

Combinatorics of non-ambiguous trees

Adrien Boussicault

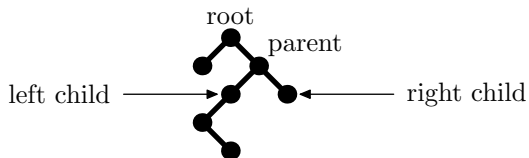
Joint work with J.C. Aval, M. Bouvel, M. Silimbani

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Université de Bordeaux, France

Université du Québec à Trois-Rivières - DMI - 6/5/2015

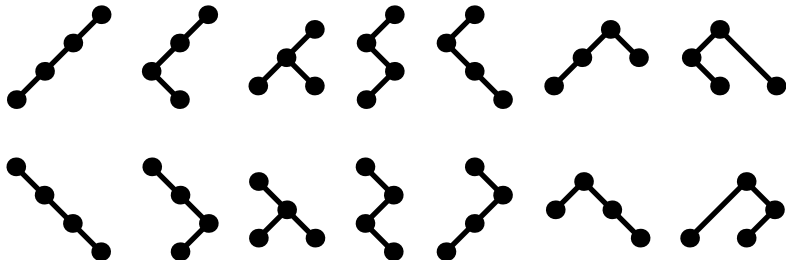
The binary trees

A binary tree is a tree where vertices have 0, 1 or 2 children.



size = nb of vertices = 7.

All the binary trees of size 4



The number of binary trees: the Catalan numbers

There are

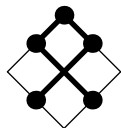
$$C_n = \frac{1}{n+1} \binom{2n}{n} \underset{+\infty}{\sim} \frac{1}{\pi n \sqrt{n}} 4^n$$

binary trees of size n .

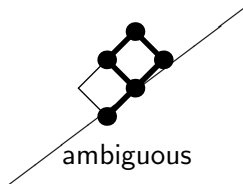
$$(C_n)_{n \geq 1} = (1, 2, 5, 14, 42, \dots)$$

Intuitive definition of non-ambiguous trees

A non-ambiguous tree is just a way of drawing a tree in a grid without "ambiguity".



non ambiguous




ambiguous

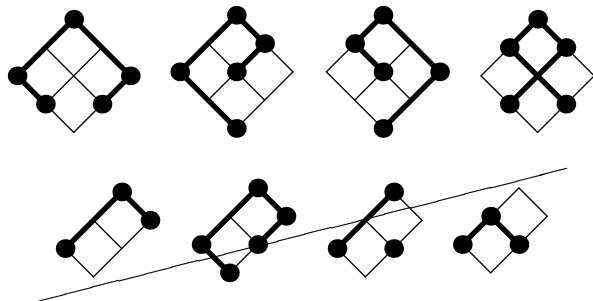
We forbid the pattern



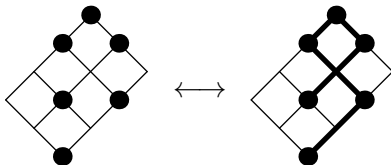
The non-ambiguous trees

A non-ambiguous tree of size n is a binary tree of size n embedded in a grid in such a way that:


- the vertices are on the grid
- the edges follow the lines of the grid
- the pattern  is forbidden
- each line contains at least one vertex



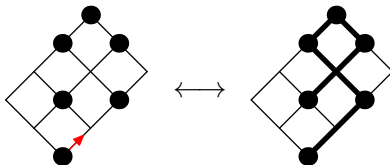
The edges of a non-ambiguous tree are not necessary




Size = nb of vertices = 6

Forbidden pattern  : a vertex can't have two parents.

