Pablo Arias

work during my Ph.D. thesis at University Pompeu Fabra with

Vicent Caselles

Journées bordelaises d'analyse mathématique des images IMB & LaBRI, Univ. Bordeaux April 6, 2014

Outline

1 Variational models for exemplar-based image inpainting

2 Convergence rate of PatchMatch, an algorithm for finding patch correspondences

Joint work with

- Vicent Caselles
- Gabriele Facciolo (ENS Cachan)
- Guillermo Sapiro (Duke University)

Exemplar-based image inpainting



Inpainting: problem statement and notation



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Definitions:

image domain: Ω inpainting domain: Oknown data: $O^c = \Omega \setminus O$ image: $\hat{u} : O^c \to \mathbb{R}$

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Objective:

visually plausible completion u

Growing a texture by copying from a sample [Efros, Leung 99], [Wei, Levoy 00].



Figure: From [Efros, Leung 99] paper

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[Demanet, Song, Chan'03] Computation of a correspondence map: $\varphi: O \rightarrow O^c$

$$E(\varphi) = \int_{O} \|p_u(x) - p_{\hat{u}}(\varphi(x))\|^2 dx$$

where $u(x) = \hat{u}(\varphi(x))$



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Hard to minimize:

- Iteration of [Efros,Leung 99] scheme
- Analysis for piece-wise roto-translations correspondences [Aujol, Ladjal, Masnou 08]



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[Demanet, Song, Chan'03] Computation of a correspondence map: $\varphi: O \rightarrow O^c$

$$E(u,\varphi) = \int_O \|p_u(x) - p_{\hat{u}}(\varphi(x))\|^2 \mathrm{d}x$$

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- Analysis for piece-wise roto-translations correspondences [Aujol, Ladjal, Masnou 08]
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- Image as additional variable [Wexler et al. 07][Peyré et al. 08][Kawai et al. 09][us too]



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Exemplar-based image inpainting



Variational models for non-local denoising

NL-means formula [Buades et al.'05;Awate Whitaker'05]

$$u(x) = \int_{\Omega} w(x, y) \hat{u}(y) \mathrm{d}y$$

Similarity weights:

$$w(x,y) \propto \exp\left(-\frac{1}{T} \|p_{\hat{u}}(x) - p_{\hat{u}}(y)\|^2\right) \quad T > 0.$$

• y

w(x,y)

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Corresponds to the minimization of [Gilboa et al.'06]

$$E_w(u) = \int_{\Omega} \int_{\Omega} w(x, y) (u(x) - \hat{u}(y))^2 \mathrm{d}y \mathrm{d}x$$

• y

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Variational models for self-similarity in denoising

NL-means formula [Buades et al.'05;Awate Whitaker'05]

$$u(x) = \int_{\Omega} w(x, y) \hat{u}(y) \mathrm{d}y$$

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Corresponds to the minimization of [Gilboa et al.'06]

$$E_w(u) = \int_O \int_{O^c} w(x,y)(u(x) - \hat{u}(y))^2 \mathrm{d}y \mathrm{d}x$$

How to compute the weights for inpainting?

• y

w(x, y)

Proposed energy for exemplar-based inpainting

Consider $w: \widetilde{O} \times \widetilde{O}^c \to \mathbb{R}^+$ as a variable,

$$\mathcal{E}_{T}(u,w) = \underbrace{\int_{\widetilde{O}} \int_{\widetilde{O}^{c}} w(x,y) \|p_{u}(x) - p_{\hat{u}}(y)\|^{2} \mathrm{d}y \mathrm{d}x}_{\text{image energy}}$$
subject to
$$\int_{\widetilde{O}^{c}} w(x,y) \mathrm{d}y = 1.$$

Proposed energy for exemplar-based inpainting

Consider $w: \widetilde{O} \times \widetilde{O}^c \to \mathbb{R}^+$ as a variable, T > 0:

$$\mathcal{E}_{T}(u,w) = \underbrace{\int_{\widetilde{O}} \int_{\widetilde{O}^{c}} w(x,y) \|p_{u}(x) - p_{\hat{u}}(y)\|^{2} \mathrm{d}y \mathrm{d}x}_{\text{image energy}} - T\mathcal{H}(w)$$
subject to $\int_{\widetilde{O}^{c}} w(x,y) \mathrm{d}y = 1.$

