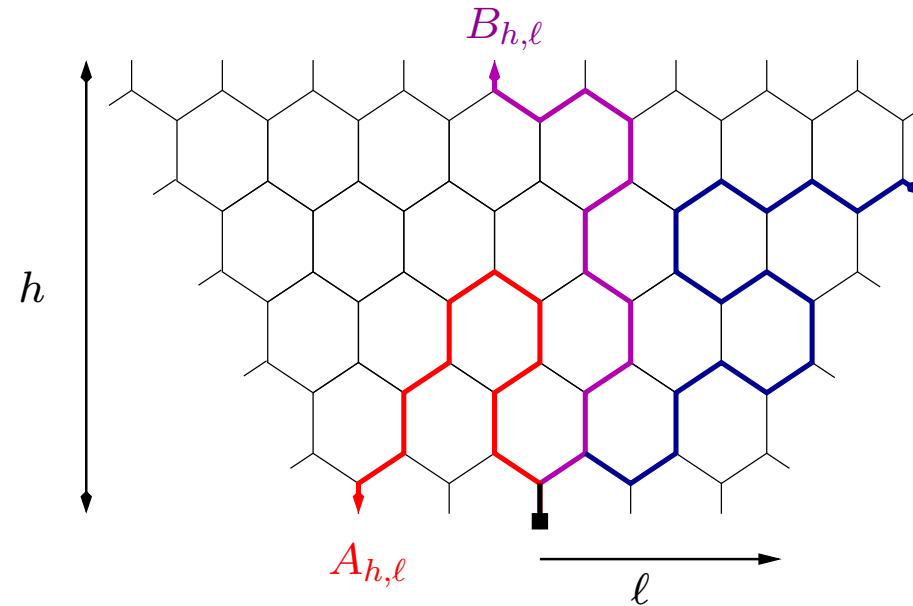
Proof of the global identity

Sum the local identity

$$(p - v)F(p) + (q - v)F(q) + (r - v)F(r) = 0$$

over all vertices v of the domain $D_{h,\ell}$.

- The inner mid-edges do not contribute.
- The winding number of walks ending on the boundary is known.
- The domain has a right-left symmetry.



Proof of the global identity

Sum the local identity

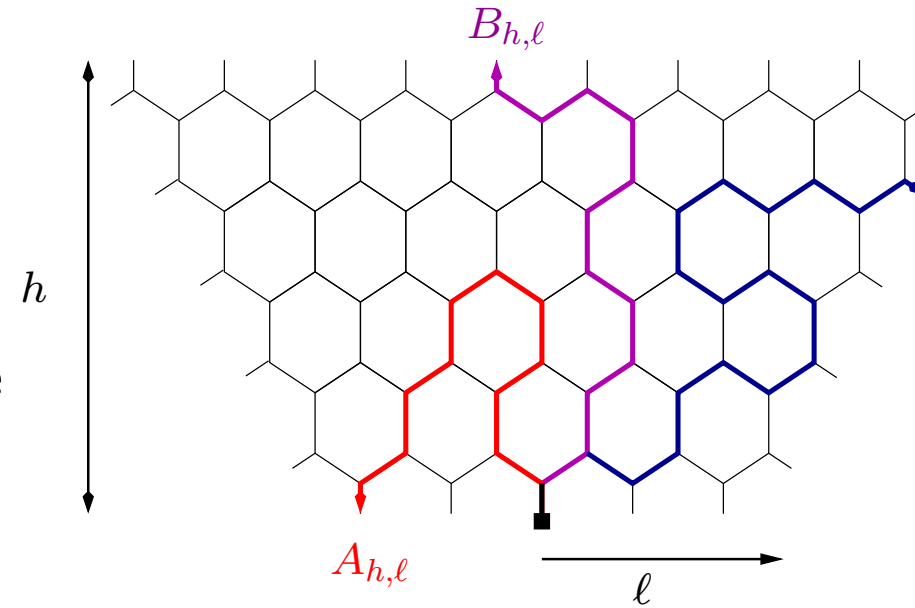
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over all vertices v of the domain $D_{h,\ell}$.

- The inner mid-edges do not contribute.
- The winding number of walks ending on the boundary is known.
- The domain has a right-left symmetry.

This gives:

$$\frac{\sqrt{2 - \sqrt{2}}}{2} A_{h,\ell}(x^*) + B_{h,\ell}(x^*) + \frac{1}{\sqrt{2}} E_{h,\ell}(x^*) = 1.$$



B3. The third ingredient: Bridges of height h

Bridges of height h

Proposition: The length generating function $B_h(x, 1)$ of bridges of height h , taken at $x^* = 1/\mu$, satisfies

$$B_h(x^*, 1) \rightarrow 0 \quad \text{as } h \rightarrow \infty.$$

Remark: the global identity implies that $B_h(x^*, 1)$ converges.