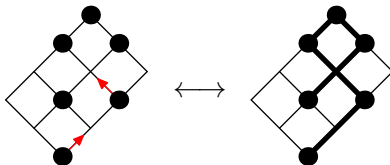
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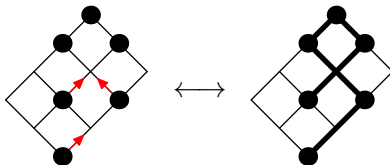
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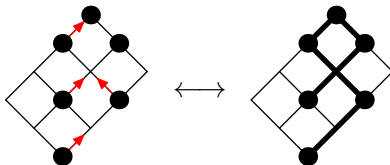
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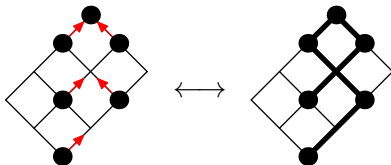
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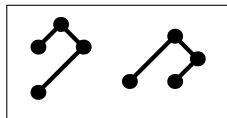
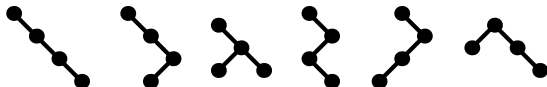
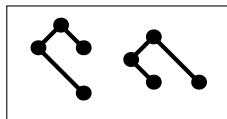
The edges of a non-ambiguous tree are not necessary



Size = nb of vertices = 6

Forbidden pattern  : a vertex can't have two parents.

All the non-ambiguous trees of size 4



The number of non-ambiguous trees

$$A136127 = (a_k)_{k \geq 1} = (1, 2, 5, 16, 63, 294, 1585, 9692, \dots)$$

The number of non-ambiguous trees

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The number a_n of non-ambiguous trees of size n is equal to the number of permutations having all its excedences placed at the beginning.

[Steingrímsson, Williams, 07] + [Burstein, 07] + [Aval, B., Nadeau, 11]

$$a_n = \sum_{k+l=n-1}^n \left(\sum_{i=1}^k (-1)^{k-i} S(k, i) i! i^{l-1} \right)$$

where $S(k, i)$ are the Stirling numbers of the second kind.

[Ehrenborg and Steingrímsson, 2000]

[Combinatorial proof with non ambiguous trees and tree-like tableaux]

The number of non-ambiguous trees

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$$a_{w,h} = \sum_{p \geq 0} (p+1)! p! S(w, p+1) S(h, p+1)$$

where $S(k, i)$ are the Stirling numbers of the second kind., $a_{w,h}$ is the number of nat with width w and height h .

Patxi Laborde-Zubieta - October/November 2015 - Trois-Rivières ?

The context

Permutation tableaux and alternative tableaux are objects used to study :

- the PASEP model in physics;
- 2 – 31 pattern inside a permutation;
- excedences and cycles of a permutation;
- Laguerre polynomials;
- ...

(Burstein, Corteel, Dasse-Harteau, Hitczenko, Josuat-Verges, Nadeau, Postnikov, Steingrímsson, Viennot, Williams, Kim, 2005-2011)

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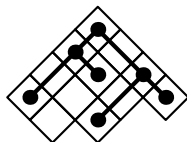
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Tree-like tableaux have been introduced to simplify and to explain lots of previous results. (Aval, B., Nadeau, 2011)

The internal structure of tree-like tableaux is described by the non-ambiguous trees.

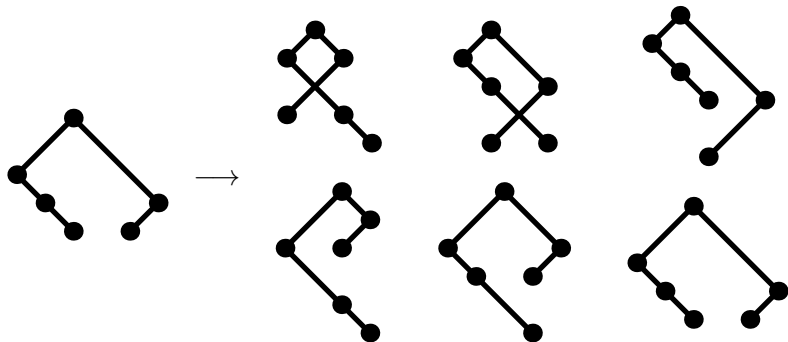


tree-like tableau

Outline

- 1 Enumeration of non-ambiguous trees with a fixed underlying tree
- 2 The complete non-ambiguous trees and the Bessel function
- 3 Bijection between parallelogram polyominoes and binary trees

