Plane lattice paths avoiding a quadrant

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Two problems

**Question:** How many $n$-step walks on the square lattice, starting from $(0,0)$, avoiding a given quadrant?

Two natural choices for the quadrant:

One can also prescribe the endpoint $(i,j)$ of the walk.
I. Counting walks in cones
Counting walks in cones

Take a starting point \( p_0 \) in \( \mathbb{Z}^d \), a (finite) step set \( S \subset \mathbb{Z}^d \) and a cone \( C \).

Questions

- What is the number \( c(n) \) of \( n \)-step walks starting at \( p_0 \), taking their steps in \( S \) and contained in \( C \)?
- For \( i = (i_1, \ldots, i_d) \in C \), what is the number \( c(i; n) \) of such walks that end at \( i \)?
Counting walks in cones

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Questions

- What is the number $c(n)$ of $n$-step walks starting at $p_0$, taking their steps in $S$ and contained in $C$?
- For $i = (i_1, \ldots, i_d) \in C$, what is the number $c(i; n)$ of such walks that end at $i$?

Example: Gessel’s walks
Why count walks in cones?

Many discrete objects can be encoded in that way:

- in combinatorics, statistical physics...
- in (discrete) probability theory: random walks, queuing theory...
Many contributions

Adan, Banderier, Bernardi, Bostan, Cori, Denisov, Duchon, Dulucq, Fayolle, Gessel, Fisher, Flajolet, Gouyou-Beauchamps, Guttmann, Guy, Janse van Rensburg, Johnson, Kauers, Koutschan, Krattenthaler, Kurkova, Kreweras, van Leeuwarden, MacMahon, Melczer, Mishna, Niederhausen, Petkovšek, Prellberg, Raschel, Rechnitzer, Sagan, Salvy, Viennot, Wachtel, Wilf, Yeats, Zeilberger...

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etc.

Specific question
Ad hoc solution

\[\rightarrow\]
Systematic approach
A too ambitious question?

• Our original question:

\[ c(n) = ? \quad c(i; n) = ? \]
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- Generating functions and their nature
  \[ C(t) = \sum_{n \geq 0} c(n) t^n, \quad C(x_1, \ldots, x_d; t) = \sum_{i, n} c(i; n) x^i t^n \]
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Remarks

• \( C(1, \ldots, 1; t) = C(t) \)
• if \( C \subset \mathbb{R}_+^d \), then \( C(0, \ldots, 0; t) \) counts walks ending at \((0, \ldots, 0)\)
• \( C(0, x_2, \ldots, x_d; t) \) counts walks ending on the hyperplane \( i_1 = 0 \)
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  Can one express these series? Are they rational? algebraic? D-finite?

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A hierarchy of formal power series

- The formal power series $C(t)$ is **rational** if it can be written

\[ C(t) = P(t)/Q(t) \]

where $P(t)$ and $Q(t)$ are polynomials in $t$. 
A hierarchy of formal power series

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- The formal power series \( C(t) \) is algebraic (over \( \mathbb{Q}(t) \)) if it satisfies a (non-trivial) polynomial equation:

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- The formal power series $C(t)$ is **D-finite** (holonomic) if it satisfies a (non-trivial) linear differential equation with polynomial coefficients:
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  - Nice and effective closure properties + asymptotics of the coefficients
A hierarchy of formal power series

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\]

- Nice and effective closure properties + asymptotics of the coefficients
- Extension to several variables (D-finite: one DE per variable)
A (very) basic cone: the full space

Rational series

If $S \subset \mathbb{Z}^d$ is finite and $C = \mathbb{R}^d$, then $C(x; t)$ is rational:

$$c(n) = |S|^n \quad \Leftrightarrow \quad C(t) = \sum_{n \geq 0} c(n)t^n = \frac{1}{1 - |S|t}$$

More generally:

$$C(x; t) = \frac{1}{1 - t \sum_{s \in S} x^s}.$$
Also well-known: a (rational) half-space

Algebraic series

If $S \subset \mathbb{Z}^d$ is finite and $C$ is a rational half-space, then $C(x; t)$ is algebraic, given by an explicit system of polynomial equations.

