

Around a curious q -analogue of Hermite
polynomials¹

Jiang Zeng

Université Claude Bernard Lyon 1

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¹Based on a Joint work with Johann Cigler

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Some well-known facts from Viennot's UQAM Lecture Notes

Given an OPS $\{p_n(x)\}$ satisfying the three term recurrence relation

$$p_{n+1}(x) = (x - b_n)p_n(x) - \lambda_n p_{n-1}(x) \quad (1)$$

with initial values $p_0(x) = 1$ and $p_{-1}(x) = 0$.
Define the connection coefficients $a(n, k)$ by

$$\sum_{k=0}^n a(n, k) p_k(x) = x^n. \quad (2)$$

These are characterized by the [Stieltjes tableau](#):

$$a(0, k) = \delta_{k,0},$$

$$a(n, 0) = b_0 a(n-1, 0) + \lambda_1 a(n-1, 1),$$

$$a(n, k) = a(n-1, k-1) + b_k a(n-1, k) + \lambda_{k+1} a(n-1, k+1).$$

Moments

Define a linear functional $\mathcal{F} : K[x] \rightarrow K$ by $\mathcal{F}(p_n(x)) = \delta_{n,0}$. Then

$$\mathcal{F}(x^n) = a(n, 0) \quad (\text{Moments}).$$

The generating function of the moments has the continued fraction expansion

$$\sum_{n \geq 0} \mathcal{F}(x^n) z^n = \frac{1}{1 - b_0 z - \frac{\lambda_1 z^2}{1 - b_1 z - \frac{\lambda_2 z^2}{1 - \dots}}}. \quad (3)$$

The Hankel determinants for the moments are

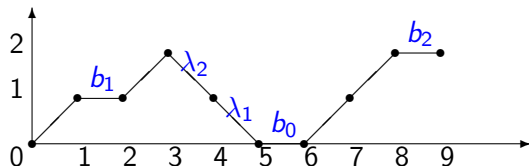
$$d(n, 0) = \det(\mathcal{F}(z^{i+j}))_{i,j=0}^{n-1} = \prod_{i=1}^{n-1} \prod_{k=1}^i \lambda_k, \quad (4)$$

and

$$d(n, 1) = \det(\mathcal{F}(z^{i+j+1}))_{i,j=0}^{n-1} = d(n, 0)(-1)^n p_n(0). \quad (5)$$

Motzkin paths

Consider the following Motzkin path γ



The **weight** is $w(\gamma) = b_0 b_1 b_2 \lambda_1 \lambda_2$.

$$a(n, k) = \sum_{\gamma: (0,0) \rightarrow (n,k)} w(\gamma).$$

Hermite polynomials

The rescaled Hermite polynomials $H_n(x, s) = s^{n/2} H_n(x/\sqrt{s}, 1)$

$$H_{n+1}(x, s) = xH_n(x, s) - snH_{n-1}(x, s). \quad (6)$$

The explicit formula

$$H_n(x, s) = \sum_{k=0}^n \binom{n}{2k} (-s)^k (2k-1)!! x^{n-2k}.$$

Let $D = \frac{d}{dx}$. Then

$$D H_n(x, s) = n H_{n-1}(x, s),$$

and

$$H_n(x, s) = (x - sD)^n \cdot 1,$$

$$H_n(x + sD, s) \cdot 1 = x^n.$$

Connection coefficients

The generating function

$$\sum_{n \geq 0} H_n(x, s) \frac{t^n}{n!} = e^{xt - st^2/2}$$

implies that

$$e^{st^2/2} \sum_{n \geq 0} H_n(x, s) \frac{t^n}{n!} = e^{xt}.$$

Comparing the coefficients of $t^n/n!$:

$$x^n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n! s^k}{2^k k! (n-2k)!} H_{n-2k}(x, s). \quad (7)$$