The non-ambiguous trees associated with a fixed tree

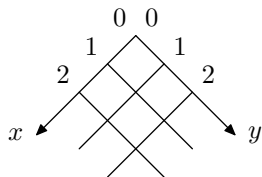


$$\text{NA} \left(\begin{array}{c} \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \end{array} \right) = 6$$

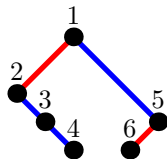
Outline

- 1 Enumeration of non-ambiguous trees with a fixed underlying tree
 - Coding a non-ambiguous tree with the coordinates of its vertices
 - All these coordinates are described by linear extensions of particular posets
 - The non-ambiguous trees are counted by a new hook formula
- 2 The complete non-ambiguous trees and the Bessel function
- 3 Bijection between parallelogram polyominoes and binary trees

Some conventions



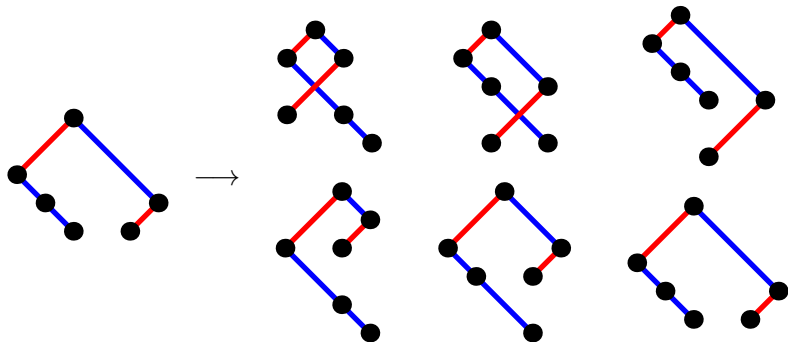
The x and y coordinates



We choose some arbitrary indices for the vertices

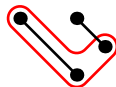
A column is a x -oriented line
A row is a y -oriented line

The non-ambiguous trees associated with a fixed tree



$$\text{NA} \left(\begin{array}{c} \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \\ \backslash \\ \bullet \end{array} \right) = 2 \times 3$$

Study of the forbidden pattern

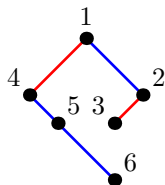


— forbidden pattern

- The ending points of two left edges can be in the same row;
- The x -coordinates of the ending points of two different edges are different;
- The x -coordinates of the ending points of the left edges form an interval $[1, k]$.

These properties are the same for the right edges.

Total order on the set of the left (resp. right) edges



$$X(3) < X(4)$$

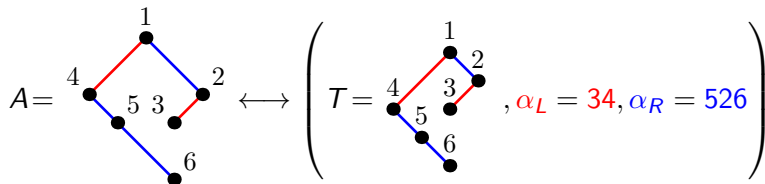
$$Y(5) < Y(2) < Y(6)$$

We get two total orders on the ending points of the left (resp. right) edges

$$\alpha_L = 34$$

$$\alpha_R = 526$$

The non-ambiguous trees are represented by (α_L, α_R)



$$T = \text{tree}(A)$$

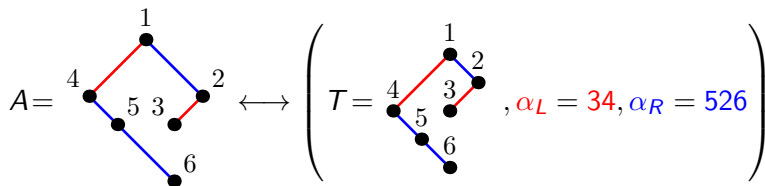
$$X(0) = Y(0) = 0$$

$\forall i$ an ending point of left edges

$\forall j$ an ending point of right edges

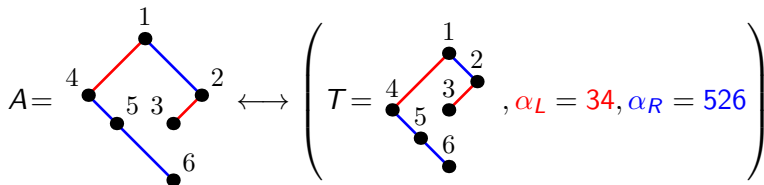
$$\begin{cases} \alpha_L^{-1}(i) = Y(i) & \text{et} & X(i) = X(\text{parent}(i)) \\ \alpha_R^{-1}(j) = X(j) & \text{et} & Y(j) = Y(\text{parent}(j)) \end{cases}$$

Question:



What are the constraints verified by α_L and α_R ?