[mbm-Petkovšek 00]; [Gessel 80], [Duchon 00]...
The “next” case: intersection of two half-spaces

- The quarter plane, with $p_0 = (0, 0)$

$$Q(x, y; t) = \sum_{i,j,n \geq 0} q(i, j; n)x^iy^jt^n = ?$$
Quadrant walks with *small* steps: classification

- $S \subset \{\bar{1}, 0, 1\}^2 \setminus \{(0, 0)\} \Rightarrow 2^8 = 256$ step sets (or: *models*)
Quadrant walks with *small* steps: classification

- \( S \subset \{\overline{1}, 0, 1\}^2 \setminus \{(0,0)\} \Rightarrow 2^8 = 256 \) step sets (or: models)

- However, some models are equivalent:
  - to a model of walks in the full or half-plane (\( \Rightarrow \) algebraic)
  - to another model in the collection (diagonal symmetry)

- One is left with 79 interesting distinct models [mbm-Mishna 09].
The series $Q(x, y; t)$ is D-finite iff a certain group associated with $S$ is finite.
The group of the model

Example. Take $S = \{\bar{1}0, 01, 1\bar{1}\}$, with step polynomial

$$S(x, y) = \frac{1}{x} + y + x \cdot \frac{1}{y} = \bar{x} + y + x\bar{y}$$
The group of the model

Example. Take \( S = \{\bar{10}, 01, 1\bar{1}\} \), with step polynomial

\[
S(x, y) = \frac{1}{\bar{x}} + y + x \cdot \frac{1}{\bar{y}} = \bar{x} + y + x\bar{y}
\]

Observation: \( S(x, y) \) is left unchanged by the rational transformations

\[
\Phi : (x, y) \mapsto (\bar{x}y, y) \quad \text{and} \quad \Psi : (x, y) \mapsto (x, x\bar{y}).
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Example. Take $S = \{\bar{1}0, 01, 1\bar{1}\}$, with step polynomial

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$$\Phi : (x, y) \mapsto (\bar{xy}, y) \quad \text{and} \quad \Psi : (x, y) \mapsto (x, x\bar{y})$$

They are involutions, and generate a finite dihedral group $G$: 

\[
\begin{array}{c}
\Phi \\
(\bar{xy}, y) \\
\psi \\
(x, y) \\
\psi \\
(x, x\bar{y}) \\
\Phi \\
(\bar{y}, \bar{x}) \\
\Phi \\
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Example. Take $S = \{ \bar{1}0, 01, 1\bar{1} \}$, with step polynomial

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They are involutions, and generate a finite dihedral group $G$:

The group can be defined for any quadrant model with small steps.
Classification of quadrant walks with small steps

Theorem
The series $Q(x, y; t)$ is D-finite iff the group $G$ is finite.

[mbm-Mishna 10], [Bostan-Kauers 10] D-finite
[Kurkova-Raschel 12] non-singular non-D-finite
[Mishna-Rechnitzer 07], [Melczer-Mishna 13] singular non-D-finite

quadrant models: 79

$|G|<\infty$: 23 $|G|\geq\infty$: 56

OS$\neq 0$: 19 OS$= 0$: 3 + 1 Not D-finite

D-finite algebraic
Conjecture: [Gessel, around 2000]

\[ g_{0,0}(2n) = 16^n \frac{(1/2)_n(5/6)_n}{(2)_n(5/3)_n} \]

with \((a)_n = a(a+1) \cdots (a+n-1)\).

• Proved in 2009 using computer algebra [Kauers, Koutschan & Zeilberger 09]

• In fact, \(Q(x, y; t)\) is algebraic! (degree 72...) [Bostan & Kauers 10]

• Two other proofs, one analytic, one “elementary” [Bostan, Kurkova and Raschel 13(a)], [mbm 15(a)]
Classification of quadrant walks with small steps

- Quadrant models: 79
  - $|G| < \infty$: 23
  - $|G| = \infty$: 56

- $\text{OS} \neq 0$: 19
- $\text{OS} = 0$: 3 + 1
- Not D-finite

- D-finite
- Algebraic

- Formal power series algebra
- Computer algebra
- Complex analysis

- Random walks in probability
- D-finite series
- Effective closure properties
- Arithmetic properties
- Asymptotics
- G-functions
II. Walks avoiding a quadrant
The “next” case: union of two half-spaces!

