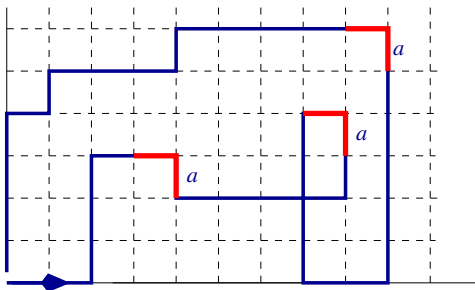


Permutations sortable by two stacks in parallel and quarter plane walks

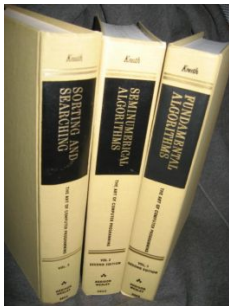
Mireille Bousquet-Mélou, CNRS, LaBRI, Bordeaux
joint work with Michael Albert, Dunedin, New Zealand



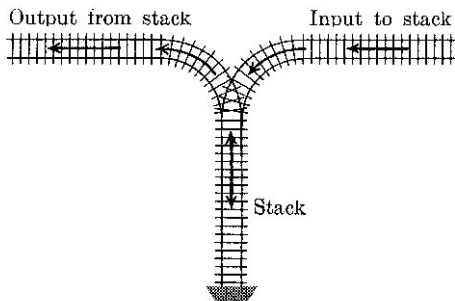
I. In the beginning was...

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The Art of Computer Programming [Knuth, 1968]

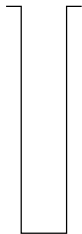


Sorting permutations through a stack [Knuth 1968]

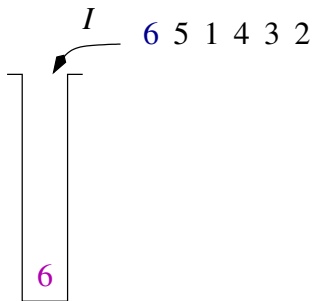


Sorting permutations through a stack [Knuth 1968]

6 5 1 4 3 2

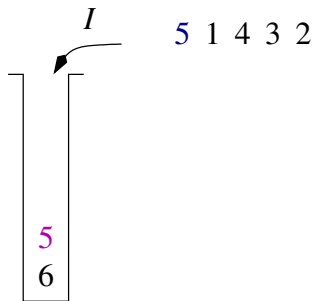


Sorting permutations through a stack [Knuth 1968]



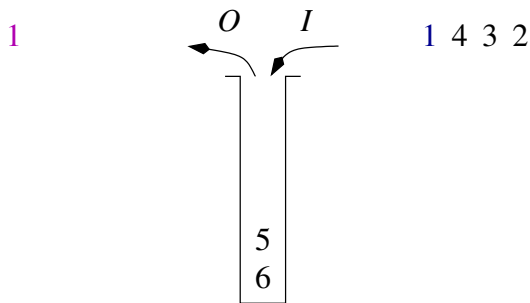
Operation sequence: $w = I$

Sorting permutations through a stack [Knuth 1968]



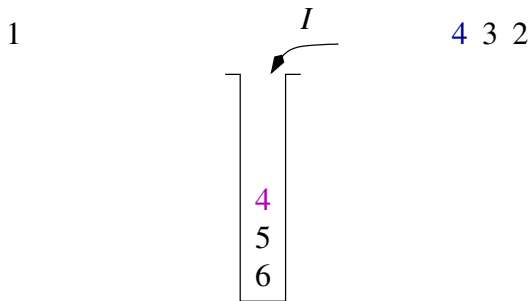
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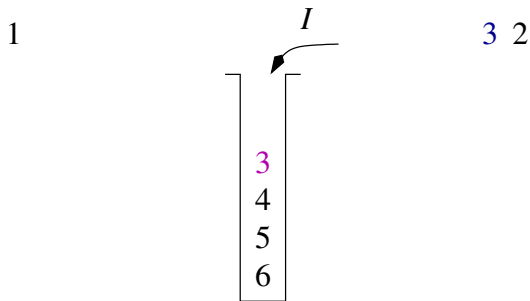
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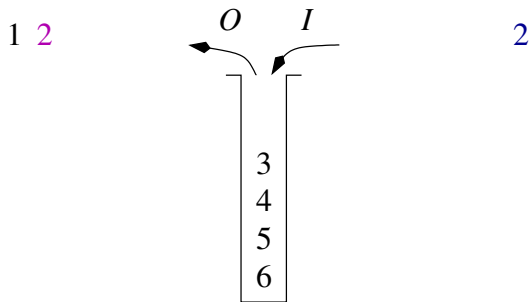
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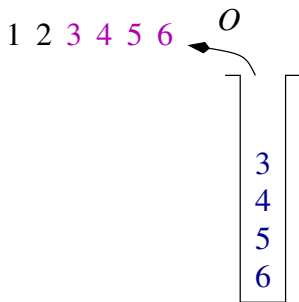
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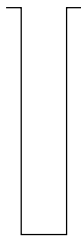
Sorting permutations through a stack [Knuth 1968]