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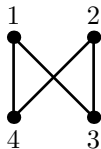
What are the constraints verified by α_L and α_R ?

Answer: there are linear extensions of two particular posets:

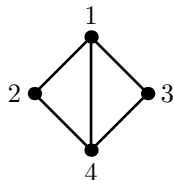
- the poset of the ending points of the left edges;
- the poset of the ending points of the right edges.

Poset and Hasse diagram

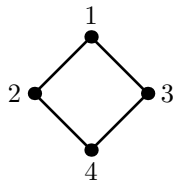
Poset = partially ordered set



$1 \leq 3$ $1 \leq 4$
 $2 \leq 3$ $2 \leq 4$
2, 1 are minimal
4, 3 are maximal



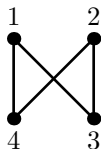
Hasse diagram :



$1 \leq 3 \leq 4$
 $1 \leq 2 \leq 4$

Linear extensions of a poset

It is a totally ordered set with respect with the oder of P



Linear extensions of the previous poset : 1234, 1243 , 2134, 2143

Notation: $\mathcal{L}(P)$ is the set of all the linear extensions of P .

The constraints verified by the ending points of the left edges (α_L)

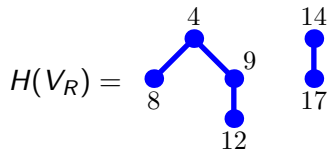
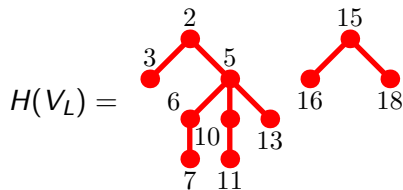
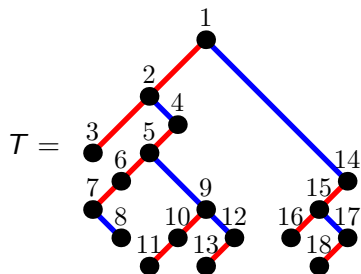


$$i = \text{ancestor}(j) \implies X(i) < X(j) \implies i <_{\alpha_L} j$$

α_h is a linear extension of the poset V_L defined on the ending points of the left edges with the following relation order:

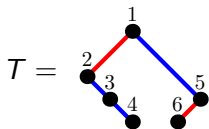
$$i = \text{ancestor}(j) \iff i <_{V_L} j$$

The Hasse diagrams of the poset V_L and V_R

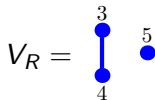
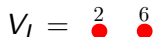
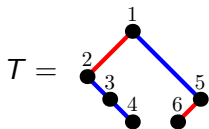


α_L (resp. α_R) are a linear extension of V_L (resp. V_R).

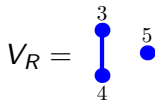
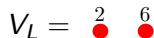
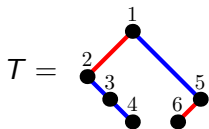
Non-ambiguous trees with a fixed underlying tree



Non-ambiguous trees with a fixed underlying tree

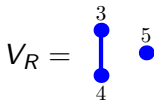
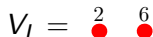
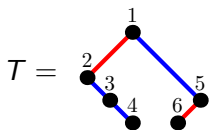


Non-ambiguous trees with a fixed underlying tree



$$\begin{aligned} & \alpha_L \times \alpha_R \\ & \in \\ & \mathcal{L}(V_L) \times \mathcal{L}(V_R) \end{aligned}$$