Is there a nice story to tell for walks avoiding a quadrant?
The “next” case: \textit{union} of two half-spaces!

Is there a nice story to tell for walks avoiding a quadrant?
The “next” case: union of two half-spaces!

- Walks avoiding the negative quadrant, with \( p_0 = (0, 0) \)

\[
C(x, y; t) = \sum_{i, j, n \geq 0} c(i, j; n)x^iy^jt^n = ?
\]

Is there a nice story to tell for walks avoiding a quadrant?
Another motivation: a new approach to Gessel’s walks

**Idea:** count Gessel’s walks by the reflection principle

If \( c(i, j; n) \) counts walks starting at \((-1, 0)\) and avoiding the negative quadrant, then for \( j \geq 0 \) and \( i < j \),

\[
c_{i,j}(n) - c_{j,i}(n) = g_{i+1,j}(n).
\]

A Gessel walk
Another motivation: a new approach to Gessel’s walks

**Idea:** count Gessel’s walks by the reflection principle

If $c(i, j; n)$ counts walks starting at $(-1, 0)$ and avoiding the negative quadrant, then for $j \geq 0$ and $i < j$,

$$c_{i,j}(n) - c_{j,i}(n) = g_{i+1,j}(n).$$

An alternative is to count walks on the **diagonal square lattice** starting from $(-2, 0)$ and avoiding the negative quadrant.
An *a posteriori* motivation

A hard (?) problem with nice numbers!

Example: the number of square lattice walks of length $n$ starting and ending at $(0,0)$ and avoiding the West quadrant is

$$c(0,0;2n) = 16n^9 - 2n^7.$$
An *a posteriori* motivation

A hard (?) problem with nice numbers!

**Example:** the number of square lattice walks of length $2n$ starting and ending at $(0, 0)$ and avoiding the West quadrant is

$$c(0, 0; 2n) = \frac{16^n}{9} \left( 3 \frac{(1/2)_n^2}{(2)_n^2} + 8 \frac{(1/2)_n(7/6)_n}{(2)_n(4/3)_n} - 2 \frac{(1/2)_n(5/6)_n}{(2)_n(5/3)_n} \right)$$

with $(a)_n = a(a + 1) \cdots (a + n - 1)$. 

![Diagram of a square lattice with a path avoiding the West quadrant](image)
An *a posteriori* motivation

A hard (?) problem with nice numbers!

Example: the number of square lattice walks of length $2n$ starting and ending at $(0, 0)$ and avoiding the West quadrant is

$$c(0, 0; 2n) = \frac{16^n}{9} \left(3 \frac{(1/2)_n^2}{(2)_n^2} + 8 \frac{(1/2)_n(7/6)_n}{(2)_n(4/3)_n} - 2 \frac{(1/2)_n(5/6)_n}{(2)_n(5/3)_n} \right)$$

with $(a)_n = a(a + 1) \cdots (a + n - 1)$.

Two known components: Quadrant walks (left) and Gessel walks (right). Conjectured before 2007...

Asymptotically,

$$c(0, 0; 2n) \sim \frac{2^5 \Gamma(2/3)}{3^2 \pi} \frac{4^{2n}}{(2n)^{5/3}}.$$
Let $Q(x, y) ≡ Q(x, y; t)$ be the generating function of square lattice walks starting from $(0, 0)$ and confined to the first quadrant:

$$Q(x, y) = \sum_{i,j,n \geq 0} \frac{(i + 1)(j + 1)}{(n + 1)(n + 2)} \left( \frac{n + 2}{n-i-j} \right) \left( \frac{n + 2}{n+i-j+2} \right) x^i y^j t^n$$
Results (square lattice, starting point (0, 0))

• Let \( Q(x, y) \equiv Q(x, y; t) \) be the generating function of square lattice walks starting from \((0, 0)\) and confined to the first quadrant:

\[
Q(x, y) = \sum_{i,j,n\geq 0} \frac{(i+1)(j+1)}{(n+1)(n+2)} \left( \frac{n+2}{n-i-j} \right) \left( \frac{n+2}{n+i-j+2} \right) x^i y^j t^n
\]

Theorem [mbm 15(a)]