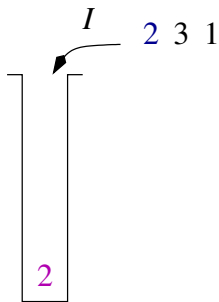
Operation sequence: $w = III0IIII000000$

Sorting permutations through a stack [Knuth 1968]

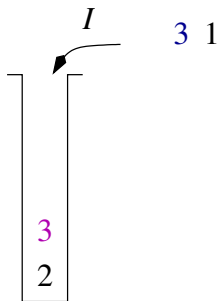
2 3 1



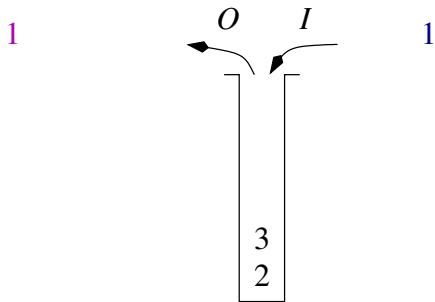
Sorting permutations through a stack [Knuth 1968]



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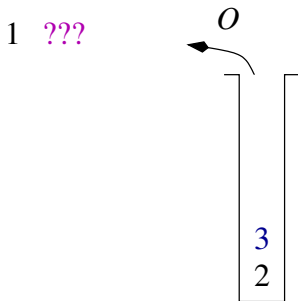


Sorting permutations through a stack [Knuth 1968]

1 ???



Sorting permutations through a stack [Knuth 1968]



Property: If $\sigma = \sigma_1 \cdots \sigma_n$ is sortable, then so are all its sub-permutations $\sigma_{i_1} \cdots \sigma_{i_k}$ with $i_1 < \cdots < i_k$

Sorting permutations through a stack [Knuth 1968]

Three main results

- Characterization: σ is sortable iff it does not contain a sub-sequence $\sigma_i\sigma_j\sigma_k$ of shape 231 (that is, with $\sigma_k < \sigma_i < \sigma_j$)

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- Enumeration?

Two equivalent viewpoints: sort, or produce

Lemma: If σ is sorted by an operation sequence $w \in \{I, O\}^*$, then w , applied to the identity, produces σ^{-1} (which is said to be **realizable**).

$$\begin{array}{ccc} 1 & 2 & \cdots & n & \xleftarrow{w} & \sigma_1 \sigma_2 \cdots \sigma_n \\ \sigma_1^{-1} \sigma_2^{-1} \cdots \sigma_n^{-1} & \xleftarrow{w} & 1 & 2 & \cdots & n \end{array}$$

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Consequence: it is equivalent to count

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- realizable permutations
- operation sequences on $\{I, O\}^*$
- discrete excursions
- arch systems

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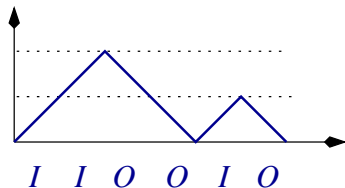
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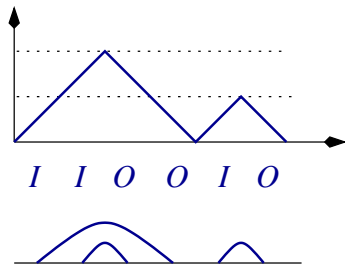


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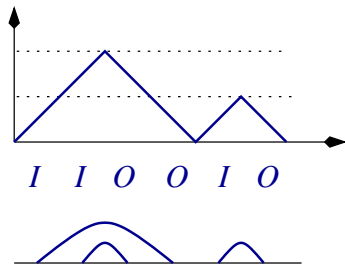


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\Rightarrow Enumeration by Catalan numbers:

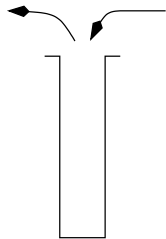
$$C_n := \frac{1}{n+1} \binom{2n}{n}$$

Networks of machines [Even & Itai, Pratt, Tarjan \simeq 1970]

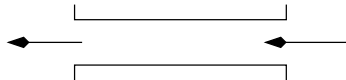
- More machines

Networks of machines [Even & Itai, Pratt, Tarjan \simeq 1970]

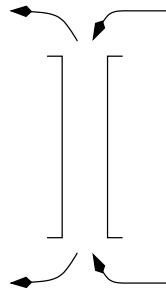
- More machines



a stack



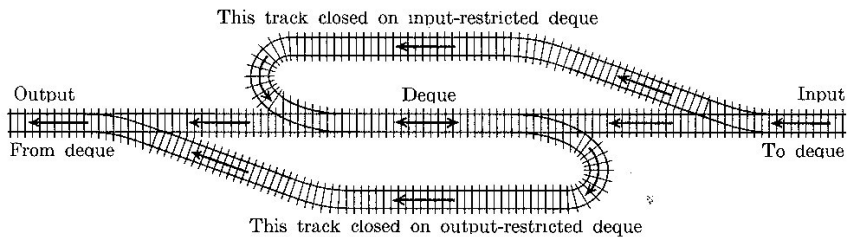
a queue



a double-ended queue

Networks of machines [Even & Itai, Pratt, Tarjan \simeq 1970]

