Inductive properties

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We have already seen how to define new datatypes by the mean of inductive types.
During this session, we shall present how *Coq*'s type system allows us to define specifications using inductive declarations.

```
Inductive even : nat -> Prop :=
|  even0 : even 0
|  evenS : forall p:nat, even p -> even (S (S p)).
```

(demo)

```
Require Import List.
Set Implicit Arguments.
Inductive is_repetition(A:Type) : list A -> Prop :=
| is_rep_nil : is_repetition nil
| is_rep_single : forall a, is_repetition (a::nil)
| is_rep_cons: forall a l, is_repetition (a::l) ->
                           is_repetition (a::a::l).
```

(demo)

Inductive predicates

Let us consider again our little programming language. We can define the predicate "the variable *v* appears in the expression *e*" :

```
Inductive Occurs  (v:toy_Var): toy_Exp -> Prop :=
|Occ_var : Occurs v (variable v)
|Occ_op1 : forall op e1 e2, Occurs v e1 ->
                            Occurs v (toy_op op e1 e2)
|Occ_op2 : forall op e1 e2, Occurs v e2 ->
                            Occurs v (toy_op op e1 e2).
```

Constructors are displayed in red.

Likewise, "The variable v may be modified by an execution of the statement s".

```
Inductive Assigned_in (v:toy_Var): toy_Statement->Prop :=
| Assigned_assign : forall e, Assigned_in v (assign v e)
| Assigned_seq1 : forall s1 s2,
                    Assigned_in v s1 ->
                    Assigned_in v (sequence s1 s2)
| Assigned_seq2 : forall s1 s2,
                    Assigned_in v s2 ->
                    Assigned_in v (sequence s1 s2)
| Assigned_loop : forall e s,
                    Assigned_in v s ->
                    Assigned_in v (simple_loop e s).
```

For proving that some given variable is assigned in some given statement, just apply (a finite number of times) the constructors.

```
Lemma Y_assigned : Assigned_in  Y factorial_Z_program.
Proof.
 unfold factorial_Z_program.
 constructor 3 (* apply Assigned_seq2 *).
 constructor 2 (* apply Assigned_seq1 *) .
 constructor 1 (* apply Assigned_assign *).
Qed.
```

```
(* Using hints and auto *)
Hint Constructors Assigned_in.

Lemma X_assigned : Assigned_in X factorial_Z_program.
Proof.
 unfold factorial_Z_program;auto.
Qed.
```

```
(* Using hints and auto *)
Hint Constructors Assigned_in.

Lemma X_assigned : Assigned_in X factorial_Z_program.
Proof.
 unfold factorial_Z_program;auto.
Qed.

Lemma Z_unassigned : ~(Assigned_in Z factorial_Z_program).
intro H.
```

1 subgoal

H : Assigned_in Z factorial_Z_program

==========

False

```
(* Using hints and auto *)
Hint Constructors Assigned_in.

Lemma X_assigned : Assigned_in X factorial_Z_program.
Proof.
 unfold factorial_Z_program;auto.
Qed.

Lemma Z_unassigned : ~(Assigned_in Z factorial_Z_program).
intro H.
1 subgoal
  H : Assigned_in Z factorial_Z_program
  ============================
   False
(* ??????????? *)
Abort.
```

Note :

OK, we are going too fast !

So . . .

Let us consider some simpler examples, learn more about inductive predicates, and the proof of the previous lemma will be a piece of cake.

A relation already used in previous lectures

The $\leq$ relation on `nat` is defined by the means of an inductive predicate :

```
Inductive le (n : nat) : nat -> Prop :=
    | le_n : le n n
    | le_S : forall m : nat, le n m -> le n (S m)
```

The proposition `(le n m)` is denoted by `n <= m`.
`n` is called a *parameter* of the previous definition.
(demos)

Reasoning with inductive predicates

Use constructors as introduction rules.

```
Lemma le_n_plus_pn : forall n p: nat, n <= p + n.
Proof.
 induction p;simpl.
2 subgoals

  n : nat
  ============================
   n <= n

subgoal 2 is:
 n <= S (p + n)
 constructor 1.
```

1 subgoal

n : nat
p : nat
IHp : n <= p + n
==========
n <= S (p + n)

```
constructor 2;assumption.
Qed.
```

The induction principle for le

```
le_ind
    : forall (n : nat) (P : nat -> Prop),
      P n ->
      (forall m : nat, n <= m -> P m -> P (S m)) ->
      forall p : nat, n <= p -> P p
```

In order to prove that for every $p \geq n$, $P\ p$, prove :

- $P\ n$
- for any $m \geq n$, if $P\ m$ holds, then $P\ (\mathtt{S}\ m)$ holds.

Use induction or destruct as elimination tactics.

```
Lemma le_plus : forall n m, n <= m ->
                exists p:nat, p+n = m
                (* P m *).
Proof.
 intros n m H.
```

1 subgoal

n : nat
m : nat
H : n <= m
==========
exists p : nat, p + n = m

induction H.

```
2 subgoals

n : nat
============================
 exists p : nat, p + n = n
 (* P n *)

subgoal 2 is:
 exists p : nat, p + n = S m
 exists 0;trivial.
```

1 subgoal

```
  n : nat
  m : nat
  H : n <= m
  IHle : exists p : nat, p + n = m  (* P m *)
  ============================
   exists p : nat, p + n = S m   (* P (S m) *)
```

```
 destruct IHle as [q Hq]; exists (S q);
   simpl;rewrite Hq;trivial.
Qed.
```

The `inversion` tactic

How to prove that :

```
Lemma foo : ~(1 <= 0).
```

The `inversion` tactic

How to prove that :

```
Lemma foo : ~(1 <= 0).
Proof.
intro h.
inversion h.
Qed.
```

The `inversion` tactic derives all the necessary conditions to an inductive hypothesis. If no condition can realize this hypothesis, the goal is proved by *ex falso quod libet*.

```
Lemma le_n_0 : forall n, n <= 0 -> n = 0.
Proof.
 intros n H;inversion H.
 1 subgoal

  n : nat
  H : n <= 0
  H0 : n = 0
  ============================
   0 = 0
trivial.
Qed.
```

Some other examples

```
Lemma Assigned_inv1 : forall v w e,
   Assigned_in v (assign w e) ->
   v=w.
Proof.
 intros v w e H; inversion H. ...

Lemma Assigned_inv2 : forall v s1 s2,
   Assigned_in v (sequence s1 s2) ->
   Assigned_in v s1 \/ Assigned_in v s2.
Proo.
intros v s1 s2 H; inversion H. ...
```

Some other examples

```
Lemma Assigned_inv1 : forall v w e,
   Assigned_in v (assign w e) ->
   v=w.
Proof.
 intros v w e H; inversion H. ...

Lemma Assigned_inv2 : forall v s1 s2,
   Assigned_in v (sequence s1 s2) ->
   Assigned_in v s1 \/ Assigned_in v s2.
Proo.
intros v s1 s2 H; inversion H. ...
```

Are we now able to prove our “piece of cake” lemma ?

Some other examples

```
Lemma Assigned_inv1 : forall v w e,
   Assigned_in v (assign w e) ->
   v=w.
Proof.
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   Assigned_in v (sequence s1 s2) ->
   Assigned_in v s1 \/ Assigned_in v s2.
Proo.
intros v s1 s2 H; inversion H. ...
```

Are we now able to prove our “piece of cake” lemma ? look at the demo !

An interesting technique : Use the type `bool` !

We first define a boolean function for testing equality on variables :

```
Require Import Bool.
Definition var_eqb (v w : toy_Var) :=
match v,w with  X, X => true
              | Y, Y => true
              | Z, Z => true
              | _, _ => false
end.
```

We define a boolean test for the “assigned” property :

```
Fixpoint assigned_inb (v:toy_Var)(s:toy_Statement) :=
 match s with
             | assign w _ => var_eqb v w
             | sequence s1 s2 =>
                 assigned_inb v s1 || assigned_inb v s2
             | simple_loop e s => assigned_inb v s
 end.
```

Bridge lemmas

```
Lemma Assigned_In_OK : forall v s,
  Assigned_in v s ->
  assigned_inb v s = true.
Proof.
 intros v s H;induction H;simpl;...

Lemma Assigned_In_OK_R :
forall v s, assigned_inb v s = true ->
            Assigned_in v s.
Proof.
 induction s;simpl.
  ...
```

```
Lemma Z_unassigned : ~(Assigned_in Z factorial_Z_program).
Proof.
intro H;assert(H0 :=  Assigned_In_OK  _ _ H).
```

1 subgoal

H : Assigned_in Z factorial_Z_program
H0 : assigned_inb Z factorial_Z_program = true
==========
False

```
simpl in H0;discriminate H0.
Qed.
```

Demos and exercises on $\leq$

Let us consider again two aspects of $\leq$:

```
Inductive le (n : nat) : nat -> Prop :=
   | le_n : n <= n
   | le_S : forall m : nat, le n m -> le n (S m)

The term  (le n m) is denoted by  n <= m.

Fixpoint leb n m : bool :=
   match n, m with
   |0, _ => true
   |S i, S j => leb i j
   | _, _ => false
end.
```

```
Eval compute in leb 5 45.
 = true : bool

Lemma L5_45 : 5 <= 45.
Proof.
 repeat constructor.
Qed.
```

```
Eval compute in leb 5 45.
 = true : bool

Lemma L5_45 : 5 <= 45.
Proof.
 repeat constructor.
Qed.
```

Just try Print L5_45. !

```
Lemma le_trans :
  forall n p q, n <= p -> p <= q -> n <= q.
Proof.
```

```
Lemma le_trans :
  forall n p q, n <= p -> p <= q -> n <= q.
Proof.
```

We recognize the scheme :

```
  p <= q -> P q where  P q is n <= q.
```

Thus, the base case is `n <= p` and the inductive step is

```
 forall q, p <= q -> n <= q  -> n <= S q.
```

```
  intros n p q H H0;induction H0.
```

2 subgoals

n : nat
p : nat
H : n <= p
==========================
n <= p ...

```
assumption.
```

1 subgoal

n : nat
p : nat
H : n <= p
m : nat
H0 : p <= m
IHle : n <= m
==========
n <= S m

```
constructor;assumption.
Qed.
```

The tactic constructor tries to make the goal progress by applying a constructor. Constructors are tried in the order of the inductive type definition.

```
Lemma le_Sn_Sp_inv:  forall n p,  S n <= S p ->  n <= p.
intros n p H;inversion H.
```

2 subgoals

n : nat
p : nat
H : S n <= S p
H1 : n = p
============================
p <= p ...

```
constructor.
```

1 subgoal

n : nat
p : nat
H : S n <= S p
m : nat
H1 : S n <= p
H0 : m = p
==========
n <= p

```
apply le_trans with (S n);
  [repeat constructor|assumption].
```

le or leb ?

We can build a bridge between both aspects by proving the following theorems :

```
Lemma le_leb_iff : forall n p, n <= p <-> leb n p=true.

Lemma lt_leb_iff : forall n p, n < p <-> leb p n = false.
(* Proofs left as exercise *)
```

```
Lemma L: 0 <= 47.
Proof.
 rewrite  le_leb_iff.
1 subgoal

  ============================
   leb 0 47 = true
trivial.
Qed.
```

```
Lemma leb_Sn_n : forall n p, leb  n (n + p)= true.
Proof.
 intros n p;rewrite <- le_leb_iff.
1 subgoal

  n : nat
  p : nat
  ============================
   n <= n + p
 SearchPattern (_ <= _ + _).
 apply le_plus_l;auto.
Qed.
```

Inductive definitions and functions

It is sometimes very difficult to represent a function `f : A -> B` as a *Coq* function, for instance because of the :

- Undecidability (or hard proof) of termination
- Undecidability of the domain characterization

This situation often arises when studying the semantic of programming languages.

In that case, describing functions as inductive relations is really efficient.

```
Definition odd n := ~ even n.

Inductive syracuse_steps : nat -> nat -> Prop :=
 done : syracuse_steps  1 1
|even_case : forall n p,even n ->
    syracuse_steps (div2 n) p ->
    syracuse_steps  n (S p)
|odd_case : forall n p , odd n ->
            syracuse_steps (S(n+n+n))  p ->
            syracuse_steps n (S p).
```

Exercise

Prove the proposition syracuse_steps 5 6. Try to improve the previous definitions !

What you think is not what you get

An odd alternative definition of le :

```
Inductive alter_le (n : nat) : nat -> Prop :=
| alter_le_n : alter_le n n
| alter_le_S : forall m : nat, alter_le n m ->
                               alter_le n (S m)
| alter_dummy : alter_le n (S n).
```

What you think is not what you get

An odd alternative definition of le :

```
Inductive alter_le (n : nat) : nat -> Prop :=
| alter_le_n : alter_le n n
| alter_le_S : forall m : nat, alter_le n m ->
                               alter_le n (S m)
| alter_dummy : alter_le n (S n).
```

The third constructor is useless ! It may increase the size of the proofs by induction.

A more abstract example

```
Section transitive_closures.
 Definition relation (A : Type) := A -> A -> Prop.
 Variables (A : Type)(R : relation A).

(* the transitive closure of R is the least
relation ... *)
Inductive clos_trans : relation A :=
  (* ... that contains R *)
  | t_step : forall x y : A, R x y -> clos_trans x y
  (* ... and is transitive *)
  | t_trans : forall x y z : A,
    clos_trans x y -> clos_trans y z
                   -> clos_trans x z.
```

If some relation R is transitive, then its transitive closure in included in R :

```
Hypothesis Rtrans :
   forall x y z, R x y -> R y z -> R x z.

Lemma trans_clos_trans :  forall a1 a2,
                           clos_trans a1 a2 -> R a1 a2.
Proof.
intros a1 a2 H; induction H.
```

2 subgoals

x : A
y : A
H : R x y
==========
R x y ...

```
exact H.
```

```
  x : A
  y : A
  z : A
  H : clos_trans x y
  H0 : clos_trans y z
  IHclos_trans1 : R x y
  IHclos_trans2 : R y z
  ============================
   R x z
apply Rtrans with y;  assumption.
Qed.
```

```
End transitive_closures.
Check trans_clos_trans.
 trans_clos_trans
     : forall (A : Type) (R : relation A),
       (forall x y z : A, R x y -> R y z -> R x z) ->
       forall a1 a2 : A, clos_trans A R a1 a2 -> R a1 a2
```

```
End transitive_closures.
Check trans_clos_trans.
 trans_clos_trans
     : forall (A : Type) (R : relation A),
       (forall x y z : A, R x y -> R y z -> R x z) ->
       forall a1 a2 : A, clos_trans A R a1 a2 -> R a1 a2


Implicit Arguments clos_trans [A].
Implicit Arguments trans_clos_trans [A].
Check (trans_clos_trans le le_trans).
trans_clos_trans nat le le_trans
     : forall a1 a2 : nat, clos_trans le a1 a2 -> a1 <= a2
```

Advice for crafting useful inductive definitions

- Constructors are “axioms” : they should be intuitively true...
- Constructors should as often as possible deal with mutually exclusive cases, to ease proofs by induction ;
- When an argument always appears with the same value, make it a parameter
- Test your predicate on negative and positive cases !

Logical connectives as inductive definitions

Most logical connectives are defined using inductive types :

- Conjunction /\
- Disjunction \/
- Existential quantification $\exists$
- Equality
- Truth and False

Notable exceptions : implication, negation.

Let us revisit the 3th and 4th lectures.

Logical connectives : conjunction

Conjunction is a pair :

```
Inductive and (A B : Prop) := conj : A -> B -> and A B.
```

- Term `(and A B)` is denoted `(A /\ B)`.
- Prove a conjunction goal with the `split` tactic (generates two subgoals).
- Use a conjunction hypothesis with the `destruct as [...]` tactic.

Logical connectives : disjunction

Disjunction is a two constructors inductive :

```
Inductive or (A B : Prop) : Prop :=
|or_introl : A -> or A B | or_intror : B -> or A B.
```

- Term `(or A B)` is denoted `(A \/ B)`.
- Prove a disjunction with the `left`, `right` tactics (choose the side to prove).
- Use a conjunction hypothesis with the `case` or `destruct as [...|...]` tactics.

Logical connectives : existential quantification

Existential quantification is a pair :

```
Inductive ex (A : Type) (P : A -> Prop) : Prop :=
    ex_intro : forall x : A, P x -> ex P.
```

- The term ex A (fun x => P x) is denoted exists x, P x.
- Prove an existential goal with the `exists` tactic.
- Use an existential hypothesis with the `destruct as [...]` tactic.

Equality

The built-in (predefined) equality relation in *Coq* is a parametric inductive type :

```
Inductive eq (A : Type) (x : A) : A -> Prop :=
   refl_equal : eq A x x.
```

- Term `eq A x y` is denoted $(x = y)$
- The induction principle is :

```
eq_ind : forall (A : Type) (x : A) (P : A -> Prop),
P x -> forall y : A, x = y -> P y
```

Equality

- Use an equality hypothesis with the `rewrite [<-]` tactic (uses `eq_ind`)
- Remember equality is computation compliant !

```
Goal 2 + 2 = 4. apply refl_equal. Qed.
```

  Because + is a program.
- Prove trivial equalities (modulo computation) using the `reflexivity` tactic.

Truth

The "truth" is a proposition that can be proved under any assumption, in any context. Hence it should not require any argument or parameter.

```
Inductive True : Prop := I : True.
```

Its induction principle is :

```
True_ind : forall P : Prop, P -> True -> P
```

which is not of much help...

Falsehood

Falsehood should be a proposition of which no proof can be built (in empty context).
In *Coq*, this is encoded by an inductive type with no constructor :

```
Inductive False : Prop :=
```

coming with the induction principle :

```
False_ind : forall P : Prop, False -> P
```

often referred to as *ex falso quod libet*.

- To prove a False goal, often apply a negation hypothesis.
- To use a H : False hypothesis, use destruct H.

Specifying programs with inductive predicates

Programs are computational objects.
Inductive types provide structured specifications.
How to get the best of both worlds ?

Specifying programs with inductive predicates

Programs are computational objects.
Inductive types provide structured specifications.
How to get the best of both worlds ?
By combining programs with inductive specifications.