The generating function \( C(x, y; t) \equiv C(x, y) \) counting walks that avoid the negative quadrant is

\[
A(x, y) + \frac{1}{3} (Q(x, y) - \bar{x}^2 Q(\bar{x}, y) - \bar{y}^2 Q(x, \bar{y}))
\]

where \( \bar{x} = 1/x, \bar{y} = 1/y \) and \( A(x, y) \) is algebraic. This series satisfies

\[
(1 - t(x + \bar{x} + y + \bar{y})) A(x, y) = (2 + \bar{x}^2 + \bar{y}^2)/3 - t\bar{y}A_-(\bar{x}) - t\bar{x}A_-(\bar{y}),
\]

where \( A_-(x) \) is a series in \( t \) with coefficients in \( \mathbb{Q}[x] \), algebraic of degree 24.
Theorem \[\text{mbm 15(a)}, \text{ cont'd}\]

\[C(x, y) = A(x, y) + \frac{1}{3} (Q(x, y) - \bar{x}^2 Q(\bar{x}, y) - \bar{y}^2 Q(x, \bar{y}))\]

where

\((1 - t(x + \bar{x} + y + \bar{y})) A(x, y) = (2 + \bar{x}^2 + \bar{y}^2)/3 - t\bar{y} A_-(\bar{x}) - t\bar{x} A_-(\bar{y}),\)

and \(A_-(x)\) is a series in \(t\) with coefficients in \(\mathbb{Q}[x]\), algebraic of degree 24.

- Moreover, \(A_-(tx)\) has a rational expression in terms of the (unique) series \(Z = 1 + O(t)\) and \(U(x) = 1 + O(t)\) satisfying:

\[Z = \sqrt{T}, \quad T = 1 + 256t^2 \frac{T^3}{(T + 3)^3},\]

\[16T^2(U^2 - T) = x(U + UT - 2T)(U^2 - 9T + 8TU + T^2 - TU^2).\]
III. Two solutions for... quadrant walks

- the (ordinary) kernel method
- the (algebraic) kernel method: the reflection principle in disguise

Starting point: a functional equation
Walks in the quadrant: a functional equation

Step by step construction:

\[ Q(x, y; t) \equiv Q(x, y) = 1 + t(x + \bar{x} + y + \bar{y})Q(x, y) - t\bar{x}Q(0, y) - t\bar{y}Q(x, 0) \]
Walks in the quadrant: a functional equation

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or

\[ (1 - t(x + \bar{x} + y + \bar{y}))Q(x, y) = 1 - t\bar{x}Q(0, y) - t\bar{y}Q(x, 0), \]
Walks in the quadrant: a functional equation

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\[ Q(x, y; t) \equiv Q(x, y) = 1 + t(x + \bar{x} + y + \bar{y})Q(x, y) - t\bar{x}Q(0, y) - t\bar{y}Q(x, 0) \]

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\[ (1 - t(x + \bar{x} + y + \bar{y}))Q(x, y) = 1 - t\bar{x}Q(0, y) - t\bar{y}Q(x, 0), \]

or

\[ (1 - t(x + \bar{x} + y + \bar{y}))xyQ(x, y) = xy - tyQ(0, y) - txQ(x, 0) \]

- The polynomial \( 1 - t(x + \bar{x} + y + \bar{y}) \) is the kernel of this equation
- The equation is linear, with two catalytic variables \( x \) and \( y \) (tautological at \( x = 0 \) or \( y = 0 \))
The ordinary kernel method: cancel the kernel

- The equation:

$$(1 - t(x + ar{x} + y + ar{y}))xyQ(x, y) = xy - tyQ(0, y) - txQ(x, 0)$$
The ordinary kernel method: cancel the kernel

- The equation:
  \[(1 - t(x + \bar{x} + y + \bar{y}))xyQ(x, y) = xy - tyQ(0, y) - txQ(x, 0)\]

- Let
  \[Y = \frac{1 - t(x + \bar{x}) - \sqrt{(1 - t(x + \bar{x}))^2 - 4t^2}}{2t} = t + (x + \bar{x})t^2 + O(t^3)\]

be the only power series in \(t\) such that \(K(x, Y) = 0\).
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be the only power series in \(t\) such that \(K(x, Y) = 0\). Then

\[xY = tYQ(0, Y) + txQ(x, 0)\]
The ordinary kernel method: cancel the kernel

- The equation:
  \[(1 - t(x + \bar{x} + y + \bar{y})) \cdot xyQ(x, y) = xy - tyQ(0, y) - txQ(x, 0)\]