$$H_n(x, s) = \sum_{k=0}^n \frac{n! (-s)^k}{2^k k! (n-2k)!} x^{n-2k}.$$

Let $p_n(z, x, s) = H_n(z - x, -s)$. Then

$$p_{n+1}(z, x, s) = (z - x)p_n(z, x, s) + ns \cdot p_{n-1}(z, x, s) \quad (8)$$

with initial values $p_0(z, x, s) = 1$ and $p_{-1}(z, x, s) = 0$.
Let $\mathcal{F} : \mathbb{C}[z] \rightarrow \mathbb{C}$ be the linear functional defined by

$$\mathcal{F}(p_n(z, x, s)) = \delta_{n,0}.$$

Since the moments μ_n of the Hermite polynomials are

$$\mu_{2n} = (2n - 1)!!, \quad \text{and} \quad \mu_{2n+1} = 0,$$

we have

$$\mathcal{F}(z^n) = (\sqrt{-s})^n \sum_{k=0}^n \binom{n}{k} (x/\sqrt{-s})^{n-k} \mu_k = H_n(x, s). \quad (9)$$

This is equivalent to the following continued fraction expansion:

$$\sum_{n \geq 0} H_n(x, s) z^n = \frac{1}{1 - xz + \frac{1}{sz^2 + \frac{1}{1 - xz + \frac{1}{2sz^2 + \frac{1}{1 - xz + \frac{1}{3sz^2 + \frac{1}{1 - xz + \dots}}}}}}}}.$$

Their Hankel determinants are

$$d(n, 0) = \det(H_{i+j}(x, s))_{i,j=0}^{n-1} = (-s)^{\binom{n}{2}} \prod_{j=0}^{n-1} j!,$$

$$d(n, 1) = \det(H_{i+j+1}(x, s))_{i,j=0}^{n-1} = (-1)^n H_n(-x, -s) (-s)^{\binom{n}{2}} \prod_{j=0}^{n-1} j!.$$

Three known q -analogues through Viennot's looking glass

We first introduce some standard q -notations. For $n \geq 1$ let

$$[n]_q = \frac{1 - q^n}{1 - q}, \quad [n]_q! = \prod_{k=1}^n [k]_q, \quad [2n - 1]_q!! = \prod_{k=1}^n [2k - 1]_q,$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}$$

for $0 \leq k \leq n$ and zero otherwise, with

$(a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})$ and

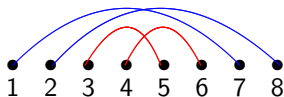
$$(a, b; q)_n = (a; q)_n (b; q)_n.$$

crossings and nestings

If α is a perfect matching of the complete graph K_n on $\{1, \dots, n\}$, a pair of edges (e, f) of α with $e = \{i, j\}$, $f = \{k, l\}$

- forms a **crossing** if $i < k < j < l$,
- forms a **nesting** if $i < k < l < j$.

Let $\text{cros}(\alpha)$ be the **crossing number** of α , and $\text{nest}(\alpha)$ be the **nesting number** of α . For instance, if $\alpha = \{1, 7\} - \{2, 8\} - \{3, 5\} - \{4, 6\}$:



$$\begin{aligned} \text{cros}(\alpha) &= 2 \\ \text{nest}(\alpha) &= 4 \end{aligned}$$

Continued fraction for crossings and nestings

Let $[n]_{p,q} = (p^n - q^n)/(p - q)$ and

$$\mu_n(p, q) = \sum_{\alpha \in M_{2n}} p^{\text{cross } \alpha} q^{\text{nest } \alpha},$$

where M_n is the set of perfect matchings of $[n]$. Then

$$\sum_{n \geq 0} \mu_n(p, q) t^n = \frac{1}{1 - \frac{1 \cdot t^2}{1 - \frac{[2]_{p,q} \cdot t^2}{\ddots \frac{[n]_{p,q} \cdot t^2}{\dots}}}}. \quad (10)$$