Entropy of
$$w$$
: $\mathcal{H}(w) = -\int_{\widetilde{O}} \int_{\widetilde{O}^c} w(x, y) \log w(x, y) dy$
$$\frac{\partial \mathcal{E}_T}{\partial w} = 0 \quad \Longleftrightarrow \quad w(x, y) = \frac{1}{q(x)} \exp\left(-\frac{1}{T} \|p_u(x) - p_{\hat{u}}(y)\|^2\right)$$

Alternating minimization

Initialization: choose u^0 (e.g. average of u on \widetilde{O}^c).

For each $k \in \mathbb{N}$ solve

$$w^{k} = \arg\min_{w} \mathcal{E}_{T}(u^{k}, w)$$
$$u^{k+1} = \arg\min_{u} \mathcal{E}_{T}(u, w^{k})$$

Alternating minimization

Initialization: choose u^0 (e.g. average of u on \widetilde{O}^c).

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$$u^{k+1} = \underset{u}{\operatorname{arg\,min}} \mathcal{E}_{T}(u, w^{k}) \leftarrow ?$$

Image update step

Minimizing $\mathcal{E}_T(u, w)$ with respect to u yields:

$$u(x) = \int_{\Omega_p} g(h) \int_{\widetilde{O}^c} w(x-h,y)\hat{u}(y+h) \mathrm{d}y \mathrm{d}h$$



Image update step

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Minimizing $\mathcal{E}_T(u, w)$ with respect to u yields:

$$\begin{split} u(x) &= \int_{\Omega_p} g(h) \int_{\widetilde{O}^c} w(x-h,y) \hat{u}(y+h) \mathrm{d}y \mathrm{d}h \\ &= \int_{O^c} m(x,y) \hat{u}(y) \mathrm{d}y \end{split}$$

where $m(x, y) = w \star g(x, y)$ averages similarity between patches around x and around y.

Additional averaging of patches overlaping x. Adds blur, but also stabilizes the algorithm



Limit when $T \rightarrow 0$: the correspondence case

As $T \rightarrow 0$, the energy \mathcal{E}_T Gamma converges to

$$\mathcal{E}_0(u,w) := \int_{\widetilde{O}} \int_{\widetilde{O}^c} \|p_u(x) - p_{\hat{u}}(y)\|_2^2 \mathrm{d}w_x(y),$$

where $x \rightarrow w_x$ is a measurable probability-valued map.

• x $\dot{\varphi(x_1)}$ $\varphi(x_2)$ • x'

 $\cdot x_2$ • x· x_1

Limit when $T \rightarrow 0$: the correspondence case

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where $x \rightarrow w_x$ is a measurable probability-valued map.

Proposition: There exist optima (u^*, w^*) of \mathcal{E}_0 with

$$w_x^* = \delta_{\varphi(x)},$$

where $\varphi: \widetilde{O} \to \widetilde{O}^c$ is a measurable map, such that

$$\varphi(x) \in \underset{y \in \widetilde{O}^c}{\arg\min} \|p_u(x) - p_{\hat{u}}(y)\|_2^2.$$

 $\varphi(x_2)$

 $\cdot x_2$ • x

• x

• x $\dot{\varphi(x_1)}$

Image update step with T = 0

For the case of patch NL-means:

$$u(x) = \int_{\Omega_p} g(h)\hat{u}(\varphi(x-h) + h)\mathrm{d}h$$



Image update step with T = 0

For the case of patch NL-means:

$$u(x) = \int_{\Omega_p} g(h)\hat{u}(\varphi(x-h) + h)\mathrm{d}h$$

When φ is a translation, *i.e.* $\varphi(x) = x + t$:

$$u(x) = \hat{u}(\varphi(x))$$

 $\Rightarrow \mathsf{rigid} \ \mathsf{copy} \Leftarrow$





Figure: Initialization

Evolution of iterates for T = 0. The red curves show the boundaries between copy regions.



Figure: Iteration # 2

Evolution of iterates for T = 0.



Figure: Iteration # 4

Evolution of iterates for T = 0.



Figure: Iteration # 6

Evolution of iterates for T = 0.



Figure: Iteration # 8

Evolution of iterates for T = 0.



Figure: Iteration # 10

Evolution of iterates for T = 0. The red curves show the boundaries between copy regions.


Figure: Iteration # 14



Figure: Iteration # 18



Figure: Iteration # 22



Figure: Iteration # 26



Figure: Iteration # 32



Figure: Iteration # 38



Figure: Iteration # 44



Figure: Convergence

Correspondence case: piece-wise translations

 $\varphi(x) = x + t(x)$, with t piece-wise constant.



In the transition band: blending by averaging

$$u(z) = \alpha_1(z)\hat{u}(z+t_1) + \alpha_2(z)\hat{u}(z+t_2) \quad \text{with } \alpha_i(z) = \int_{\Omega_p} g(y)\chi_{R_i}(z-y)\mathrm{d}y.$$

Copy regions: maximal sets where φ is a translation.

T = 0: existence of mildly regular optimal correspondences

Proposition: There exist optima (u^*, w^*) of \mathcal{E}_0 with

$$w_x^* = \delta_{\varphi(x)},$$

where $\varphi:\widetilde{O}\rightarrow\widetilde{O}^c$ is a measurable map, such that

$$\varphi(x) \in \arg\min_{y \in \widetilde{O}^c} U(x, y),$$

and such that φ which are uniform limit of BV maps.

$$\mathcal{E}_T(u,w) = \int_{\widetilde{O}} \int_{\widetilde{O}^c} w(x,y) U(x,y) \mathrm{d}y \mathrm{d}x - T\mathcal{H}(w)$$

Patch NL-means

$$U(x,y) = \|p_u(x) - p_{\hat{u}}(y)\|_2^2$$

$$u(x) = \int_{O^c} m(x, y) u(y) \mathrm{d}y$$

$$\mathcal{E}_T(u,w) = \int_{\widetilde{O}} \int_{\widetilde{O}^c} w(x,y) U(x,y) \mathrm{d}y \mathrm{d}x - T\mathcal{H}(w)$$

Patch NL-medians

$$U(x,y) = \|p_u(x) - p_{\hat{u}}(y)\|_1$$

$$u(x) = \text{median} \left\{ \left[u(y); m(y) \right], y \in O^c \right\}$$

$$\mathcal{E}_T(u,w) = \int_{\widetilde{O}} \int_{\widetilde{O}^c} w(x,y) U(x,y) \mathrm{d}y \mathrm{d}x - T\mathcal{H}(w)$$

Patch NL-Poisson

$$\begin{split} U(x,y) &= (1-\lambda) \| p_{\nabla u}(x) - p_{\nabla \hat{u}}(y) \|_2^2 + \lambda \| p_u(x) - p_{\hat{u}}(y) \|_2^2 \\ \text{With } \lambda &= 0 :\implies \begin{cases} \Delta u(x) = \nabla \cdot \boldsymbol{v}(x), & x \in O \\ \\ u(x) = \hat{u}(x), & x \in \partial O \end{cases} \\ \text{Guiding vector field: } \boldsymbol{v}(x) &= \int_{O^c} m(x,y) \nabla \hat{u}(y) \mathrm{d}y \end{split}$$

$$\mathcal{E}_T(u,w) = \int_{\widetilde{O}} \int_{\widetilde{O}^c} w(x,y) U(x,y) \mathrm{d}y \mathrm{d}x - T\mathcal{H}(w)$$

Patch NL-Gradient Medians

$$U(x,y) = (1-\lambda) \|p_{\nabla u}(x) - p_{\nabla \hat{u}}(y)\|_1 + \lambda \|p_u(x) - p_{\hat{u}}(y)\|_2^2$$

Convex optimization problem. We solve it by adapting Chambolle's dual minimization of the $TV - \mathcal{L}^2$.

Comparison: four inpainting schemes

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inpainting problem



Patch NL-means

Patch NL-medians

Patch NL-Poisson

Patch NL-GM



Comparison: four inpainting schemes



inpainting problem

initialization



Profile for a horizontal line through the center of the image, T > 0.

T > 0 performs denoising.

Comparison: four inpainting schemes



inpainting problem

initialization



Profile for a horizontal line through the center of the image, T = 0.

The relevant case for inpainting is T = 0.

Exemplar-based image inpainting





Image and inpainting domain

Pab	lo Aria	as (U	IPE)













Results with non-variational scheme



Departure from variational model:

- 1. Synthesize image using $\lambda_u << 1$.
- 2. Compute weights using λ_w . Parameter to be set for each image.

Greater flexibility, but it is not variational.

Questions

- Further analysis on the appearance of copy regions (regularity of φ).
- Can we impose regularity to the correspondence maps [Aujol, Ladjal, Masnou 08] ?
- Ability of exemplar-based methods to interpolate geometry [Aujol, Ladjal, Masnou 08] .
- A better modelling should include the multiscale scheme.
- Interpretation of the energy in the space of patches.

Convergence of PatchMatch
Efficient estimation of the nearest neighbor field $\theta: \widetilde{O} \to \widetilde{O}^c$.

$$\theta_x = \underset{\theta_x \in \widetilde{O}^c - x}{\arg\min} U_x(\theta_x) = \|p_u(x) - p_{\hat{u}}(x + \theta_x)\|^2$$

- 1. for k = 0 to kmax,
- 2. random_search
- 3. if k is even, forward_propagation
- if k is odd, backward_propagation

5. end

C. Barnes, E. Shechtman, A. Finkelstein, D.B. Goldman [SIGGRAPH'09]











Forward propagation



Forward propagation



Forward propagation



After propagation: $U_x(\theta_x) > \epsilon \Rightarrow U_y(\theta_y) > \epsilon - d_{y,x}$



$$d_{y,x} = \|U_y - U_x\|_{\infty}$$

 $c \in \mathcal{P}(z, x)$: path from z to x

$$d_{z,x} = \sum_{i=2}^{n_c} d_{c_{i-1},c_i}$$



$$d_{y,x} = \|U_y - U_x\|_{\infty}$$

 $c \in \mathcal{P}(z, x)$: path from z to x

Graph geodesic distance
$$\ell_{z,x} = \min_{c \in \mathcal{P}(z,x)} \sum_{i=2}^{n_c} d_{c_{i-1},c_i}$$



$$d_{y,x} = \|U_y - U_x\|_{\infty}$$

 $c \in \mathcal{P}(z, x)$: path from z to x

Graph geodesic distance $\ell_{z,x} = \min_{c \in \mathcal{P}(z,x)} \sum_{i=2}^{n_c} d_{c_{i-1},c_i}$

After propagation: $U_x(\theta_x) > \epsilon \Rightarrow U_z(\theta_z) > \epsilon - \ell_{z,x}$

Convergence of the PatchMatch Algorithm

Assume: For each (y, x), $d_{y,x} = ||U_y - U_x||_{\infty} < \infty$.

Then:
$$P(U_x(\theta_x^{n+1}) > \epsilon) \leq \prod_{z \in \mathcal{A}(x)} C(z, \epsilon - \ell_{z,x}) P(U_x(\theta_x^n) > \epsilon),$$

where

$$C(z,\beta) = \sup_{\eta \in \{U_z > \beta\}} P(U_z(\theta_z^{n+1/2}) > \beta \mid \theta_z^n = \eta)$$

is a non-increasing function of β .

Merci Beaucoup

An example of reconstruction of a sparsely sampled image

Obtained with a cooling scheme $T \rightarrow 0$ (Deterministic Annealing [Rose 98]). Sampling rate: 5%



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Convergence of the PatchMatch Algorithm



Collaborative searches in parallel: smaller ℓ s means more collaboration

If $\epsilon - \ell_{z,x} < 0$, $C(z, \epsilon - \ell_{z,x}) = 1$: no contribution to convergence rate

Merci Beaucoup