- Let
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  be the only power series in \(t\) such that \(K(x, Y) = 0\). Then
  \[xY = tYQ(0, Y) + txQ(x, 0)\]

- Since \(Y\) is symmetric in \(x\) and \(\bar{x}\),
  \[\bar{x}Y = tYQ(0, Y) + t\bar{x}Q(\bar{x}, 0).\]
The ordinary kernel method: cancel the kernel

- The equation:
  \[
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  \]

- Let
  \[
  Y = \frac{1 - t(x + \bar{x}) - \sqrt{(1 - t(x + \bar{x}))^2 - 4t^2}}{2t} = t + (x + \bar{x})t^2 + O(t^3)
  \]
  be the only power series in \( t \) such that \( K(x, Y) = 0 \). Then
  \[
  xY = tYQ(0, Y) + txQ(x, 0)
  \]

- Since \( Y \) is symmetric in \( x \) and \( \bar{x} \),
  \[
  \bar{x}Y = tYQ(0, Y) + t\bar{x}Q(\bar{x}, 0).
  \]

- Thus
  \[
  txQ(x, 0) - t\bar{x}Q(\bar{x}, 0) = (x - \bar{x})Y,
  \]
  and \( txQ(x, 0) \) can be obtained by extracting positive powers of \( x \) in \((x - \bar{x})Y\).
The algebraic kernel method: use the full group

- The equation reads (with $K(x, y) = 1 - t(x + \bar{x} + y + \bar{y})$):
  $$K(x, y)xyQ(x, y) = xy - txQ(x, 0) - tyQ(0, y)$$

- The kernel is invariant by $x \mapsto \bar{x}$ and $y \mapsto \bar{y}$. Hence
  $$K(x, y) \quad xyQ(x, y) \quad = \quad xy - txQ(x, 0) - tyQ(0, y)$$
  $$K(x, y) \quad \bar{x}yQ(\bar{x}, y) \quad = \quad \bar{x}y - t\bar{x}Q(\bar{x}, 0) - tyQ(0, y)$$
  $$K(x, y) \quad \bar{x}\bar{y}Q(\bar{x}, \bar{y}) \quad = \quad \bar{x}\bar{y} - t\bar{x}Q(\bar{x}, 0) - t\bar{y}Q(0, \bar{y})$$
  $$K(x, y) \quad x\bar{y}Q(x, \bar{y}) \quad = \quad x\bar{y} - txQ(x, 0) - t\bar{y}Q(0, \bar{y})$$
The algebraic kernel method: use the full group

• The equation reads (with $K(x, y) = 1 - t(x + \bar{x} + y + \bar{y})$):

$$K(x, y)xyQ(x, y) = xy - txQ(x, 0) - tyQ(0, y)$$

• The kernel is invariant by $x \mapsto \bar{x}$ and $y \mapsto \bar{y}$. Hence

$$K(x, y) \ xyQ(x, y) = xy - \ txQ(x, 0) - \ tyQ(0, y)$$

$$K(x, y) \ \bar{x}yQ(\bar{x}, y) = \bar{x}y - \ t\bar{x}Q(\bar{x}, 0) - \ tyQ(0, y)$$

$$K(x, y) \ \bar{x}\bar{y}Q(\bar{x}, \bar{y}) = \bar{x}\bar{y} - \ t\bar{x}Q(\bar{x}, 0) - \ t\bar{y}Q(0, \bar{y})$$

$$K(x, y) \ x\bar{y}Q(x, \bar{y}) = x\bar{y} - \ txQ(x, 0) - \ t\bar{y}Q(0, \bar{y})$$

⇒ Form the alternating sum of the four equations:

$$K(x, y) \left( xyQ(x, y) - \bar{x}yQ(\bar{x}, y) + \bar{x}\bar{y}Q(\bar{x}, \bar{y}) - x\bar{y}Q(x, \bar{y}) \right) =$$

$$xy - \bar{x}y + \bar{x}\bar{y} - x\bar{y}$$

( the orbit sum).
Why is this interesting?