Continuous q -Hermite polynomials

Consider the moment sequence

$$\mu_n(q, 1) = \sum_{\alpha \in M_n} q^{\text{cros } \alpha}.$$

It follows that $\mu_{2n+1}(q, 1) = 0$ and the **Touchard-Riordan formula**

$$\mu_{2n}(q, 1) = \frac{1}{(1-q)^n} \sum_{k=-n}^n \binom{2n}{n+k} (-1)^k q^{\binom{k}{2}}.$$

Moments of the rescaled continuous q -Hermite polynomials

$$H_{n+1}(x|q) = 2xH_n(x|q) - (1-q^n)H_{n-1}(x|q). \quad (11)$$

(rescaled) $\hat{H}_{n+1}(x|q) = x\hat{H}_n(x|q) - [n]_q\hat{H}_{n-1}(x|q).$

the generating function ($x = \cos \theta$)

$$\sum_{n \geq 0} H_n(x|q) \frac{t^n}{(q; q)_n} = \frac{1}{(te^{i\theta}, te^{-i\theta}; q)_\infty}. \quad (12)$$

Discrete q -Hermite polynomials

Let

$$\mu_n(q, q^2) = \sum_{\alpha \in M_n} q^{\text{cross } \alpha + 2 \text{nest } \alpha}.$$

Then $\mu_{2n+1} = 0$ and $\mu_{2n} = [2n-1]_q [2n-3]_q \cdots [1]_q$.
Moments of (rescaled) discrete q -Hermite polynomials

$$H_{n+1}(x; q) = xH_n(x; q) - q^{n-1}(1 - q^n)H_{n-1}(x; q).$$

$$\text{(rescaled)} \quad \hat{H}_{n+1}(x; q) = x\hat{H}_n(x; q) - q^{n-1}[n]_q \hat{H}_{n-1}(x; q).$$

The generating function is

$$\sum_{n=0}^{\infty} H_n(x; q) \frac{t^n}{(q; q)_n} = \frac{(t^2; q^2)_{\infty}}{(xt; q)_{\infty}}. \quad (13)$$

$$H_n(x; q) = \sum_{k=0}^{n/2} \begin{bmatrix} n \\ 2k \end{bmatrix}_q (q; q^2)_k (-1)^k q^{k(k-1)} x^{n-2k}. \quad (14)$$

A strange q -Hermite polynomials (V. V. Borzov)

Let

$$\{n\}_q = \frac{q^{-n} - q^n}{q^{-1} - q}.$$

The rescaled q -Hermite polynomials have the moment sequence

$$\mu_n = \sum_{\alpha \in M_n} q^{\text{cros } \alpha - \text{nest } \alpha} = ?.$$

The three term recurrence

$$HH_{n+1}(x; q) = xHH_n(x; q) - (q^{-n} - q^n)HH_{n-1}(x; q). \quad (15)$$

$$\text{(rescaled)} \quad \hat{H}H_{n+1}(x; q) = x\hat{H}H_n(x; q) - \{n\}_q \hat{H}H_{n-1}(x; q).$$

Almost nothing is known about these polynomials!

Cigler's question—a curious q -analogue

Let

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}.$$

What can we say about

$$H_n(x, s|q) = (x - s D_q)^n \cdot 1?$$

The first terms of the sequence $H_n(x, s|q)$ are

$$\begin{aligned} &1, \quad x, \quad -s + x^2, \\ &\quad - (2 + q)sx + x^3, \\ &(1 + q + q^2)s^2 - (3 + 2q + q^2)sx^2 + x^4, \\ &(3 + 4q + 4q^2 + 3q^3 + q^4)s^2x - (4 + 3q + 2q^2 + q^3)sx^3 + x^5. \end{aligned}$$

+several conjectures...