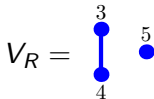
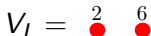
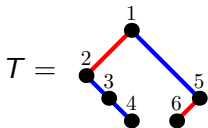
Non-ambiguous trees with a fixed underlying tree



$$(26, 534) \quad (26, 354) \quad (26, 345)$$

$$\begin{array}{c} \alpha_L \times \alpha_R \\ \in \\ \mathcal{L}(V_L) \times \mathcal{L}(V_R) \end{array} \implies (62, 534) \quad (62, 354) \quad (62, 345)$$

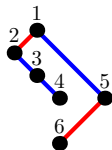
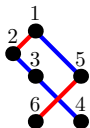
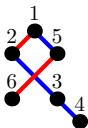
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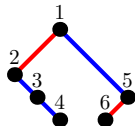
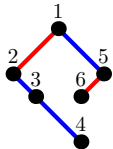
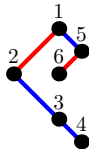
$$\alpha_L \times \alpha_R \in \mathcal{L}(V_L) \times \mathcal{L}(V_R)$$

\implies

$(26, 534)$ $(26, 354)$ $(26, 345)$



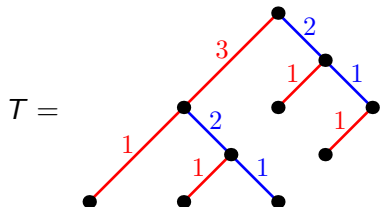
$(62, 534)$ $(62, 354)$ $(62, 345)$



The number of non-ambiguous trees with a fixed underlying tree

$$\begin{aligned}
 \mathcal{NA} \left(\begin{array}{c} 1 \\ \swarrow \quad \searrow \\ 2 \quad 3 \quad 4 \quad 5 \\ \swarrow \quad \searrow \\ 6 \end{array} \right) &= \left| \mathcal{L} \left(\begin{array}{c} 2 \quad 6 \\ \bullet \quad \bullet \end{array} \right) \right| \times \left| \mathcal{L} \left(\begin{array}{c} 3 \quad 5 \\ \bullet \quad \bullet \\ \bullet \\ 4 \end{array} \right) \right| \\
 &= \mathit{hook} \left(\begin{array}{c} 2 \quad 6 \\ \bullet \quad \bullet \end{array} \right) \times \mathit{hook} \left(\begin{array}{c} 3 \quad 5 \\ \bullet \quad \bullet \\ \bullet \\ 4 \end{array} \right) \\
 &= \frac{2!}{1.1} \times \frac{3!}{1.1.2} \\
 &= 2 \times 3
 \end{aligned}$$

A hook formula for the non-ambiguous trees



We associate each edge with an integer n_e . Given a left (resp. right) edge e , the integer n_e is the number of left (resp. right) edges contained in the subtree whose root is the ending point of e , plus 1.

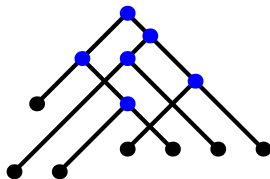
$$\begin{aligned}
 \mathcal{NA}(T) &= \frac{\#\{\text{left edges}\}! \cdot \#\{\text{right edges}\}!}{\prod\{\text{left labels}\} \cdot \prod\{\text{right labels}\}} \\
 &= \frac{5! \cdot 4!}{1 \cdot 1 \cdot 1 \cdot 1 \cdot 3 \cdot 1 \cdot 1 \cdot 2 \cdot 2} \\
 &= 240
 \end{aligned}$$

Outline

- 1 Enumeration of non-ambiguous trees with a fixed underlying tree
- 2 The complete non-ambiguous trees and the Bessel function
 - The complete non-ambiguous trees are related with the formal power series of the logarithm of the Bessel function of order 0
 - The root suppression gives a recursive enumeration
 - Another identities
- 3 Bijection between parallelogram polyominoes and binary trees

The complete non-ambiguous trees

A complete non-ambiguous tree is a non-ambiguous tree whose vertices have only 0 or 2 children.



size = nb of internal vertices
= nb of left edges
= nb of right edges
= 6

The number of non-ambiguous trees

The number b_n of complete non-ambiguous trees of size n is referenced on OEIS:

$$A136127 = (b_n)_{n \geq 0} = (1, 1, 4, 33, 456, 9460, \dots)$$

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$$-\ln(J_0(x)) = \sum_{k \geq 0} b_k \frac{x^{2(k+1)}}{((k+1)!2^{k+1})^2}$$

where J_0 is the Bessel function of order 0.