- More machines

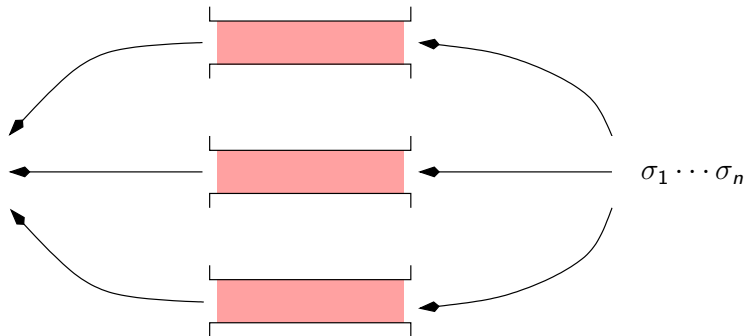


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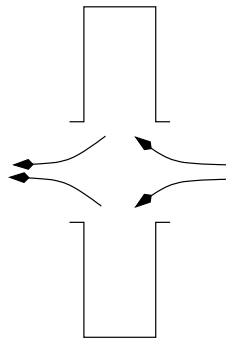
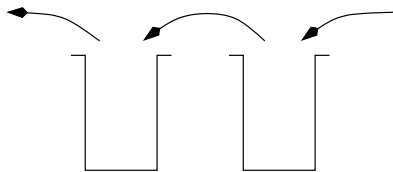
- More machines
- Networks of machines
 - ▶ m queues in series?



\Rightarrow Permutations with no descending sub-sequence of length $m + 1$

Networks of machines [Even & Itai, Pratt, Tarjan \simeq 1970]

- More machines
- Networks of machines
 - ▶ m queues in series?
 - ▶ m stacks in series? in parallel?

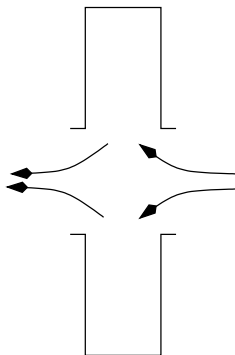


II. Two stacks

Two stacks

In parallel:

- **Characterization:** infinitely many forbidden patterns [Pratt 73]
- **Decidability:** in polynomial time [Even & Itai 71]
- **Enumeration** [Albert-MBM 2013]



Two stacks

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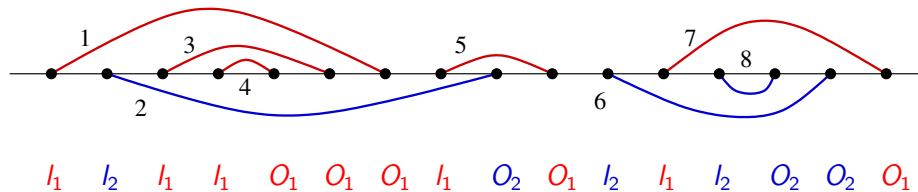
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In series:

- **Characterization:** infinitely many forbidden patterns [Murphy 02]
- **Decidability:** in polynomial time [Pierrot-Rossin 13]
- **Enumeration:** ???

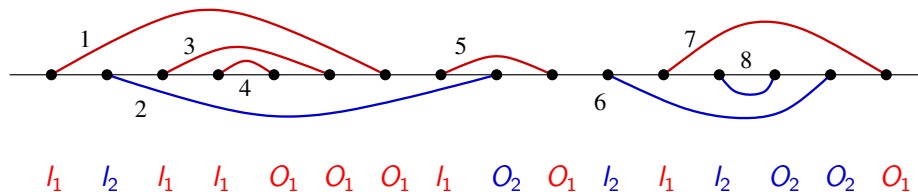
Permutations produced by two parallel stacks

