

# Discrete and continuous time models: one variable

4TTV313U - Introduction to simulation of dynamic process

University of Bordeaux - 2022

## Regarding plots

Additional information: it is better practice to open and close the figure files you are using, and to save them instead of just showing them.

```
import matplotlib.pyplot as plt

fig=plt.figure()
...
plt.savefig('yourfile.pdf',bbox_inches="tight")
plt.close(fig)
```

## World population

Now that we have an easy access to the world population census, we can build several basic models of the population growth. All models are defined by a series of parameters which quantify the way processes impact the state of the system from one time step to another. We will estimate the values of the parameters with the data from 1950 to 2016.

To assess the quality of each model, we will compute two of the most used diagnostics of a model, the bias  $B$  and the residual sum of squares ( $RSS$ ). The bias can tell you if, on average, models tend to overestimate or underestimate the values. The  $RSS$  measures the distance between observed and simulated data, whether the difference is positive or negative.

$$B = \sum_t (\hat{N}_t - N_t)$$
$$RSS = \sum_t (\hat{N}_t - N_t)^2$$

where  $N_t$  is the observed population at year  $t$  and  $\hat{N}_t$  is the simulated population.

Once we know the values of the model parameters, we can try and predict the size of the population in 2020. This can also be used as a diagnostic for the model.

### For each model described below, you will

- compute necessary parameters
- simulate the evolution of the population size
- make a figure with both simulated and observed population size over time
- output the predicted population for 2020 (the population of 2020 has been estimated around 7.693 billions)
- output  $RSS$  and bias

## Discrete time model

**Constant growth model** The simplest model we can build is a discrete time model which assumes that the population increase has remained the same since the beginning of the census, in 1950. In this case, the system follows eq. 1.

$$N_{t+1} = N_t + g_a \quad (1)$$

where  $g_a$  is the average annual growth (hint: you can compute  $g_a$  for each year based on the observations, and take the mean).

Of course, this purely phenomenological model is highly unrealistic. There is no reason to consider that annual growth will remain the same for all population sizes.

**Linear model** We can consider simple processes behind population dynamics, such as birth and death, which are proportional to the current population size. Instead, the more people there are, the more deaths and births there are.

$$N_{t+1} = N_t + (b - d)N_t = N_t(1 + g_r) \quad (2)$$

where  $b$  and  $d$  are the birth and death rates, which can be summarized into the net annual growth rate,  $g_r = (b - d)$ , which you can compute in a similar way as above.

⇒ In addition to the comparison between observed and simulated populations counts, we can compare the observed growth rate to the computed one. Plot both growth rates as a function of the population size (see Fig. 1). Do you think the model fits the data?

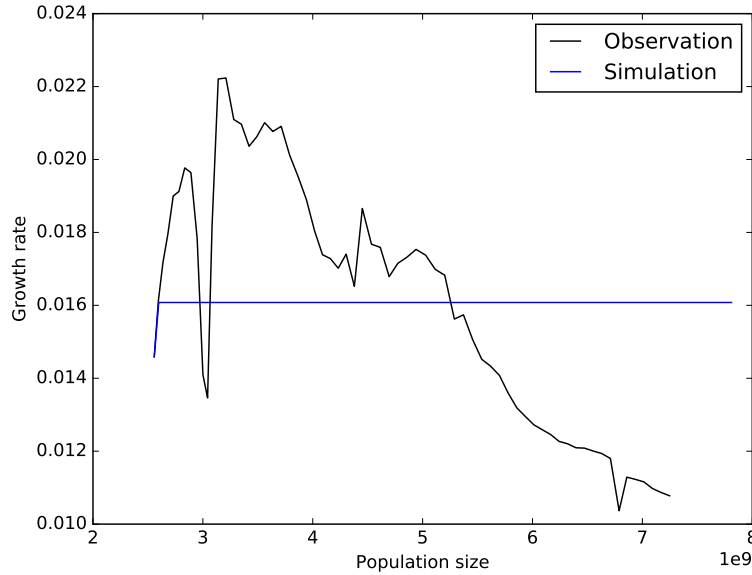


Figure 1: Observed growth rate as a function of the population size

**Quadratic model** According to Fig 1, a linear growth rate is not sufficient to model the data. We can try and add a quadratic component to the growth rate. This can represent the saturation of the growth when the population is too large if  $\beta < 0$ .

$$N_{t+1} = N_t + \alpha N_t + \beta N_t^2 \quad (3)$$

In this case, you can use  $\alpha = 0.025$  and  $\beta = -1.8 \cdot 10^{-12}$ .

**Bonus:** Could you compute the exact population size for which the population is at equilibrium, i.e. will remain at the same size?

## Continuous time model

**When we can solve the system** Population growth is a continuous process and can be modeled as such. We still consider that the growth rate is proportional to the current population.

$$\begin{aligned}\frac{dN}{dt} &= \delta N \\ \Leftrightarrow \frac{1}{N} \frac{dN}{dt} &= \delta \\ \Leftrightarrow \frac{1}{N} dN &= \delta dt \\ \Leftrightarrow \ln(N) &= \delta t + N_0 \\ \Leftrightarrow N &= N_0 \exp(\delta t)\end{aligned}$$

From this equation, you can simulate a new population (i.e., find the values of  $\delta$  and  $N_0$ ).

**When we cannot solve the system** In the previous case, the function is easy to integrate and we do not need to use its differential expression. However, it is sometimes much harder, even impossible, to do so for other dynamic systems. We will therefore use an integrator. There are several ways to code such integral calculator, depending on the used scheme, but there are now efficient packages which can be used instead of having to code everything by oneself. We can use the SciPy package which proposes the function `scipy.integrate.odeint`, an adaptative scheme for stiff and non-stiff systems.

We will apply this method with yet another model. As opposed to exponential growth, we can explore logistic growth, in the form  $\frac{dN}{dt} = g_{\max} N_t (1 - \frac{N_t}{K})$  where  $K$  is called the carrying capacity of the environment. This model assumes that the growth cannot be infinite and will saturate for a certain size of the population, given by  $K$ .

```
from scipy import integrate
import numpy as np
import matplotlib.pyplot as plt

#Definition of the function to integrate
def f1(x,t,g_max,K):
    dxdt=g_max*x*(1-x/K)
    return dxdt

#Import data. Note the use of genfromtxt
#which is another way of extracting data from a text file
tab=np.genfromtxt("World_population.csv",
...dtype=['int','float'],delimiter=";",skip_header=1)

year=[a[0] for a in tab[0:(len(tab)-1)]]
pop=[a[1] for a in tab[0:(len(tab)-1)]]
t_list=list(range(1950,2016))

#Launch simulations
```

```
g_max=0.025
K=14e9
N_odeint=integrate.odeint(f1,pop[0],t_list,args=(g_max,K))
```

**Bonus:** Based on  $RSS$ , try and find a value of  $K$  which would allow for a better fit (hint: you should use a loop structure over several values of  $K$ ).

⇒ Compare all your models. Based on the three diagnostic metrics (prediction for 2020, bias and  $RSS$ ), which model would you use? Predict the population in 2100 based on this model<sup>1</sup>.

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<sup>1</sup>Warning: This prediction assumes that the model remains correct when we extrapolate on the long-term. Keep in mind that this is a strong assumption that should be justified in a 'real-life' context.