[Carlitz, 1963]

Proof

The Bessel function of order 0 is the solution of the differential equation:

$$\frac{d^2y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + y = 0,$$

such that the first coefficients in its series expansion are $j_0 = 1$ and $j_1 = 0$.

We try

$$B(x) = \exp \left(- \sum_{k \geq 0} b_k \frac{x^{2(k+1)}}{((k+1)!2^{k+1})^2} \right).$$

Proof

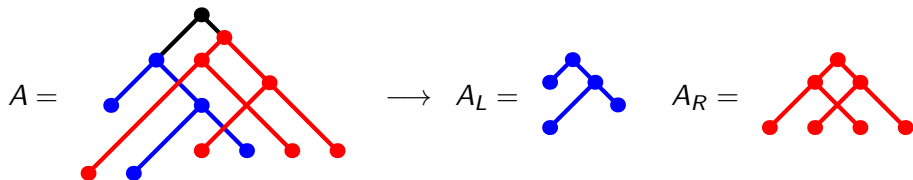
The function

$$B(x) = \exp \left(- \sum_{k \geq 0} b_k \frac{x^{2(k+1)}}{((k+1)!2^{k+1})^2} \right)$$

satisfies the Bessel differential equation if and only if

$$b_{n+1} = \sum_{u+v=n} \binom{n+1}{u} \binom{n+1}{v} b_u b_v.$$

The root suppression gives a recursive enumeration



$$b_{n+1} = \sum_{u+v=n} \binom{n+1}{u} \binom{n+1}{v} b_u b_v$$

The coefficients b_k are the number of complete non-ambiguous trees of size k .

Another identity involving complete non-ambiguous trees

$$\sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} \binom{n-1}{k} b_k = 1$$

[Carlitz, 1963]

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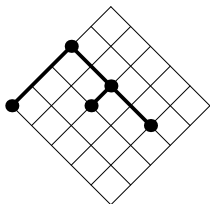
Combinatorial proof:

Another identity involving complete non-ambiguous trees

$$\sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} \binom{n-1}{k} b_k = 1$$

[Carlitz, 1963]

Combinatorial proof:



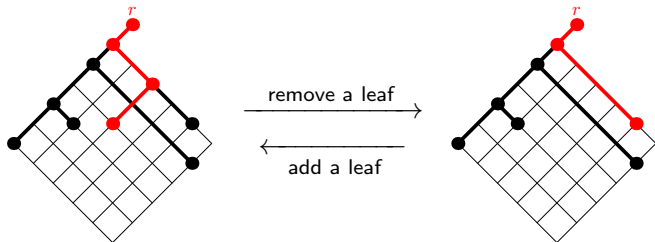
A gridded tree

Another identity involving complete non-ambiguous trees

$$\sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} \binom{n-1}{k} b_k = 1$$

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Combinatorial proof:

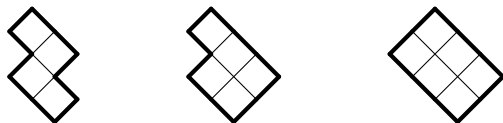


Outline

- 1 Enumeration of non-ambiguous trees with a fixed underlying tree
- 2 The complete non-ambiguous trees and the Bessel function
- 3 **Bijection between prallelogram polyominoes and binary trees**
 - Definition of a prallelogram polyomino
 - Parallelogram polyominoes are the shapes of particular non-ambiguous trees

The parallelogram polyominoes

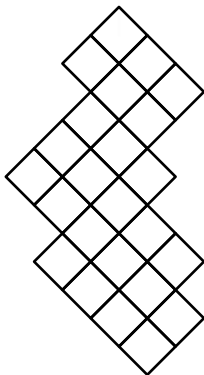
A parallelogram polyomino of size n is a pair of lattice paths of lengths $n + 1$ with south-west and south-east steps starting at the same point, ending at the same point and never meeting each other.