Theorem (Ex-conjecture of Cigler)

Let $H_n(x, s|q) = (x - sD_q)^n \cdot 1$, then

$$\sum_{n \geq 0} H_n(x, s|q)t^n = \frac{1}{1 - b_0 t - \frac{\lambda_1 t^2}{1 - b_1 t - \frac{\lambda_2 t^2}{\ddots}}}, \quad (16)$$

where

$$b_n = q^n x, \quad \lambda_n = -[n]_q s.$$

This means that $H_n(x, s|q)$'s are moments.

Al-Salam-Chihara polynomials

The Al-Salam-Chihara polynomials $Q_n(x) := Q_n(x; \alpha, \beta)$ satisfy the three term recurrence:

$$Q_{n+1}(x) = (2x - (\alpha + \beta)q^n)Q_n(x) - (1 - q^n)(1 - \alpha\beta q^{n-1})Q_{n-1}(x),$$

with $Q_0(x) = 1$ and $Q_{-1}(x) = 0$. They have the following explicit formulas:

$$Q_n(x; \alpha, \beta | q) = \frac{(\alpha\beta; q)_n}{a^n} {}_3\phi_2 \left(\begin{matrix} q^{-n}, & \alpha e^{i\theta}, & \alpha e^{-i\theta} \\ \alpha\beta, & 0 \end{matrix} \middle| q; q \right), \quad (17)$$

where $x = \cos \theta$.

A rescaled version

From a combinatorial point of view it is more convenient to take the following rescaled version:

$$P_{n+1}(x; a, b, c) = (x - aq^n)P_n(x; a, b, c) - (c + bq^{n-1})[n]_q P_{n-1}(x; a, b, c) \quad (18)$$

with initial values $P_{-1}(x; a, b, c) = 0$ and $P_0(x; a, b, c) = 1$.

Definition

Let $\mathcal{F}_{a,b,c}$ be the linear functional on the polynomials in z which satisfy

$$\mathcal{F}_{a,b,c}(P_n(z; a, b, c)) = \delta_{n,0}. \quad (19)$$

q -Hermite polynomials

Note that the **continuous q -Hermite polynomials** are

$$\tilde{H}_n(x, s|q) = P_n(x; 0, 0, s) \quad (20)$$

and are also the moments

$$\tilde{H}_n(x, s|q) = \mathcal{F}_{x, -s, 0}(z^n). \quad (21)$$

The **discrete q -Hermite polynomials I** are

$$\tilde{h}_n(x, s; q) = P_n(x; 0, (1 - q)s, 0). \quad (22)$$

The **discrete q -Hermite polynomials II** are

$$\tilde{h}_n(x; q) = (-i)^n \tilde{h}_n(ix, 1; q^{-1}). \quad (23)$$

The curious q -analogue of Hermite polynomials

The new q -Hermite polynomials are defined by

$$H_n(x, s|q) := \mathcal{F}_{x,0,-s}(z^n). \quad (24)$$

and

$$h_n(x, s; q) := P_n(0; -x, 0, s) = q^{\binom{n}{2}} s^{n/2} \tilde{h}_n(x/\sqrt{s}; q). \quad (25)$$

A proof by brute force

Let $f(n, x) = H_n(x, (q-1)s|q) = (x + (1-q)s\mathcal{D}_q)^n \cdot 1$. Then

$$f(n, x) = \left(x + \frac{s}{x}\right)f(n-1, x) - \frac{s}{x}f(n-1, qx).$$

Hence $G(x, t) = \sum_{n \geq 0} f(n, x)t^n$ satisfies the functional equation:

$$\left(1 - \frac{x^2 + s}{x}t\right)G(x, t) = 1 - \frac{s}{x}tG(qx, t). \quad (26)$$

Suppose that

$$G(x, t) = \frac{1}{1 - \frac{(g_1 - 1)At}{1 - \frac{(g_2 - 1)g_1At}{1 - \frac{(g_3 - 1)g_2At}{1 - \dots}}}},$$

where $A := A(x) = -\frac{x^2 + s}{x}$ and $g_i := g_i(x)$.