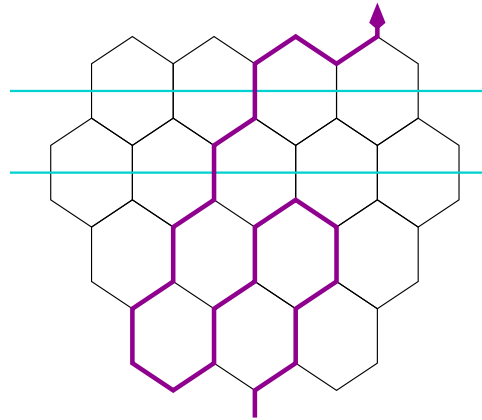
Conjecture (from SLE):

$$B_h(x^*, 1) \simeq h^{-1/4}$$

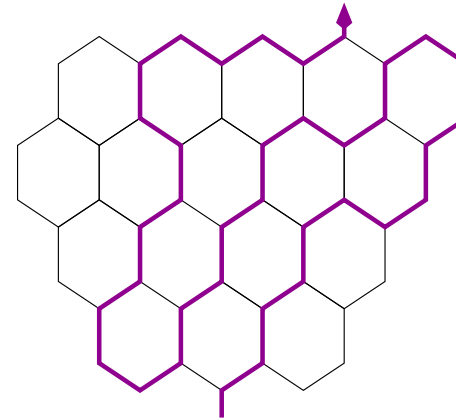
Inspired by [Duminil-Copin & Hammond], “The self-avoiding walk is sub-ballistic”,
ArXiv 2012

Some tools and steps in the proof

- Irreducible bridges



Reducible



Irreducible

Proposition [Kesten 63]:

$$\sum_{\omega \in iB} x_c^{|\omega|} = 1.$$

(Related to $B(x) = 1/(1 - IB(x))$)

One can thus define **random irreducible bridges** by $\mathbb{P}_{iB}(\omega) = x_c^{|\omega|}$.

- **Lemma:** (renewal theory) As $h \rightarrow \infty$,

$$B_h(x_c, 1) \rightarrow \frac{1}{\mathbb{E}_{iB}(H(\omega))}$$

where $H(\omega)$ is the height of ω .

- **Lemma:** (renewal theory) As $h \rightarrow \infty$,

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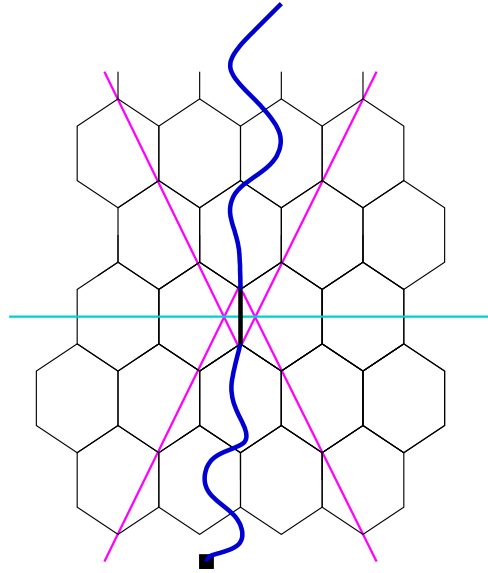
- **Proposition:** If $E_{i_B}(H(\omega)) < \infty$, then $E_{i_B}(W(\omega)) < \infty$.

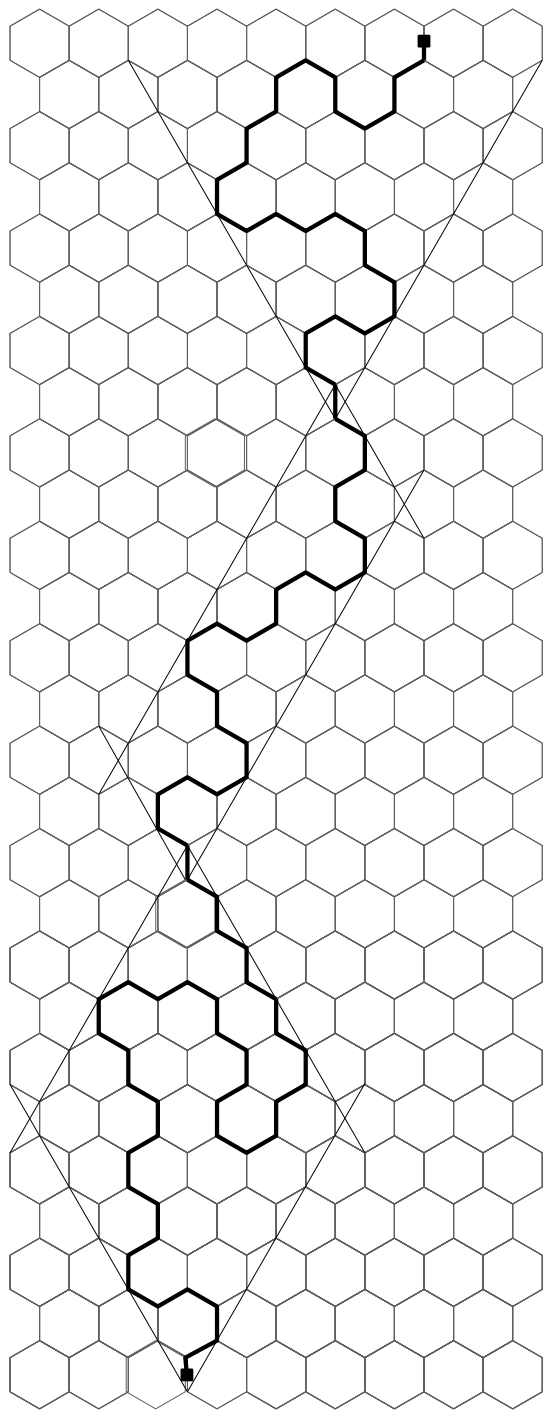
The proof uses a global identity on a rectangle.