- Operation sequences on $\{l_1, l_2, O_1, O_2\}$



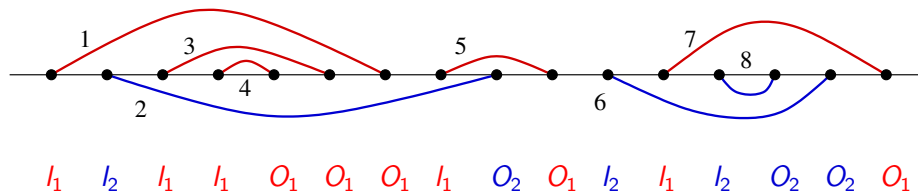
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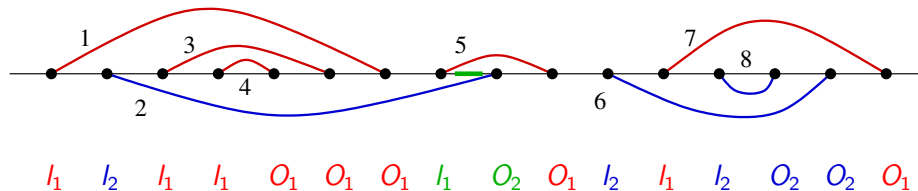
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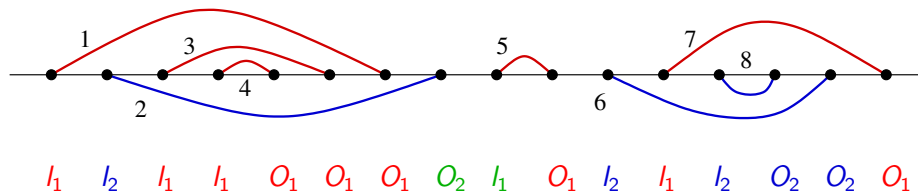
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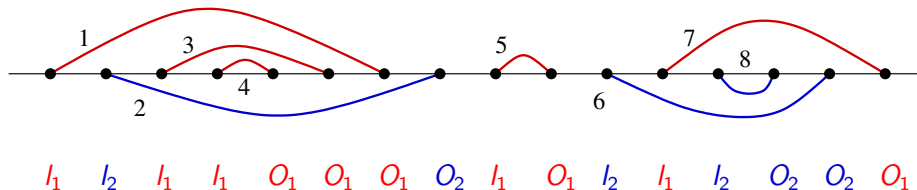
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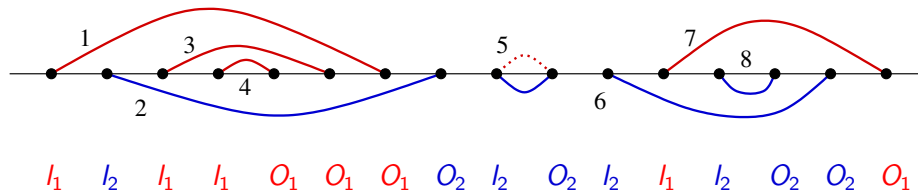
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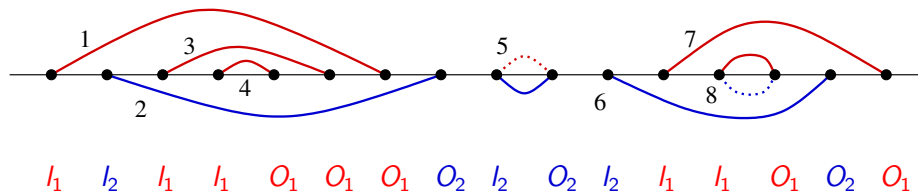
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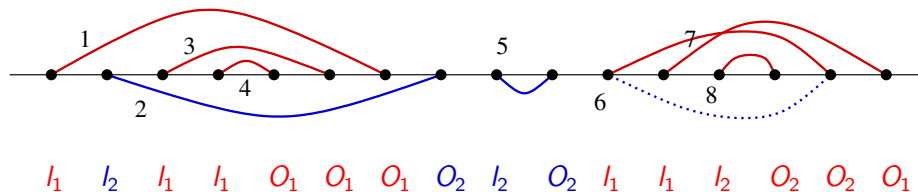
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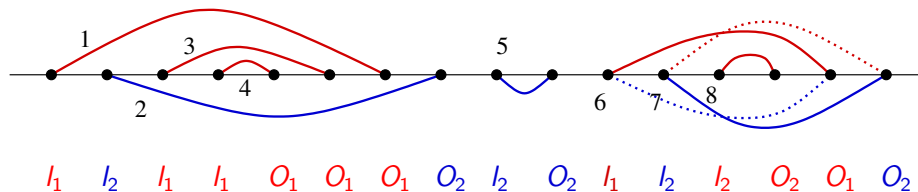
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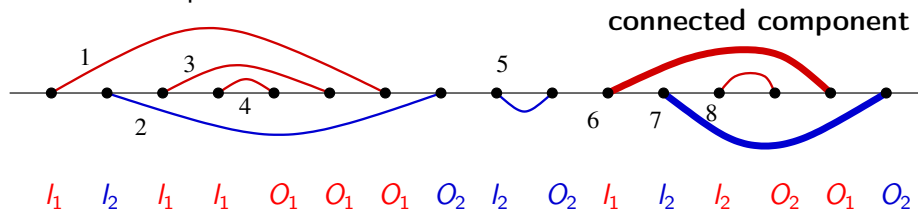
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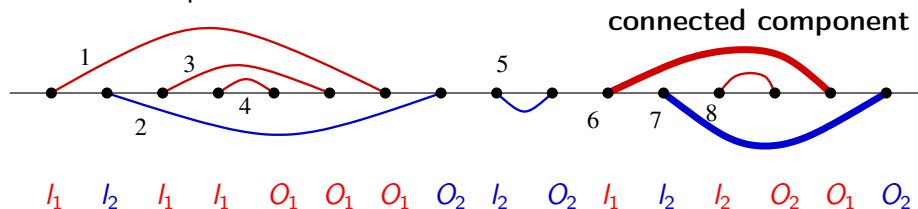
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- $l_1 O_2 \equiv O_2 l_1$ and $l_2 O_1 \equiv O_1 l_2 \Rightarrow$ **Eager** outputs: no $l_1 O_2$ nor $l_2 O_1$
- In each **connected component**, the first move is l_1 .