Solving this using Wall's formula we obtain Let $L_n(x) := h_n(x, (1 - q)s; q)$. Then

$$\sum_{n \geq 0} H_n(x, (q - 1)s|q)t^n = \frac{1}{1 - \frac{c_1 t}{1 - \frac{c_n t}{1 - \dots}}}, \quad (27)$$

where $c_1 = x$, $c_2 = (1 - q)s/x$ and

$$c_{2n} = (1 - q^n)s \frac{L_{n-1}(x)}{L_n(x)}, \quad \text{for } n \geq 1; \quad (28)$$

$$c_{2n+1} = \frac{L_{n+1}(x)}{L_n(x)}, \quad \text{for } n \geq 0.$$

where $L_n(x) := h_n(x, (1 - q)s; q)$. The theorem follows then by applying the *contraction formula*, which transforms a S-continued fraction to a J-continued fraction.

Connection with non commutative operators

A. Varvak (2005): if

$$DU - qUD = 1$$

then (**Normal ordering**)

$$(D + U)^n = \sum_{m,k \geq 0} \left\{ \begin{matrix} n \\ m \end{matrix} \right\}_{k,q} U^{n-m-k} D^{m-k}.$$

where

$$\left\{ \begin{matrix} n \\ m \end{matrix} \right\}_{k,q} = \left[\begin{matrix} n-2k \\ m-k \end{matrix} \right]_q \cdot \sum_{\alpha \in M(n, n-2k)} q^{c(\alpha) + \text{cros}(\alpha)}.$$

where $M(n, k)$ is the set of matchings of $[n]$ with k isolated points and

$$c(\alpha) = \sum_{a: \text{isolated vertices}} |\{\text{edges } i < j : i < a < j\}|.$$

Varvak proved an equivalent form of the following result.

Proposition

We have

$$\begin{aligned} & \sum_{n \geq 0} t^n \sum_{\sigma \in M(n)} x^{\text{fix}(\sigma)} q^{c(\sigma) + \text{cr}(\sigma)} \\ &= \frac{1}{1 - xt - \frac{t^2}{1 - qxt - \frac{[2]_q t^2}{1 - q^2 xt - \frac{[3]_q t^2}{\dots}}}}. \end{aligned}$$

where $\text{fix}(\sigma)$ is the number of fixed points of σ .

Varvak's proof has three steps:

- Interpreting the normal order coefficients as the number of non-taking rook placements in some tableau,
- Passing from rook placements to involutions,
- Encoding involutions by weighted Motzkin paths ([Flajolet, Viennot's theory](#)).

Varvak did not give a formula for the polynomial

$$\sum_{\sigma \in M(n)} x^{\text{fix}(\sigma)} q^{c(\sigma) + \text{cr}(\sigma)} \quad \text{or} \quad H_n(x, s|q),$$

which would generalize Touchard-Riordan's formula when $x = 0$.

[Corteel, Josuat-Vergès, Prellberg, Rubey \(2009\)](#): PASEP,

Let

$$H_n(x, s|q) = (x - sD_q)^n \cdot 1 = \sum_k c(n, k, q) x^k (-s)^{\frac{n-k}{2}},$$

and the continuous q -Hermite polynomials

$$\tilde{H}_n(x, s|q) = \sum_k b(n, k, q) x^k (-s)^{\frac{n-k}{2}}. \quad (29)$$

Theorem (Cigler-Z.)

The matrices $(c(i, j, q))_{i,j=0}^{n-1}$ and $(b(i, j, q)(-1)^{\frac{i-j}{2}})_{i,j=0}^{n-1}$ are mutually inverse.

Proof.