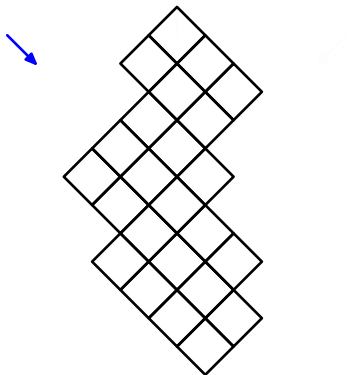
Parallelogram polyominoes and binary trees are in bijection.

[Delest and Viennot, 1984]

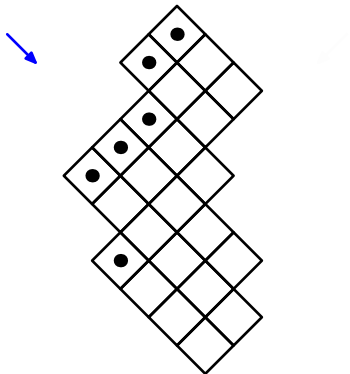
Bijection between prallelogram polyominoes and binary trees



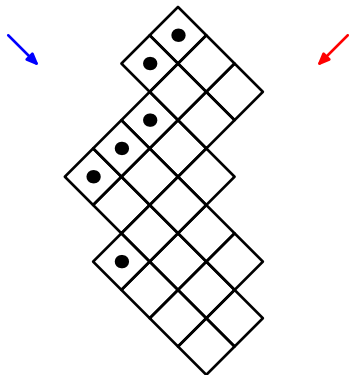
Bijection between prallelogram polyominoes and binary trees



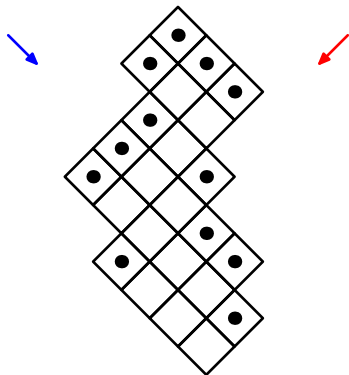
Bijection between prallelogram polyominoes and binary trees



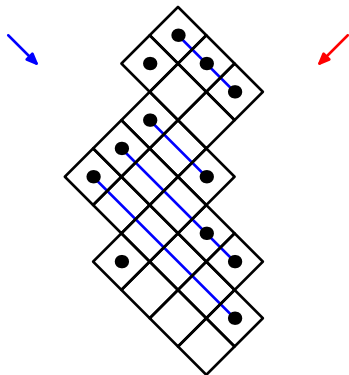
Bijection between prallelogram polyominoes and binary trees



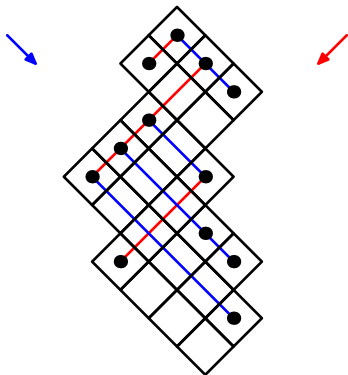
Bijection between prallelogram polyominoes and binary trees



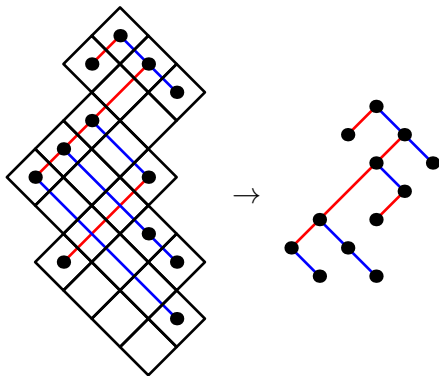
Bijection between prallelogram polyominoes and binary trees



Bijection between prallelogram polyominoes and binary trees



Bijection between parallelogram polyominoes and binary trees



In the future ...

We can try to

- find a combinatorial interpretation of the Bessel functions for any order;
- study the link between tree-like tableau background and the non-ambiguous trees.
- define some Hopf Algebras for non-ambiguous trees.
- generalize in other dimensions (in progress ...)
(Patxi Laborde-Zubieta in October 2015 - Trois-Rivières ?)

Thank you for your attention!