Canonical operation sequences

Proposition

The permutations that can be produced with two parallel stacks are in bijection with operation sequences such that

- there is no factor $l_1 O_2$ nor $l_2 O_1$
- the first letter (arch) in each connected component is l_1 (red).

[...] Enumeration

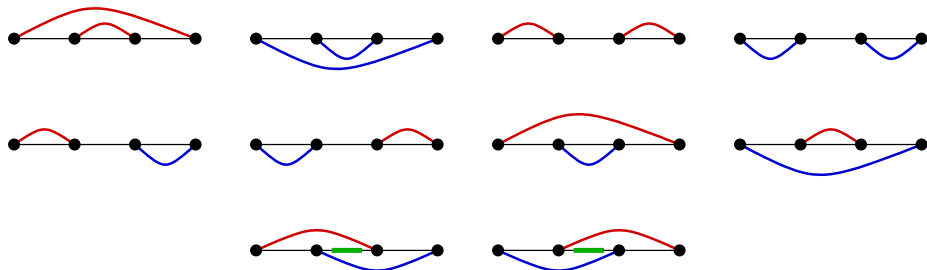
- Let $Q(a, u)$ be the generating function of arch systems counted by the number of arches (u) and the number of l_1O_2 and l_2O_1 factors (a):

$$Q(a, u) = 1 + 2u + (8 + 2a)u^2 + (44 + 24a + 2a^2)u^3 + O(u^4).$$

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- Then the generating function of permutations produced with two stacks is

$$S(t) = 1 + t + 2t^2 + 6t^3 + 23t^4 + 103t^5 + O(t^6) = \frac{1}{1 - S^\bullet(t)},$$

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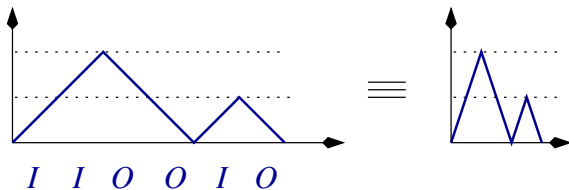
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with $S^\bullet \equiv S^\bullet(t)$ the unique solution of

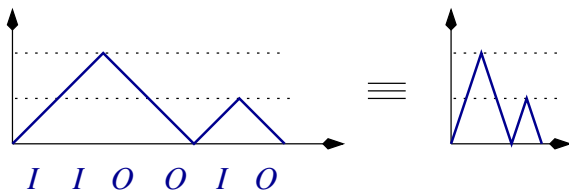
$$Q\left(-S^\bullet, \frac{t}{(1 + S^\bullet)^2}\right) = \frac{1 + S^\bullet}{1 - S^\bullet}.$$

III. Arch systems, operation sequences and quarter plane walks

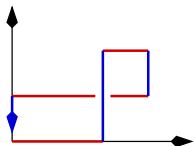
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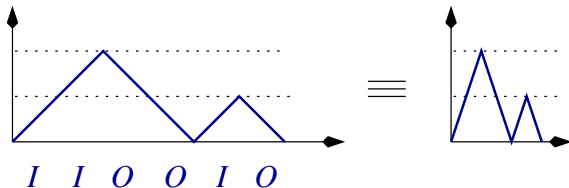


- A bi-coloured operation sequence is a walk from $(0,0)$ to $(0,0)$ (also called **loop**) in the quadrant \mathbb{N}^2

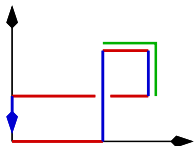


l_1 l_1 l_2 l_2 l_1 O_2 O_1 O_1 O_1 O_2

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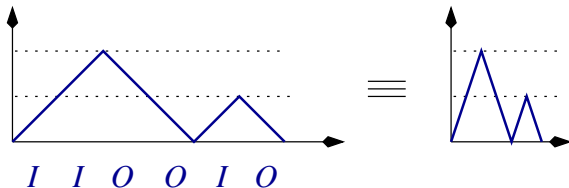


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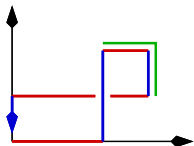


l_1 l_1 l_2 l_2 l_1 O_2 O_1 O_1 O_1 O_2
 corner

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 corner

- Can we determine

$$Q(a, u) = \sum_{w \text{ loop}} u^{|w|/2} a^{\text{corners}(w)} \quad ?$$

Consider all quarter plane walks (starting from $(0, 0)$)

- Generating function $Q(a, u; x, y) \equiv Q(x, y)$, where u counts steps, a corners, and x, y the coordinates of the endpoint:

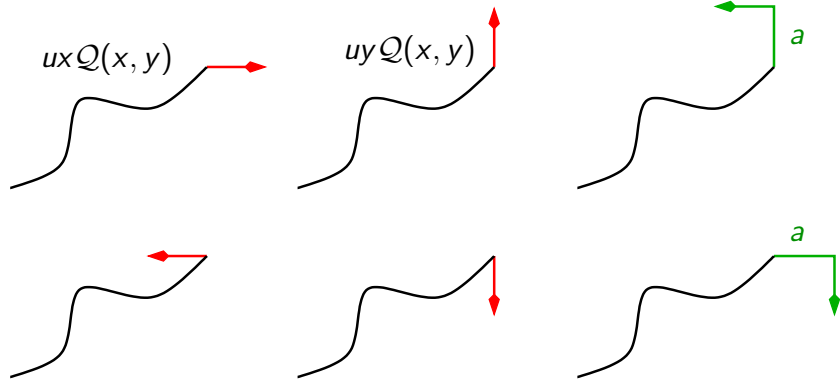
$$Q(a, u; x, y) = 1 + (x + y)u + (2 + xy + x^2 + y^2)u^2 \\ + ((a + 4)(x + y) + 3x^2y + 3xy^2 + x^3 + y^3) u^3 + O(u^4).$$

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- Recursive construction



Functional equation for quadrant walks

- Generating function $Q(a, u; x, y) \equiv Q(x, y)$ of all quadrant walks:

$$(1 - u(x + \bar{x} + y + \bar{y}) - u^2(a - 1)(x\bar{y} + y\bar{x}))Q(x, y) = \\ 1 - u\bar{y}(1 + ux(a - 1))Q(x, 0) - u\bar{x}(1 + uy(a - 1))Q(0, y)$$

with $\bar{x} = 1/x$ and $\bar{y} = 1/y$.

- Loops in the quadrant:

$$Q(a, u) = Q(a, \sqrt{u}; 0, 0)$$

\Rightarrow Compute the first coefficients of $Q(a, u)$... and discover and test conjectures

The enumeration problem is solved

- All walks in the quadrant: $Q(a, u; x, y) \equiv Q(x, y)$ with

$$(1 - u(x + \bar{x} + y + \bar{y}) - u^2(a - 1)(x\bar{y} + y\bar{x}))Q(x, y) = \\ 1 - u\bar{y}(1 + ux(a - 1))Q(x, 0) - u\bar{x}(1 + uy(a - 1))Q(0, y)$$

- Loops in the quadrant:

$$Q(a, u) = Q(a, \sqrt{u}; 0, 0)$$

- Sortable permutations:

$$S(t) = \frac{1}{1 - S^\bullet(t)}$$

with

$$Q\left(-S^\bullet, \frac{t}{(1 + S^\bullet)^2}\right) = \frac{1 + S^\bullet}{1 - S^\bullet}.$$

Did you say “solved”?

- What is the asymptotic behaviour of the number s_n of sortable permutations of length n ?

$$s_n = [t^n]S(t)$$

- What is the nature of the generating function $S(t)$? is it algebraic, D-finite, D-algebraic?
- What is the nature of the generating function $Q(a, u)$?

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Asymptotics: the main source of difficulty

- Generating function of quarter plane loops:

$$Q(a, u) = \sum_{w \text{ loop}} u^{|w|/2} a^{\text{corners}(w)} \in \mathbb{N}[a][[u]]$$

- Characterization of $S(t) = 1/(1 - S^\bullet(t))$:

$$Q\left(-S^\bullet, \frac{t}{(1 + S^\bullet)^2}\right) = \frac{1 + S^\bullet}{1 - S^\bullet}.$$

⇒ To find the first positive singularity t_c of S^\bullet (that is, its radius) we must explore the singularities of $Q(a, u)$ when $a < 0$.

Conjecture 1: $(a + 1)$ -Positivity

Conjecture

The generating function of quarter plane loops is $(a + 1)$ -positive :

$$Q(a, u) = \sum_{n \geq 0} u^n p_n(a + 1)$$

with $p_n(b) \in \mathbb{N}[b]$.

Refinement

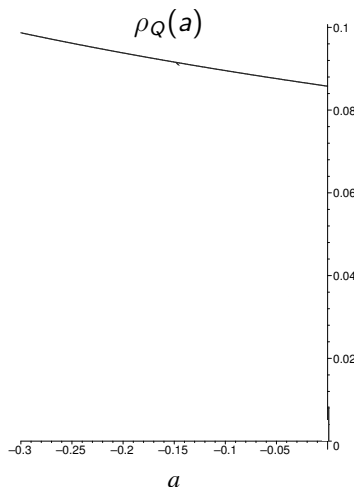
- One can count separately vertical and horizontal steps:

$$Q(a, s, t) = \sum_{i, j \geq 0} s^i t^j p_{i, j}(a + 1)$$

- True up to $n = 200$.

The general picture: what happens on $[0, t_c]$

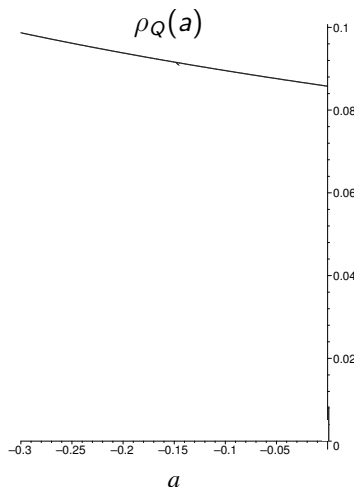
Let $\rho_Q(a)$ be the radius of convergence of $u \mapsto Q(a, u)$ (a non-increasing function of $a \geq -1$).