\[ xyQ(x, y) - \bar{xy}Q(\bar{x}, y) + \bar{x}\bar{y}Q(\bar{x}, \bar{y}) - x\bar{y}Q(x, \bar{y}) = \frac{xy - \bar{xy} + \bar{x}\bar{y} - x\bar{y}}{1 - t(x + \bar{x} + y + \bar{y})} \]

- Both sides are power series in \( t \), with coefficients in \( \mathbb{Q}[x, \bar{x}, y, \bar{y}] \).
Why is this interesting?

\[ \begin{align*}
xyQ(x, y) - \bar{x}yQ(\bar{x}, y) + \bar{x}\bar{y}Q(\bar{x}, \bar{y}) - x\bar{y}Q(x, \bar{y}) &= \frac{xy - \bar{x}y + \bar{x}\bar{y} - x\bar{y}}{1 - t(x + \bar{x} + y + \bar{y})} \\
\end{align*} \]

- Both sides are power series in \( t \), with coefficients in \( \mathbb{Q}[x, \bar{x}, y, \bar{y}] \).
- Extract the part with positive powers of \( x \) and \( y \):

\[ xyQ(x, y) = [x>0 \ y>0] \frac{xy - \bar{x}y + \bar{x}\bar{y} - x\bar{y}}{1 - t(x + \bar{x} + y + \bar{y})} \]

is a D-finite series.  

\[ \text{[Lipshitz 88]} \]

\[ \Rightarrow Q(x, y) = \sum_{i,j,n\geq 0} \frac{(i + 1)(j + 1)}{(n + 1)(n + 2)} \left( \frac{n + 2}{2} \right) \left( \frac{n + 2}{n + i - j + 2} \right) x^i y^j t^n \]
IV. Some ingredients for the three quadrant problem

- the (ordinary) kernel method
- the (algebraic) kernel method
- polynomial equations with one catalytic variable

Starting point: a functional equation again
Walks avoiding a quadrant: a functional equation

Step by step construction:

\[ C(x, y; t) \equiv C(x, y) = 1 + t(x + \bar{x} + y + \bar{y})C(x, y) - t\bar{x}C_-(\bar{y}) - t\bar{y}C_-(\bar{x}) \]

with

\[ C_-(\bar{x}) = \sum_{i < 0, n \geq 0} c(i, 0; n)x^i t^n. \]
Walks avoiding a quadrant: a functional equation

Step by step construction:

\[ C(x, y; t) \equiv C(x, y) = 1 + t(x + \bar{x} + y + \bar{y})C(x, y) - t\bar{x}C_-(\bar{y}) - t\bar{y}C_-(\bar{x}) \]

with

\[ C_-(\bar{x}) = \sum_{i<0, n\geq 0} c(i, 0; n)x^i t^n. \]

or

\[ (1 - t(x + \bar{x} + y + \bar{y}))C(x, y) = 1 - t\bar{x}C_-(\bar{y}) - t\bar{y}C_-(\bar{x}), \]
Walks avoiding a quadrant: a functional equation

Step by step construction:

\[ C(x, y; t) \equiv C(x, y) = 1 + t(x + \bar{x} + y + \bar{y})C(x, y) - t\bar{x}C_-(\bar{y}) - t\bar{y}C_-(\bar{x}) \]

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or

\[ (1 - t(x + \bar{x} + y + \bar{y})) C(x, y) = 1 - t\bar{x}C_-(\bar{y}) - t\bar{y}C_-(\bar{x}), \]

or

\[ (1 - t(x + \bar{x} + y + \bar{y})) xyC(x, y) = xy - tyC_-(\bar{y}) - txC_-(\bar{x}) \]

Compare with

\[ (1 - t(x + \bar{x} + y + \bar{y})) xyQ(x, y) = xy - tyQ(0, y) - txQ(x, 0) \]
Where the series $A(x, y)$ comes from