- **Corollary:** If $E_{i_B}(H(\omega)) < \infty$, a **random infinite bridge** is tall and skinny.

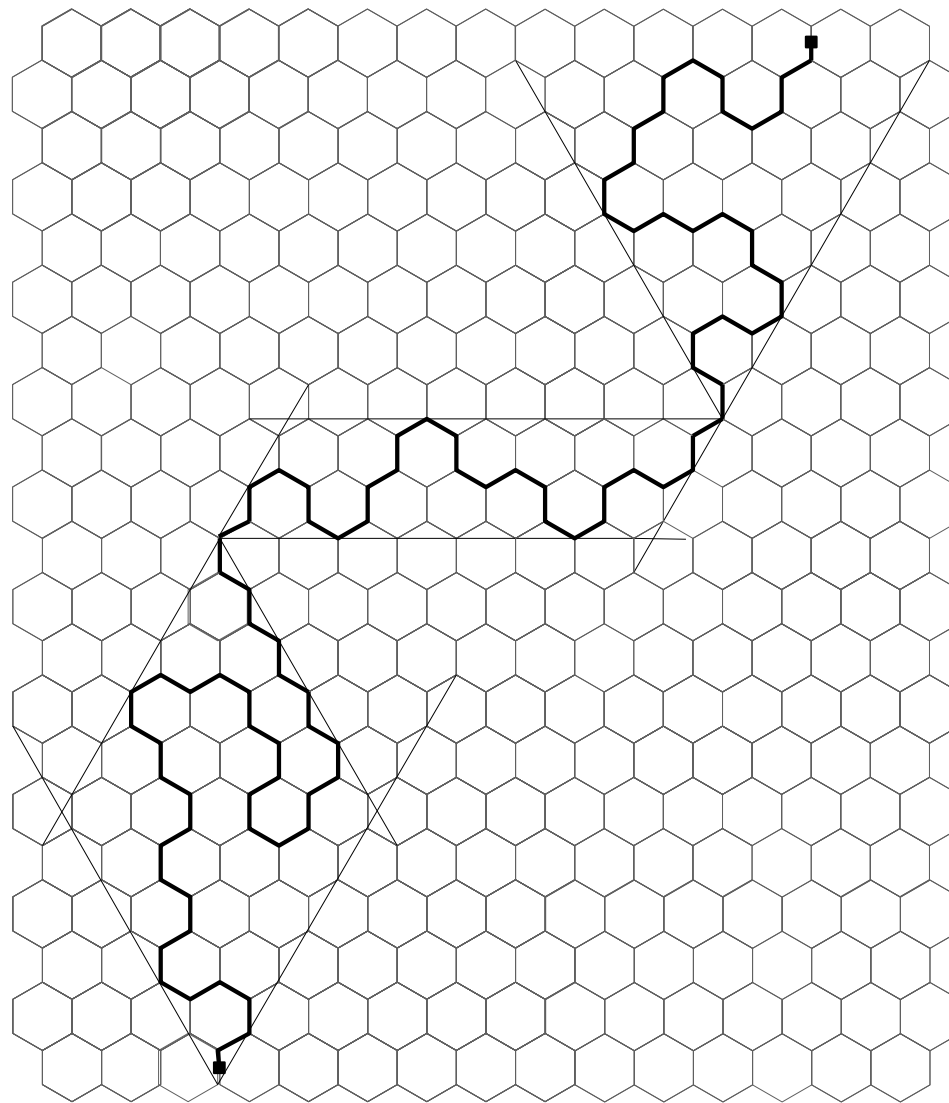
Proof: law of large numbers

- **Proposition:** If $E_{iB}(H(\omega)) < \infty$, a positive fraction of the renewal points of a random infinite bridge are **diamond points**.





- Stickbreaking in bridges



Contradicts the fact that $E_{iB}(H(\omega)) < \infty$, and thus

$$B_h(x_c, 1) \rightarrow \frac{1}{E_{iB}(H(\omega))} = 0.$$