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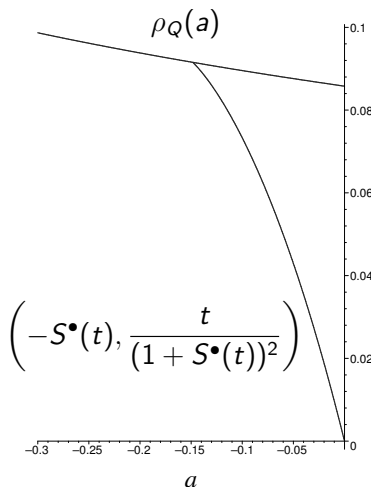
$$Q\left(-s^\bullet, \frac{t}{(1+s^\bullet)^2}\right) = \frac{1+s^\bullet}{1-s^\bullet}$$



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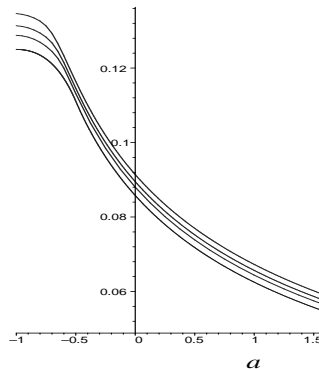


Conjecture 2: The radius of convergence of $Q(a, \cdot)$

Conjecture

For $a \geq -1$, the radius of $Q(a, \cdot)$ is

$$\rho_Q(a) = \begin{cases} \frac{1}{(2 + \sqrt{2 + 2a})^2} & \text{if } a \geq -1/2, \\ -\frac{a}{2(1 - a)^2} & \text{if } a \in [-1, -1/2] \end{cases}$$



- In particular: $\rho_Q(1) = 1/16$ and $\rho_Q(-1) = 1/8$
- Allows us to estimate $t_c \sim 0.12$, so that $s_n^{1/n} \rightarrow 8.2 \dots$

Conjecture 3: The growth of the coefficients of $Q(a, u)$

Conjecture

The series $Q'_2(a, u) := \frac{\partial Q}{\partial u}(a, u)$ is convergent at $u = \rho_Q(a)$ for $a \geq -1/3$.

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- Expected:

$$q_n(a) := [u^n]Q(a, u) \sim \rho_Q(a)^{-n} n^{\gamma(a)}$$

with $\gamma(a) < -2$ for $a > -1/3$.

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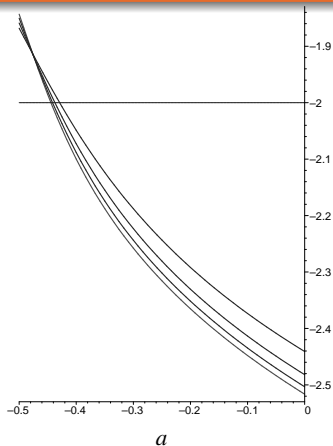
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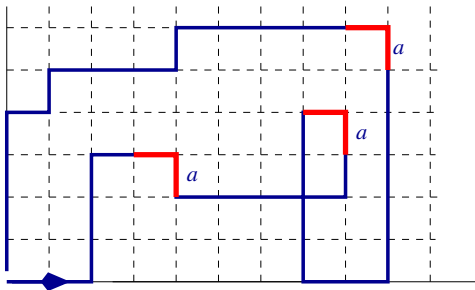
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with $\gamma(a) < -2$ for $a > -1/3$.



The curves $n^2(1 - q_{n-1}(a)q_{n+1}(a)/q_n(a)^2)$, for $n = 40, 60, 80$ and 100 .

IV. What we know about $Q(a, u)$



$$Q(a, u) = \sum_{w \text{ loop}} u^{|w|/2} a^{\text{corners}(w)}$$

A functional equation for $Q(a, u; x, y) \equiv Q(x, y)$

The generating function $Q(a, u; x, y) \equiv Q(x, y)$ of all quadrant walks, counted by the length (u), the number of corners (a) and the coordinates of the endpoint (x, y) satisfies:

$$(1 - u(x + \bar{x} + y + \bar{y}) - u^2(a - 1)(x\bar{y} + y\bar{x}))Q(x, y) = \\ 1 - u\bar{y}(1 + ux(a - 1))Q(x, 0) - u\bar{x}(1 + uy(a - 1))Q(0, y)$$

Alas... $Q(a, u; x, y)$ is **not** $(a + 1)$ -positive.

When $a = 1$

- It's simple!

$$Q(\mathbf{1}, u) = \sum_{i, j \geq 0} \binom{2i + 2j}{2i} C_i C_j u^{i+j}$$

- Refinement with the number of \rightarrow steps (variable s) and \uparrow steps (variable t):

$$Q(\mathbf{1}, s, t) = \sum_{i, j \geq 0} \binom{2i + 2j}{2i} C_i C_j s^i t^j.$$

- Asymptotic behaviour:

$$[u^n]Q(\mathbf{1}, u) \sim 16^n n^{-3}$$

so that Conjectures 2 (radius) and 3 (growth of the coefficients) hold.