- In particular, $C(x, y)$ and $Q(x, y)$ have the same orbit sum:

  $$xyQ(x, y) - \bar{x}yQ(\bar{x}, y) + \bar{x}\bar{y}Q(\bar{x}, \bar{y}) - x\bar{y}Q(x, \bar{y}) =$$

  $$xyC(x, y) - \bar{x}yC(\bar{x}, y) + \bar{x}\bar{y}C(\bar{x}, \bar{y}) - x\bar{y}C(x, \bar{y}) = \frac{xy - \bar{x}y + \bar{x}\bar{y} - x\bar{y}}{1 - t(x + \bar{x} + y + \bar{y})}$$

  $$-\bar{x}^2Q(\bar{x}, y)$$ and $$-\bar{y}^2Q(x, \bar{y})$$ (and $$\bar{x}^2\bar{y}^2Q(\bar{x}, \bar{y})$$) as well!
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$-\bar{x}^2Q(\bar{x}, y)$ and $-\bar{y}^2Q(x, \bar{y})$ (and $\bar{x}^2\bar{y}^2Q(\bar{x}, \bar{y})$) as well!

- This leads us to introduce

$$A(x, y) := C(x, y) - \frac{1}{3} (Q(x, y) - \bar{x}^2Q(\bar{x}, y) - \bar{y}^2Q(x, \bar{y})),$$

which satisfies

$$(1 - t(x + \bar{x} + y + \bar{y})) A(x, y) = (2 + \bar{x}^2 + \bar{y}^2)/3 - t\bar{y}A_-(\bar{x}) - t\bar{x}A_-(\bar{y}),$$

with

$$A_-(\bar{x}) = \sum_{i<0, n\geq 0} a(i, 0; n)x^i t^n.$$
The algebraic kernel method for $A(x, y)$?

The equation reads (with $K(x, y) = 1 - t(x + \bar{x} + y + \bar{y})$):

$$K(x, y)A(x, y) = (2 + \bar{x}^2 + \bar{y}^2)/3 - t\bar{y}A_-(\bar{x}) - t\bar{x}A_-(\bar{y})$$

The orbit sum vanishes:

$$xyA(x, y) - \bar{x}yA(\bar{x}, y) + \bar{x}\bar{y}A(\bar{x}, \bar{y}) - x\bar{y}A(x, \bar{y}) = 0$$

and this does not characterize $A$. 

The ordinary kernel method for $A(x, y)$?

The equation reads (with $K(x, y) = 1 - t(x + \bar{x} + y + \bar{y})$):

$$K(x, y)A(x, y) = (2 + \bar{x}^2 + \bar{y}^2)/3 - t\bar{y}A_-(\bar{x}) - t\bar{x}A_-(\bar{y})$$

But none of the roots $Y$ of $K(x, Y)$ can be legally substituted for $y$ in $A(x, y)$ (due to positive and negative powers of $y$).
Back to series with *polynomial coefficients* in $x$ and $y$

- The equation (with $K(x, y) = 1 - t(x + \bar{x} + y + \bar{y})$):

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Back to series with *polynomial coefficients* in $x$ and $y$

- The equation (with $K(x, y) = 1 - t(x + \bar{x} + y + \bar{y})$):
  $$K(x, y)A(x, y) = \frac{(2 + \bar{x}^2 + \bar{y}^2)}{3} - t\bar{y}A_-(\bar{x}) - t\bar{x}A_-(\bar{y})$$

- Split $A(x, y)$ into three parts:
  $$A(x, y) = P(x, y) + \bar{x}M(\bar{x}, y) + \bar{y}M(\bar{y}, x)$$
Back to series with *polynomial coefficients* in $x$ and $y$

- The equation (with $K(x, y) = 1 - t(x + \bar{x} + y + \bar{y})$):
  \[
  K(x, y)A(x, y) = (2 + \bar{x}^2 + \bar{y}^2)/3 - t\bar{y}A_-(\bar{x}) - t\bar{x}A_-(\bar{y})
  \]
- Split $A(x, y)$ into three parts:
  \[
  A(x, y) = P(x, y) + \bar{x}M(\bar{x}, y) + \bar{y}M(\bar{y}, x)
  \]
- Then the equation for $A$ implies
  \[
  P(x, y) = \bar{x}(M(x, y) - M(0, y)) + \bar{y}(M(y, x) - M(0, x)),
  \]
  and
  \[
  (1 - t(x + \bar{x} + y + \bar{y})) (2M(x, y) - M(0, y)) = 2x/3 - 2t\bar{y}M(x, 0) + t(x - \bar{x})M(0, y) + t\bar{y}M(y, 0).
  \]

