

## Short vertex labels for connectivity checking in planar graphs with forbidden parts

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Let  $P(x, y, Z)$  be the graph property, called a *connectivity query*, that holds iff  $Z$  does not separate  $x$  from  $y$ , i.e., iff there a path from  $x$  to  $y$  that avoids  $Z$ .

**Main result** : An algorithm  $A$  can define for each vertex  $u$  of a connected planar graph  $G$  a bit sequence  $J(u)$  of length  $O(\log(n))$  called a *label*, and another algorithm  $B$  can decide if  $P(x, y, Z)$  is true by using only  $J(x), J(y)$  and  $J(Z)$ , the sets of labels  $J(z)$  for  $z$  in  $Z$ . This algorithm has no other knowledge of  $G$  than these labels.

There are other results of this type that construct *labeling schemes*, i.e., pairs  $(A, B)$  as above, for logically specified graph properties and particular graph classes :

(1) For monadic second-order properties and graphs of bounded tree-width or clique-width (Courcelle and Vanicat [CV]),

(2) for first-order properties (with set arguments) and graphs that can be covered by graphs of bounded tree-width or clique-width with limited overlaps (Courcelle, Gavaille and Kanté [CGK]),

(3) for connectivity queries and the associated distance computations, in graphs of bounded tree-width or clique-width (Courcelle and Twigg [CT]); this is a special case of (1) but with a proof that does not use the huge automata constructed from logical formulas .

The connectivity query is monadic second-order expressible but not first-order expressible, and the class of planar graphs has unbounded clique-width. Hence, this result is not a consequence of the above cited ones. Here is the structure of the construction. A labeling scheme for connectivity queries in 3-connected planar graphs (and even in 2-connected planar graphs such that any two vertices belong to at most  $m$  faces for some fixed  $m$ ) is first constructed. It uses a *straightline embedding* of the given graph with positive integer coordinates of the vertices of value  $O(n)$ . Such values are encoded by bit sequences of length  $\log(n)$ . The label  $J(u)$  of a vertex  $u$  consists of its coordinates and of those of a bounded number of neighbour vertices and of points in the plane that are "in the middle" of certain faces. The decoding by  $B$  uses algorithms of computational geometry for the *planar point location problem* because the validity of  $P(x, y, Z)$  is equivalent to the fact that the points of the plane corresponding to  $x$  and  $y$  are not separated by certain line segments linking vertices to points representing faces, the coordinates of which are in  $J(x), J(y)$  and  $J(Z)$ .

For general connected planar graphs, this geometric method cannot detect all cases where  $Z$  separates  $x$  from  $y$ . We decompose the given graph into a tree of 3-connected components. On this tree, one can detect *some cases of separation*, for instance the case where  $Z$  contains a separating vertex  $s$  of the graph  $G$  such that every path from  $x$  to  $y$  goes through  $s$ . A labeling  $K(u)$  using the result of [CV] can be constructed to detect this case, and also some cases where  $x$  and  $y$  are separated by a pair of vertices in  $Z$ . By combining this labeling and the one based on a straightline embedding of  $G$ , one obtains the desired labeling.

This construction can be extended to graphs that are planar gluings of graphs of bounded clique-width with limited overlaps (Kanté [K]).

All results presented in the lecture are available in the following publications.

[CGK] B. Courcelle, C. Gavaille, M.M. Kanté : Efficient First-Order Model-Checking using Short Labels, Proceedings of Frontiers of Algorithms Workshop, June 2008, Changsha, China, Lec. Notes Comput. Sci. 5059 (2008) 159-170

[CGKT08] B. Courcelle, C. Gavaille, M.M. Kanté, A. Twigg : Connectivity check in 3-connected planar graphs with obstacles, Electronic Notes in Discrete Mathematics 31 (2008) 151-155

[CGKT09] B. Courcelle, C. Gavaille, M.M. Kanté, A. Twigg : Optimal labeling for connectivity checking in planar networks with obstacles, Preprint, December 2008.

[CV] B. Courcelle, R. Vanicat : Query efficient implementations of graphs of bounded clique-width, Discrete Applied Mathematics 131 (2003) 129-150.

[CT] B. Courcelle, A. Twigg : Compact forbidden-set routing, Proceedings STACS 2007, Lec. Notes. Comput. Sci. 4393 (2007) 37-48.

[K] : M. M. Kanté : Graph structurings : some algorithmic applications, Ph.D., December 2008, Bordeaux 1 University (written in English).