Canonical Graph Decompositions
for
algorithms and graph structure

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Graph decompositions

1. Essential tool for constructing algorithms.
2. Give structural descriptions that help to prove properties (e.g., the Graph Minor Theorem).

-- Tree-decompositions,

-- Rank-decompositions and clique-width expressions yield FPT algorithms.

These decompositions are *not canonical* (hence “difficult” to construct).
Canonical decompositions

Useful as preprocessing in many cases, and also to describe certain “features”.

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<td>Directed</td>
<td>$O(n + m)$</td>
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Many authors: Hopcroft, Tarjan,
Most recent algorithms: Charbit, Corneil, Gioan, Habib, Joeris, Lundberg, McConnell, de Montgolfier, Paul, Raffinot, Rao, Tedder, …
Descriptions via canonical decompositions

Example: All planar embeddings of a planar graph.

1. The components of the Tutte decomposition are 3-connected graphs, cycles and “bonds” (sets of parallel edges).

2. 3-connected planar graphs and cycles have unique planar embeddings.

3. Bonds have planar embeddings described by permutations.

4. The embeddings of the components can be combined into one of the given graph.
Similar situations:

1. All transitive orientations of a comparability graph:
   using the modular decomposition:
   a prime module has a unique transitive orientation
   (up to reversal); the other components are edgeless
   and cliques (all linear orders fit).

2. All chord diagrams of a circle graph:
   using the split decomposition:
   a prime component has a unique chord diagram
   (up to flipping, rotation); the others are as above (their
   chord diagrams are defined from linear orders).
The atomic decomposition of a strongly connected graph

Different from the split decomposition of the same graphs by Cunningham (1982).

Close to the Tutte decomposition.

The subgraphs $G_i$ are its atoms.
Motivation: Gauss words describing intersections of curves

word: aabb

word: abcabc

multiword: (abcd, bfde, aecf)
Question: Can one describe all (tuples of) curves (up to homeomorphism) corresponding to a given Gauss (multi)word?

An ambiguous multiword \((abcd, bc, ad)\) (the circle “bc” is “flipped”). It is not ambiguous for nonoriented curves (invariance under flipping).
**Method**: A multiword $W$ yields a (2,2)-regular graph $\text{Gra}(W)$ with transitions.

**Ex.**: $\text{Gra}(Z)$ for $Z = (abcd, akcbkd)$

What are the planar embeddings of $\text{Gra}(W)$ that respect its transitions?

Planarity of (2,2)-regular $t$-graphs (with transitions) can be checked in linear time. (Easy reduction to the usual planarity test).
Circular composition of graphs, t-graphs and maps

\[ G = H \boxplus_{e,f} K \]

**Atoms**: those that cannot be decomposed (as G)

**Theorem**: A planar t-atom has a unique planar embedding.

By “flipping” the embedding of K, one gets a different embedding of the t-graph G. (Flipping is like turning over a pancake.)
Examples of $t$-atoms:

not planar
(as $t$-atom;
the graph is planar).
Circular composition satisfies some laws (circular associativity).

Graphs and not trees, represent “normal forms” of terms.

Circular composition $\boxplus e_1, e_2, e_3, e_4 (G_1, G_2, G_3, G_4)$ defined as

$(((G_1 \boxplus e_1, e_2 G_2) \boxplus e_2, e_3 G_3) \boxplus e_3, e_4 G_4$
Two vertices belong to an atom $\iff$ they are not separated by an “edge-cut” of 2 edges.

The quotient graph is a **cactus**:

It defines the **atomic decomposition**.

**Def. of cactus**: The 2-connected components are circuits.

**Equiv.**: for all $x,y$, there is a unique path from $x$ to $y$  

(the directed version of one characterization of undirected trees).
The Tutte decomposition of a 2-connected graph:

The blocks have vertex sets \{c,d\}, \{a,b,i,j,k\}, \{e,f,g,h\}, \{l,m,n\}.
Atoms determined from the Tutte decomposition

- u and v are separated at block B by \{ e, f \}

- v and w are not separated at block B by \{ e, f \}

An atom is not in general a (subgraph of a) block of the Tutte decomposition.

The atomic decomposition can be computed in linear time from the Tutte decomposition.
What about undirected graphs?

The 2-edge connected graphs (equiv.: connected without bridges) are those having a strongly connected orientation.

The atomic decomposition is (essentially) invariant under a change of strongly connected orientation. Because so are the notions of separation of 2 vertices and of atom.

$$G = H \bigoplus_{e,f} K$$

Idea: If one reverses $e$, one must also reverse $f$ to keep a strongly connected orientation.
Application: Number of strongly connected (or of Eulerian orientations), denoted by \( #(G) \).

Fact:

\[ G = H \boxplus_{e,f} K \quad \Rightarrow \quad #(G) = #(H) \cdot #(K) / 2 \]

Hence if \( G \) has atoms \( H_1, \ldots, H_p \), then:

\[ #(G) = #(H_1) \cdot \ldots \cdot #(H_p) / 2^{p-1} \]

oOo

Relation to tree-width:

\[ \text{twd}(G) = \text{Maximum tree-width of the atoms of } G \]

(if \( \text{twd}(G) \geq 2 \))
Application to Gauss multiwords

1. For multiword $W$, construct the $t$-graph $\text{Gra}(W)$.

$Ex.: \text{Gra}(Z)$ for

$Z = (abcd, akcbkd)$

2. Check if $\text{Gra}(W)$ is planar; check if it is a $t$-atom.
3. If planar, build the atomic decomposition (trivial if atom).

4. Build the unique planar embedding of each t-atom.

5. By flipping each of the $p$ t-atoms, we get the $2^{p-1}$ different planar embeddings of $\text{Gra}(W)$.

All these tasks can be done by linear time algorithms.
Gauss multiwords representing tuples of nonoriented curves

The multiword \((abcd, bc, ad)\) is ambiguous for oriented curves but not for nonoriented ones. The problem comes (only) from small curves: those with 2 intersections.
Both orientations of the small red curve give the same multiword.

\[ W = (ab, w_1, \ldots, w_n) \rightarrow W' = (w'_1, \ldots, w'_n) \quad \text{(we remove } a, b) \]

\[ \downarrow \]

\[ N \text{ tuples of nonoriented } \leftrightarrow N \text{ planar embeddings of } \text{Und}(\text{Gra}(W')) \text{ curves for } W. \]
General case for nonoriented curves

Theorem: If $W$ is a Gauss multiword of the form $W = (ab, \ldots, cd, w_1, \ldots, w_n) \rightarrow W' = (w'_1, \ldots, w'_n)$ (removing $a,b,\ldots,c,d$), $w_1,\ldots,w_n$ have length $\geq 3$ (actually $\geq 4$ by planarity), if $W$ has $q$ small components : $ab,\ldots,cd$, then $\text{Gra}(W')$ has $p$ $t$-atoms, and

there are $2^{p-q-1}$ $(q+n)$-tuples of nonoriented curves with Gauss multiword $W$.

Remark: Each small component yields a $t$-atom, whence the number $p-q$. 
There are 6 small components (in red).
The remaining curves (in black and blue) represent Gra(W’) that has 7 atoms $\rightarrow$ 64 different 11-tuples of curves
Extensions and open questions

Curves with multiple intersections,
Knot diagrams,
Decompositions based on circular composition of directed graphs,
Gauss multiwords on other surfaces,
Forbidden configurations (minors ?) for planar 4-regular graphs with transitions,
Algorithmic applications ? Thanks for suggestions ! Artistic applications: Possible.
Find grammar for the Gauss words of *kolams*?