We first show by induction that

$$\tilde{H}_n(x + s\mathcal{D}_q, s|q)1 = x^n. \quad (30)$$

On the other hand we have

$$\begin{aligned} \tilde{H}_n(x + s\mathcal{D}_q, s|q)1 &= \sum_{k=0}^n b(n, k, q) (-s)^{\frac{n-k}{2}} (x + s\mathcal{D}_q)^k 1 \\ &= \sum_{j=0}^n s^{\frac{n-j}{2}} x^j \sum_{k=j}^n b(n, k, q) (-1)^{\frac{n-k}{2}} c(k, j, q). \end{aligned}$$

□

A new approach

It's equivalent to compute the coefficients $C_{n,k}$ in

$$x^n = \sum_{k \geq 0} C_{n,k} H_{n-2k}(x|q).$$

We will deduce the coefficients in two steps:

$$x^n = \sum_{k \geq 0} b_{n,k} U_{n-2k}(x),$$

and

$$U_n(x) = \sum_{k \geq 0} a_{n,k} H_{n-2k}(x|q).$$

Tchebychev polynomials

The Tchebychev polynomials $U_n(x) = \frac{\sin((n+1)\theta)}{\sin\theta}$, $x = \cos\theta$ have

1 Recurrence

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x) \quad (n \geq 0),$$

with $U_{-1}(x) = 0$ and $U_0(x) = 1$.

2 Generating function

$$\sum_{n \geq 0} U_n(x) w^n = \frac{1}{1 - 2xw + w^2}.$$

3 Explicit formula

$$U_n(x) = \sum_{k \geq 0} (-1)^k \binom{n-k}{k} (2x)^{n-2k}.$$

Now

$$(2x)^n = \sum_{k=0}^{\lfloor n/2 \rfloor} b_{n,k} U_{n-2k}(x),$$

where $b_{n,k}$ is the number of Dyck paths from $(0,0)$ to $(n, n - 2k)$.

Using **André's reflection principle** we see that

$$b_{n,k} = \binom{n}{k} - \binom{n}{k-1}, \quad 0 \leq k \leq \frac{n}{2}.$$

Lemma

We have

$$H_n(x|q) = \sum_{k=0}^{\lfloor n/2 \rfloor} \left(\begin{bmatrix} n \\ k \end{bmatrix}_q - \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \right) U_{n-2k}(x), \quad (31)$$

$$U_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k q^{\binom{k+1}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix}_q H_{n-2k}(x|q). \quad (32)$$

Remark. A killing involution can be established to prove the above identities. When $q = 1$, $H_n(x|1) = (2x)^n$. Tchebyshev inverse relation:

$$b_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} a_{n-2k} \iff a_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left[\binom{n}{k} - \binom{n}{k-1} \right] b_{n-2k}.$$

It follows that

$$(2x)^n = \sum_{l \geq 0} H_{n-2l}(x|q) \sum_{j=0}^l (-1)^j q^{\binom{j+1}{2}} b_{n,l-j} \begin{bmatrix} n-2l+j \\ j \end{bmatrix}_q. \quad (33)$$

Clearly the coefficient of $H_0(x|q)$ is 0 if n is odd, if n is even, replacing n by $2n$ we see that the corresponding coefficient is

$$T_n(q)(1-q)^n = \sum_{j=0}^n (-1)^j q^{\binom{j+1}{2}} b_{2n,n-j}. \quad (34)$$

This is the Touchard-Riordan formula.

We derive the following result. An equivalent form of this result has been obtained with different means by [Corteel, Josuat-Vergès, Prellberg, Rubey \(2009\)](#).

