

Kempe recoloring version of Hadwiger's conjecture

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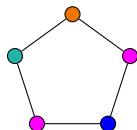
October 15, 2021

Joint work with Marthe Bonamy, Marc Heinrich and Jonathan Narboni

Recoloring with Kempe changes

Kempe chain (1879)

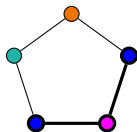
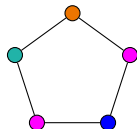
Maximal bichromatic connected component in G



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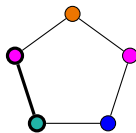
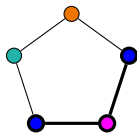
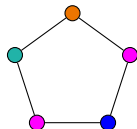
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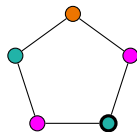
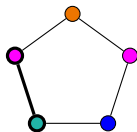
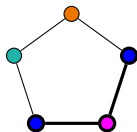
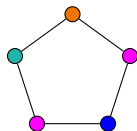
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Recoloring with Kempe changes

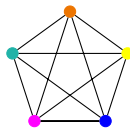
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Graph minor

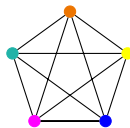
H is a minor of G if H can be obtained by deleting vertices, edges and contracting edges of G



K_t is a minor of G if and only if $V_1 \sqcup \dots \sqcup V_t \subseteq V(G)$, with each V_i connected and $G[V_1, \dots, V_t] = K_t$

Graph minor

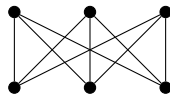
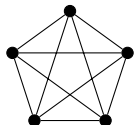
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Wagner, Kuratowski 1930

A graph is planar iff K_5 -minor and $K_{3,3}$ -minor free



Hadwiger's conjecture

Appel, Haken 1976

If G is planar, then $\chi(G) \leq 4$

Robertson, Sanders, Seymour, Thomas 1997

Much simpler proof, but still computer assisted

Hadwiger's conjecture 1943

If G is K_t -minor free then $\chi(G) \leq t - 1$

Proved for $1 \leq t \leq 6$, widely open for $t > 6$

Reconfiguration counterpoint

Meyniel 1978

All 5-colorings of a planar graph are Kempe-equivalent (tight)

Las Vergnas and Meyniel 1981

All 5-colorings of a K_5 -minor free graph are Kempe-equivalent

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Conjecture 1 [Las Vergnas and Meyniel 1981]

All the t -colorings of a K_t -minor free graph are Kempe-equivalent

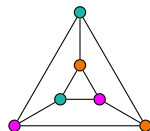
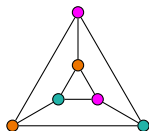
Conjecture 2 [Las Vergnas and Meyniel 1981]

All the t -colorings of a K_t -minor free graph are Kempe-equivalent to a $(t - 1)$ -coloring

Frozen colorings

Frozen coloring

α is frozen if $\forall i, j \leq k$, the graph induced by colors i and j is connected



Quasi-minor

K_t is a quasi-minor of G if there exists $V_1 \sqcup \dots \sqcup V_t$ such that $\forall i \neq j, G[V_i \cup V_j]$ is connected and $G[V_1, \dots, V_t] = K_t$

Frozen t -coloring \Rightarrow quasi- K_t -minor

K_t -minor \Rightarrow quasi- K_t -minor

Conjecture 3 [Las Vergnas and Meyniel 1981]

No K_t -minor \Rightarrow No quasi- K_t -minor

Motivation

If G has no K_t minor and all its t -colorings are Kempe equivalent then

- either no frozen t -coloring
- or only one t -coloring up to color permutation

Holds for:

Las Vergnas, Meyniel 1981 $t \leq 5$

Jørgensen 1994 $t \leq 8$

Song, Thomas 2000 $t \leq 9$

Kriesell 2020 $t \leq 10$

Conjecture 1 [Las Vergnas and Meyniel 1981]

All the t -colorings of a K_t -minor free graph are Kempe-equivalent

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No K_t -minor \Rightarrow No quasi- K_t -minor

Bonamy, Heinrich, L., Narboni '21+

- Strongly disproved for large t : for every $\varepsilon > 0$ and large enough t , there exists a graph G with a frozen t -coloring but no $K_{(\frac{2}{3}+\varepsilon)t}$ -minor
- G admits another non-frozen t -coloring, so its t -colorings are not all Kempe equivalent

Probabilistic proof

- Build a random graph G_t with a frozen t -coloring
- Show that $\mathbb{P}(G_t \text{ admits a } K_{(\frac{2}{3}+\varepsilon)_t}\text{-minor}) \xrightarrow[t \rightarrow \infty]{} 0$

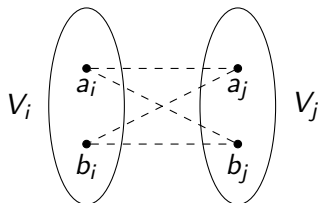
Construction

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Construction

- Vertex set : $\bigsqcup_{i \leq t} V_i$, where $V_i = \{a_i, b_i\}$
- For all $i \neq j$ independently, pick uniformly at random and add three of the edges $\{a_i a_j, a_i b_j, b_i a_j, b_i b_j\}$



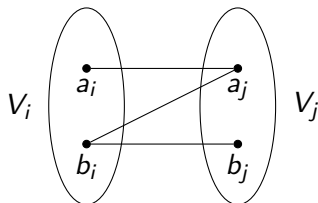
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$$\mathbb{P}(G_t \text{ admits a } K_{(\frac{2}{3}+\varepsilon)t}\text{-minor}) \xrightarrow[t \rightarrow \infty]{} 0$$

Three types of K_p -minor:

- bags of size 1 \rightarrow simple K_{p_1} -minor
- bags of size 2 \rightarrow double K_{p_2} -minor
- bags of size at least 3 \rightarrow triple K_{p_3} -minor

$$\mathbb{P}(G_t \text{ admits a simple } K_{\varepsilon_1 t}\text{-minor}) \xrightarrow[t \rightarrow \infty]{} 0$$

$$\mathbb{P}(G_t \text{ admits a double } K_{\varepsilon_2 t}\text{-minor}) \xrightarrow[t \rightarrow \infty]{} 0$$

G_t does not contain a triple $K_{\frac{2}{3}t+1}$ -minor

$$\mathbb{P}(G_t \text{ admits a simple } K_{\varepsilon_1 t}\text{-minor}) \xrightarrow[t \rightarrow \infty]{} 0$$

Given $S \subset V(G)$ of size k ,

$$\mathbb{P}(S \text{ induces a clique}) \leq \left(\frac{3}{4}\right)^{\binom{k}{2}}$$

So

$$\begin{aligned} \mathbb{P}(\exists S \subset V(G) \text{ of size } \varepsilon_1 t \text{ that induces a clique}) &\leq \binom{2t}{\varepsilon_1 t} \cdot \left(\frac{3}{4}\right)^{\binom{\varepsilon_1 t}{2}} \\ &\leq 2^{2t} \cdot \left(\frac{3}{4}\right)^{\binom{\varepsilon_1 t}{2}} \\ &\xrightarrow[t \rightarrow \infty]{} 0 \end{aligned}$$

$$\mathbb{P}(G_t \text{ admits a double } K_{\varepsilon_2 t}\text{-minor}) \xrightarrow[t \rightarrow \infty]{} 0$$

Consider S' set of pairwise disjoint pairs of vertices such that $\forall i, a_i$ or b_i not involved in S'

G_t/S' : remove the vertices not in S' and identify each pair

$$\mathbb{P}(G_t \text{ admits a double } K_{\varepsilon_2 t}\text{-minor}) \xrightarrow[t \rightarrow \infty]{} 0$$

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G_t/S' : remove the vertices not in S' and identify each pair

$$\forall (x_1, y_1), (x_2, y_2) \in S', \mathbb{P}(\text{edge between them}) = 1 - \left(\frac{1}{4}\right)^4$$

So

$$\mathbb{P}(S' \text{ yields a quasi-} K_{|S'|}\text{-minor}) = \left(1 - \left(\frac{1}{4}\right)^4\right)^{\binom{|S'|}{2}}$$

Number of candidates for S' of size $\varepsilon'_2 t$: at most $\binom{2t}{2\varepsilon'_2 t} \cdot (2\varepsilon'_2 t)! \leq (2t)^{2\varepsilon'_2 t}$

$$\mathbb{P}(G_t \text{ admits a double } K_{\varepsilon_2 t}\text{-minor}) \xrightarrow[t \rightarrow \infty]{} 0$$

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If G_t admits a double $K_{\varepsilon_2 t}$ -minor, extract from S a set S' such that a_i or b_i not involved in S' . We have $|S'| \geq |S|/3$, so take $\varepsilon'_2 = \varepsilon_2/3$

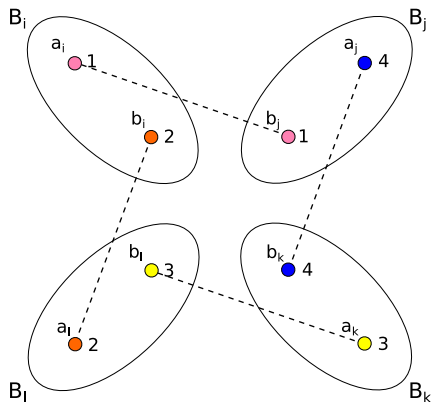
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Otherwise $|V(G_t)| \geq 3(\frac{2}{3}t + 1) > 2t$

Bonamy, Heinrich, L., Narboni '21+

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- G admits another non-frozen t -coloring, so its t -colorings are not all Kempe equivalent

$\mathbb{P}(G_t \text{ contains an other non-frozen coloring}) \xrightarrow[t \rightarrow \infty]{} 1$



H appears with high probability when $t \rightarrow \infty$

What next ?

Question 1

What is the infimum c such that G admits a quasi- K_t -minor $\Rightarrow G$ admits a K_{ct} -minor ?

- $c \leq \frac{2}{3}$
- If G admits a quasi K_{2t} -minor, then it admits a K_t minor, so $c \geq \frac{1}{2}$

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Question 2

Is there a constant $c' \geq \frac{3}{2}$ such that for all t , all the $c' \cdot t$ -colorings of a graph with no K_t -minor form a single equivalence class?