When $a = -1$

- Not so simple! But still very nice...

$$Q(-1, u) = \sum_{i,j \geq 0} \binom{i+j}{i} C_i C_j u^{i+j}$$

- Refinement with the number of \rightarrow steps (variable s) and \uparrow steps (variable t):

$$Q(-1, s, t) = \sum_{i,j \geq 0} \binom{i+j}{i} C_i C_j s^i t^j.$$

- Asymptotic behaviour:

$$[u^n]Q(-1, u) \sim 8^n n^{-3}$$

so that Conjecture 2 (radius) holds.

A lower bound on the radius of $Q(a, \cdot)$ for $a \geq 0$

- Count all NSEW walks by the number of corners and the length:

$$\mathcal{W}(a, u) = \frac{1}{1 - 4u - 2(a - 1)u^2}.$$

This series has radius

$$\frac{1}{2 + \sqrt{2 + 2a}}.$$

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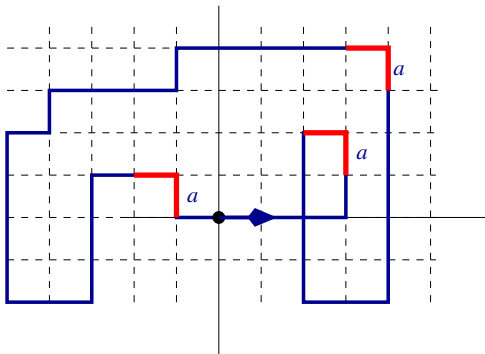
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- For $a \geq 0$, $Q(a, u)$ is dominated (coefficient by coefficient) by $\mathcal{W}(a, \sqrt{u})$, and thus has radius at least

$$\frac{1}{(2 + \sqrt{2 + 2a})^2}.$$

Conjecture 2: quarter plane loops have the same exponential growth as general walks.

V. More evidence: The case of unconfined loops



$(a + 1)$ -Positivity

Proposition

The generating function of general plane loops is $(a + 1)$ -positive :

$$W(a, u) = \sum_{n \geq 0} u^n w_n(a + 1)$$

with $w_n(b) \in \mathbb{N}[b]$.

Refinement

- One can count separately vertical and horizontal steps:

$$W(a, s, t) = \sum_{i, j \geq 0} s^i t^j w_{i, j}(a + 1)$$

When $a = 1$ or $a = -1$

- The series $W(a, s, t)$ that counts general loops satisfies

$$W(1, s, t) = \sum_{i, j \geq 0} \binom{2i+2j}{2i} \binom{2i}{i} \binom{2j}{j} s^i t^j$$

and

$$W(-1, s, t) = \sum_{i, j \geq 0} \binom{i+j}{i} \binom{2i}{i} \binom{2j}{j} s^i t^j$$

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- Combinatorial explanation when $a = -1$?

The radius of convergence of $W(a, \cdot)$

Proposition

For $a \geq -1$, the radius of $W(a, \cdot)$ is

$$\rho_W(a) = \begin{cases} \frac{1}{(2 + \sqrt{2 + 2a})^2} & \text{if } a \geq -1/2, \\ -\frac{a}{2(1 - a)^2} & \text{if } a \in [-1, -1/2]. \end{cases}$$

The exponent for general loops

Proposition

For $a \geq -1$,

$$w_n(a) := [u^n]W(a, u) \sim \rho_W(a)^{-n} n^{-1}$$

except when $a = -1/2$ where the exponent is $-3/4$.

What makes unconfined loops easier?

- Their generating function is the constant term in a **rational** series:

$$W(a, s^2, t^2) = [x^0 y^0] \frac{1}{1 - s(x + \bar{x}) - t(y + \bar{y}) - st(a - 1)(x\bar{y} + \bar{x}y)}.$$

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- Let v be a vertical 1D loop of length $2j$. The generating function of loops that project vertically on v only depends on j .

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It suffices to prove the $(a+1)$ -positivity of

$$\sum_{j \geq 0} t^j W_j(a, s),$$

which is an **algebraic** (quartic, in fact biquadratic) series.

Loops confined to the upper half-plane

Generating function of general loops:

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The generating function of half-plane loops is

$$H(a, s, t) = \sum_{j \geq 0} \frac{1}{j+1} \binom{2j}{j} t^j W_j(a, s).$$

It is also $(a+1)$ -positive, and $H(a, u)$ has the same radius of $W(a, u)$.

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It is also $(a+1)$ -positive, and $H(a, u)$ has the same radius of $W(a, u)$.

Remark: the exponent is -2 , except at $a = -1/2$ where it is $-7/4$.

Final remarks

- Prove the conjectures!
- Combinatorial proofs for the $(a + 1)$ -positivity, for the values at $a = -1...$
- Nature of the series $S(t)$ and $Q(a, u)$
- More detailed asymptotics of $[t^n]S(t)$ (sub-exponential terms?)
- Other sorting machines: the deque (double-ended queue), 3 parallel stacks...