This looks (remotely) like a quadrant problem. Two main differences:
- $M(y, 0)$ is a new ingredient
- the r.h.s. cannot be decoupled into $F(x) + G(y)$
Let $\Delta(x) := (1 - t(x + \bar{x}))^2 - 4t^2$,

$$S(x) = txM(0, x), \quad S_1 = [x]S(x) = tM(0, 0)$$

and

$$P_0 := [x^0] (\Delta(x)S(x)S(\bar{x})).$$
Let \( \Delta(x) := (1 - t(x + \bar{x}))^2 - 4t^2 \),

\[
S(x) = txM(0, x), \quad S_1 = [x]S(x) = tM(0, 0)
\]

and

\[
P_0 := [x^0] (\Delta(x)S(x)S(\bar{x})).
\]

Then

\[
\Delta(x) \left( S(x)^3 + (2x + \bar{x})S(x)^2 + x(x + \bar{x})S(x) \right) = t^2(x - \bar{x})(1 + S_1)^2 + (2t^2S_1^2 + 2t (tx^2 + t\bar{x}^2 - x - \bar{x} + t) S_1 - P_0 + t^2(x^2 + \bar{x}^2)) (S(x) + x).
\]

A polynomial equation with one catalytic variable \( x \), solvable using \[mbm-Jehanne 06\]. The solution is (invariably) algebraic.
Results (square lattice, starting point (0, 0))

- Let \( Q(x, y) \equiv Q(x, y; t) \) be the generating function of square lattice walks starting from (0, 0) and confined to the first quadrant:

\[
Q(x, y) = \sum_{i,j,n \geq 0} \frac{(i+1)(j+1)}{(n+1)(n+2)} \left( \frac{n+2}{n-i-j} \right) \left( \frac{n+2}{n+i-j+2} \right) x^i y^j t^n
\]

**Theorem [mbm 15(a)]**

The generating function \( C(x, y; t) \equiv C(x, y) \) counting walks that avoid the negative quadrant is

\[
A(x, y) + \frac{1}{3} (Q(x, y) - \bar{x}^2 Q(\bar{x}, y) - \bar{y}^2 Q(x, \bar{y}))
\]

where \( \bar{x} = 1/x, \bar{y} = 1/y \) and \( A(x, y) \) is algebraic. This series satisfies

\[
(1 - t(x + \bar{x} + y + \bar{y})) A(x, y) = (2 + \bar{x}^2 + \bar{y}^2)/3 - t\bar{y}A_-(\bar{x}) - t\bar{x}A_-(\bar{y}),
\]

where \( A_-(x) \) is a series in \( t \) with coefficients in \( \mathbb{Q}[x] \), algebraic of degree 24.
Walks on the diagonal square lattice avoiding the negative cone
Walks on the square lattice starting at $(-1, 0)$ (entirely algebraic!)
Walks on the diagonal square lattice starting at $(-2, 0)$
The solution of Gessel’s model in a $135^\circ$ cone then comes for free
More results

- Walks on the diagonal square lattice avoiding the negative cone
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- Walks on the diagonal square lattice starting at \((-2, 0)\)
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More results

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More results

- Walks on the diagonal square lattice avoiding the negative cone
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Perspectives: classification

Walks with small steps confined to the positive quadrant:

- models: 79
  - $|G| < \infty$: 23
  - $|G| = \infty$: 56
    - OS $\neq 0$: 19
    - OS $= 0$: 3 + 1
      - Not D-finite
      - D-finite
      - algebraic
Perspectives: classification

Walks with small steps avoiding the negative quadrant:

models: 74

$|G| < \infty$: 23  

$G = \infty$: 51

$\text{OS} \neq 0$: 19  

$\text{OS} = 0$: 3 + 1  

Not D-finite?

D-finite?  

algebraic?
Perspectives: classification

Walks with small steps avoiding the negative quadrant:

models: 74

| \(|G|<\infty\): 23 | \(|G|=\infty\): 51 |
|------------------|------------------|
| OS\(\neq0\): 19 | OS\(=0\): 3 + 1  | Not D-finite? |
| D-finite?        | algebraic?       |

Conjecture: the generating function of walks starting and ending at \((0, 0)\) and avoiding the negative quadrant is algebraic for the four models for which \(Q(x, y)\) (the quadrant GF) is algebraic: