Kempe recoloring version of Hadwiger’s conjecture

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Joint work with Marthe Bonamy, Marc Heinrich and Jonathan Narboni
Recoloring with Kempe changes

Kempe chain (1879)

Maximal bichromatic connected component in $G$
Recoloring with Kempe changes

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Usual questions

• Are any two $k$-colorings of a graph $G$ equivalent?
• Are all $k$-colorings equivalent to a $\chi(G)$-coloring?
• How many Kempe changes separate any two $k$-colorings?
• Application to sampling: Does the corresponding Markov chain mix well?
Recoloring with Kempe changes

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Graph minor

$H$ is a minor of $G$ if $H$ can be obtained by deleting vertices, edges and contracting edges of $G$

$K_t$ is a minor of $G$ if and only if $V_1 \sqcup \cdots \sqcup V_t \subseteq V(G)$, with $V_i$ connected and $G[V_1, \ldots, V_t] = K_t$
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**Wagner, Kuratowski 1930**

A graph is planar iff $K_5$-minor and $K_{3,3}$-minor free
Hadwiger’s conjecture

Appel, Haken 1976
If $G$ is planar, then $\chi(G) \leq 4$

Robertson, Sanders, Seymour, Thomas 1997
Much simpler proof, but still computer assisted

Hadwiger’s conjecture 1943
If $G$ is $K_t$-minor free then $\chi(G) \leq t - 1$
Proved for $1 \leq t \leq 6$, widely open for $t > 6$
Meyniel 1978
All 5-colorings of a planar graph are Kempe-equivalent (tight)

Las Vergnas and Meyniel 1981
All 5-colorings of a \(K_5\)-minor free graph are Kempe-equivalent
Meyniel 1978
All 5-colorings of a planar graph are Kempe-equivalent (tight)

Las Vergnas and Meyniel 1981
All 5-colorings of a $K_5$-minor free graph are Kempe-equivalent

Conjecture 1 [Las Vergnas and Meyniel 1981]
All the $t$-colorings of a $K_t$-minor free graph are Kempe-equivalent

Conjecture 2 [Las Vergnas and Meyniel 1981]
All the $t$-colorings of a $K_t$-minor free graph are Kempe-equivalent to a $(t - 1)$-coloring
Frozen colorings

Frozen coloring

α is frozen if ∀i, j, the graph induced by colors i and j is connected

Quasi-minor

K_t is quasi-minor of G if there exists V_1 ⊔ ... ⊔ V_t such that ∀i ≠ j, G[V_i ∪ V_j] is connected and G[V_1, ..., V_t] = K_t

K_t-minor ⇒ quasi K_t-minor Frozen

t-coloring ⇒ quasi K_t-minor
Frozen colorings

Frozen coloring

\( \alpha \) is frozen if \( \forall i, j \), the graph induced by colors \( i \) and \( j \) is connected

Quasi-minor

\( K_t \) is quasi-minor of \( G \) if there exists \( V_1 \cup \cdots \cup V_t \) such that \( \forall i \neq j \), \( G[V_i \cup V_j] \) is connected and \( G[V_1, \ldots V_t] = K_t \)
Frozen colorings

Frozen coloring

$\alpha$ is frozen if $\forall i, j$, the graph induced by colors $i$ and $j$ is connected

Quasi-minor

$K_t$ is quasi-minor of $G$ if there exists $V_1 \sqcup \cdots \sqcup V_t$ such that $\forall i \neq j$, $G[V_i \cup V_j]$ is connected and $G[V_1, \ldots, V_t] = K_t$

$K_t$-minor $\Rightarrow$ quasi $K_t$-minor

Frozen $t$-coloring $\Rightarrow$ quasi $K_t$-minor
Motivation

If $G$ has no $K_t$ minor and all its $t$-colorings are Kempe equivalent then either

- no frozen $t$-coloring
- only one $t$-coloring up to color permutation $\leadsto$ Hadwiger’s conjecture is false
Last conjecture

Motivation

If $G$ has no $K_t$ minor and all its $t$-colorings are Kempe equivalent then either

- no frozen $t$-coloring
- only one $t$-coloring up to color permutation $\leadsto$ Hadwiger’s conjecture is false

Conjecture 3 [Las Vergnas and Meyniel 1981]

No $K_t$-minor $\Rightarrow$ No quasi $K_t$-minor $\Rightarrow$ No frozen $t$-coloring
Motivation

If $G$ has no $K_t$ minor and all its $t$-colorings are Kempe equivalent then either

- no frozen $t$-coloring
- only one $t$-coloring up to color permutation $\Leftrightarrow$ Hadwiger’s conjecture is false

Conjecture 3 [Las Vergnas and Meyniel 1981]

No $K_t$-minor $\Rightarrow$ No quasi $K_t$-minor $\Rightarrow$ No frozen $t$-coloring

Conjecture 3 holds for

- [Las Vergnas and Meyniel ’81] $t \leq 5$
- [Jørgensen ’94] $t = 8$
- [Song and Thomas ’06] $t = 9$
- [Kriesell ’21] $t = 10$
No $K_t$-minor implies ...

1. $t$-recolorable
2. Every $t$-coloring is equivalent to a $(t-1)$-coloring
3. No quasi-$K_t$-minor

$\iff$ Hadwiger’s conjecture is false

Assuming Hadwiger

$\forall \varepsilon > 0$ and large enough $t$, $\exists G$ with a frozen $t$-coloring but no $K(2t + \varepsilon)$ $t$-minor. This graph admits another $t$-coloring.

Any graph with a quasi-$K_t$-minor has a $K_{t^2}$-minor.
To sum up

No $K_t$-minor implies ...

1. $t$-recolorable

2. Every $t$-coloring is equivalent to a $(t-1)$-coloring

3. No quasi-$K_t$-minor

Assuming Hadwiger's conjecture is false

Bonamy, Heinrich, L., Narboni '23

- Strongly disproved for large $t$: $\forall \varepsilon > 0$ and large enough $t$, $\exists G$ with a frozen $t$-coloring but no $K_{(2/3+\varepsilon)t}$-minor. This graph admits another $t$-coloring.
No $K_t$-minor implies ...

1. $t$-recolorable \hspace{1cm} \land \hspace{1cm} 3. No quasi-$K_t$-minor \rightarrow \text{Hadwiger’s conjecture is false}

Assuming Hadwiger

2. Every $t$-coloring is equivalent to a $(t-1)$-coloring

Bonamy, Heinrich, L., Narboni ’23

• Strongly disproved for large $t$: $\forall \varepsilon > 0$ and large enough $t$, $\exists G$ with a frozen $t$-coloring but no $K_{(\frac{2}{3}+\varepsilon)t}$-minor. This graph admits another $t$-coloring.

• Any graph with a quasi-$K_t$-minor has a $K_{\frac{t}{2}}$-minor
Random construction of $G_t$

• Start with a clique on $V = \{a_1, b_1, \ldots, a_t, b_t\}$
Random construction of $G_t$

- Start with a clique on $V = \{a_1, b_1, \ldots, a_t, b_t\}$
- For all $i$, remove $a_ib_i$
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- Start with a clique on $V = \{a_1, b_1, \ldots a_t, b_t\}$
- For all $i$, remove $a_i b_i$
- For all $i, j$, pick independently at random an edge in $\{a_i, b_i\} \times \{a_j, b_j\}$ and remove it
Sketch of proof

Random construction of $G_t$

• Start with a clique on $V = \{a_1, b_1, \ldots, a_t, b_t\}$
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Properties of $G_t$

• has a frozen $t$-coloring
• $\mathbb{P}(G_t \text{ has another } t\text{-coloring}) \xrightarrow{t \to \infty} 1$
• $\mathbb{P}(G_t \text{ is } K_{(\frac{2}{3}+\varepsilon)t}\text{-minor free}) \xrightarrow{t \to \infty} 1$
\[ P(G_t \text{ is } K_{(\frac{2}{3} + \epsilon)t}-\text{minor free}) \xrightarrow{t \to \infty} 1 \]

Sort the bags in a \( K_{(\frac{2}{3} + \epsilon)t}-\text{minor}\)

- Bags of size 1 \( \rightarrow K_{p_1} \) simple minor
- Bags of size 2 \( \rightarrow K_{p_2} \) double minor
- Bags of size at least 3 \( \rightarrow K_{p_3} \) triple minor
\[ \mathbb{P}(G_t \text{ is } K_{\left(\frac{2}{3} + \varepsilon\right)t\text{-minor free}}) \xrightarrow{t \to \infty} 1 \]

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\[ \mathbb{P}(G_t \text{ is } K_{\left(\frac{2}{3} + \varepsilon\right)t\text{-minor free}}) \xrightarrow{t \to \infty} 1 \text{ because} \]

- For all \( \varepsilon_1 > 0 \), \( \mathbb{P}(G_t \text{ has no simple } K_{\varepsilon_1 t\text{-minor}}) \xrightarrow{t \to \infty} 1 \)
- For all \( \varepsilon_2 > 0 \), \( \mathbb{P}(G_t \text{ has no double } K_{\varepsilon_2 t\text{-minor}}) \xrightarrow{t \to \infty} 1 \)
- \( G_t \) has no triple \( K_{\frac{2}{3} t + 1}\text{-minor} \)
For all $\varepsilon > 0$, $\mathbb{P}(G_t \text{ has no simple } K_{\varepsilon t} \text{-minor}) \xrightarrow{t \to \infty} 1$

- Simple $K_p$-minor = induced $K_p$
- Given $S \subset V$ of size $p$,

$$
\mathbb{P}(S \text{ induces a } K_p) \leq \left( \frac{3}{4} \right)^p
$$
For all $\epsilon > 0$, $\mathbb{P}(G_t \text{ has no simple } K_{\epsilon t}\text{-minor}) \xrightarrow{t \to \infty} 1$

- Simple $K_p$-minor = induced $K_p$
- Given $S \subset V$ of size $p$,
  \[ \mathbb{P}(S \text{ induces a } K_p) \leq \left(\frac{3}{4}\right)^{\binom{p}{2}} \]
- By Union-Bound:
  \[ \mathbb{P}(G_t \text{ has an induced } K_{\epsilon t}) \leq \left(\frac{2t}{\epsilon t}\right) \left(\frac{3}{4}\right)^{\binom{\epsilon t}{2}} \]
  \[ \leq 2^{2t} \left(\frac{3}{4}\right)^{\binom{\epsilon t}{2}} \]
  \[ \xrightarrow{t \to \infty} 0 \]
For all $\varepsilon > 0$, $\mathbb{P}(G_t \text{ has no double } K_{\varepsilon t}-\text{minor}) \xrightarrow{t \to \infty} 1$

A special case of double-minor

- Let $S'$ be a set of pairwise disjoint pairs of vertices, such that $\forall i$, at most one of $a_i, b_i$ is involved in $S'$.
- $G_t \setminus S'$ : contract pairs in $S'$ and remove the rest
For all $\varepsilon > 0$, $\mathbb{P}(G_t \text{ has no double } K_{\varepsilon t}\text{-minor}) \xrightarrow{t \to \infty} 1$

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- $G_t \setminus S'$: contract pairs in $S'$ and remove the rest
- $\forall (x_1, y_1), (x_2, y_2) \in S'$, $\mathbb{P}(\exists \text{ an edge between } \{x_1, y_1\} \text{ and } \{x_2, y_2\}) = 1 - (\frac{1}{4})^4$
- $\mathbb{P}(G_t \setminus S' \text{ is a clique}) = (1 - (\frac{1}{4})^4)^{|S'|\choose 2}$
For all $\varepsilon > 0$, $\mathbb{P}(G_t \text{ has no double } K_{\varepsilon t}-\text{minor}) \xrightarrow{t \to \infty} 1$

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- $\mathbb{P}(G_t \setminus S' \text{ is a clique}) = \left(1 - \left(\frac{1}{4}\right)^4\right)^{\binom{|S'|}{2}}$
- For $|S'| = \varepsilon't$, at most $\binom{2t}{2\varepsilon't} \cdot (2\varepsilon t)! \leq (2t)^{2\varepsilon t}$ possibilities
- By Union-Bound:

$$\mathbb{P}(\exists \text{ special } S', G_t \setminus S' = K_{\varepsilon't}) \leq (2t)^{2\varepsilon't} \left(1 - \frac{1}{4^4}\right)^{\binom{\varepsilon't}{2}} \xrightarrow{t \to \infty} 0$$
For all $\varepsilon > 0$, $\mathbb{P}(G_t \text{ has no double } K_{\varepsilon t}\text{-minor}) \xrightarrow{t \to \infty} 1$

Reducing to the special case

- Let $S$ be a double $K_{\varepsilon t}\text{-minor}$
- Greedy special $S' \subset S$: $\forall i$, if $a_i$ and $b_i$ are involved in $S$, remove the pair containing $b_i$
For all $\varepsilon > 0$, $\mathbb{P}(G_t \text{ has no double } K_{\varepsilon t}\text{-minor}) \xrightarrow{t \to \infty} 1$

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- Greedy special $S' \subset S$: $\forall i$, if $a_i$ and $b_i$ are involved in $S$, remove the pair containing $b_i$
- $|S'| \geq \frac{\varepsilon}{3}t$ so take $\varepsilon' = \frac{\varepsilon}{3}$:

\[
\mathbb{P}(\exists S \text{ a double } K_{\varepsilon t}\text{-minor}) \leq \mathbb{P}(\exists \text{ a special } S', G_t \setminus S' = K_{\varepsilon' t}) \xrightarrow{t \to \infty} 0
\]
Open questions

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• What is the infimum \( c \) such that for \( t \) large enough, there is \( G \) with a quasi \( K_t \)-minor but no \( K_{ct} \)-minor?

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\frac{1}{2} \leq c \leq \frac{2}{3}
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- Is there $c'$ such that for every $t$, all the $c't$-colourings of a graph with no $K_t$-minor are equivalent?
  \[ \frac{3}{2} \leq c' \quad \text{and all } O(t\sqrt{\log(t)})\text{-colorings are equivalent} \]
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- What is the maximum $t$ for which any graph with no $K_t$ minor is $t$-recolorable? $t \geq 5$
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Thanks!