Theorem

If $k \equiv n \pmod{2}$ then

$$\begin{aligned} c(n, k, q) &= \sum_{\alpha \in M(n, k)} q^{c(\alpha) + \text{cr}(\alpha)} \\ &= (1 - q)^{-\frac{n-k}{2}} \sum_{j \geq 0} \left(\binom{n}{\frac{n-k-2j}{2}} - \binom{n}{\frac{n-k-2j-2}{2}} \right) \\ &\quad \times (-1)^j q^{\binom{j+1}{2}} \begin{bmatrix} k+j \\ k \end{bmatrix}. \end{aligned}$$

- When $k = 0$, we recover a formula of Touchard-Riordan:

$$\begin{aligned}c(2n, 0, q) &= \sum_{\alpha \in M(2n, 0)} q^{\text{cros}(\alpha)} \\ &= \frac{1}{(1-q)^n} \sum_{k=0}^n \left(\binom{2n}{n-k} - \binom{2n}{n-k-1} \right) q^{\frac{k(k+1)}{2}}.\end{aligned}$$

- **Peanud** (1995) gave a combinatorial proof of this formula. Corteel-Rubey conjectured their formula by generalizing Penaud's method.

Connection with A.A. Kirillov's formula

Let $A_n(\mathcal{F}_q)$ be the set of solutions to the equation

$$X^2 = 0$$

in the $n \times n$ upper-triangular matrices with elements from \mathcal{F}_q . Denote by $A_n(q)$ the cardinality of this set. Let $A_n^r(q)$ be the number of solutions consisting of matrices of a given rank r . Then $A_n(q) = \sum_{r \geq 0} A_n^r(q)$. Kirillov proved

$$(2z)^n = \sum_r A_n^r(q) \cdot q^{r(r-n)} \cdot H_{n-2r}(z|q^{-1}).$$

Kirillov and Melnikov (1996?) made several conjectures about $A_n(q) = \sum_{r \geq 0} A_n^r(q)$, one of which was proved by Ekhad and Zeilberger (1996), Krattenthaler, Warnaar, Cigler \longrightarrow Rogers (1893).

Connection to Rogers' formulae

Gegenbauer polynomials $C_n^\nu(x)$ may be defined by the generating function

$$(1 - 2xt + t^2)^{-\nu} = \sum_{n=0}^{\infty} C_n^\nu(x) t^n. \quad (35)$$

Set $(x)_n = x(x+1)\cdots(x+n-1)$ for any non negative integer n .
Then

$$C_n^\nu(x) = \sum_{k \geq 0} \frac{(-1)^k (\nu)_{n-k} (2x)^{n-2k}}{k! (n-2k)!}. \quad (36)$$

The inversion of (36) is given by the following relation

$$x^n = \frac{n!}{2^n} \sum_{k \geq 0} \frac{(\nu + n - 2k) C_{n-2k}^\nu(x)}{k! (\nu)_{n+1-k}}. \quad (37)$$

q -ultraspherical polynomials

The generating function is

$$\sum_{n \geq 0} C_n(\cos \theta; \beta | q) t^n = \frac{(t\beta e^{i\theta}, t\beta e^{-i\theta}; q)_\infty}{(te^{i\theta}, te^{-i\theta}; q)_\infty}.$$

Note that

$$\frac{H_n(x|q)}{(q; q)_n} = C_n(x; 0|q), \quad (38)$$

$$U_n(x) = C_n(x; 0|0) = H_n(x|0), \quad (39)$$

$$T_n(x) = \lim_{\beta \rightarrow 1} \frac{1 - q^n}{2(1 - \beta)} C_n(x; q|q), \quad (40)$$

$$C_n^\nu(x) = \lim_{q \rightarrow 1} C_n(x; q^\nu|q). \quad (41)$$

Roger's connection formula

From

$$C_n(x; \gamma | q) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{\beta^k (\gamma/\beta; q)_k (\gamma; q)_{n-k} (1 - \beta q^{n-2k})}{(q; q)_k (q\beta; q)_{n-k} (1 - \beta)} C_{n-2k}(x; \beta | q),$$

one can deduce a connection formula

$$x^n = \sum_{k \geq 0} a(n, k) C_{n-2k}(x; \gamma | q)$$

This formula generalizes the previous lemma.

PB: A combinatorial analysis?