



Canonical Graph Decompositions
for
algorithms and graph structure

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Graph decompositions

1. Essential tool for constructing algorithms.
2. Give structural descriptions that help to prove properties (e.g., the Graph Minor Theorem).

-- Tree-decompositions,

-- Rank-decompositions and clique-width expressions yield FPT algorithms.

These decompositions are *not canonical* (hence “difficult” to construct).

Canonical decompositions

Useful as preprocessing in many cases,
and also to describe certain “features”.

	Construction time	Algos / Descriptions
Modular decomp. Undirected Directed	 $O(n + m)$ $O(n + m)$	Clique-width $\leq k$, interval graphs and related ones, cluster editing, genomics, opt. problems for comparability graphs. All transitive orientations of a comparability graph,
Split decomposition Undirected Directed	 $O(n + m \cdot \alpha(n+m))$ $O(m \cdot \log(n))$	Rank-width $\leq k$, circle graphs, 3-leaf powers, distance labelling, coloring and optimization problems. All chord diagrams of a circle graph.

Tutte decomp. (3-connected comp.)	$O(n+m)$	Tree-width $\leq k$. All planar embeddings of a planar graph.
Atomic decomposition	$O(n+m)$	Number of strongly connected orientations, Gauss words, Knots, Matroids ???

Many authors : Hopcroft, Tarjan,

Most recent algorithms : Charbit, Corneil, Gioan, Habib, Joeris, Lundberg, McConnell, de Montgolfier, Paul, Raffinot, Rao, Tedder, ...

Descriptions *via* canonical decompositions

Example : All planar embeddings of a planar graph.

1. The components of the **Tutte decomposition** are 3-connected graphs, cycles and “bonds”
(sets of parallel edges).
2. 3-connected planar graphs and cycles have **unique** planar embeddings.
3. Bonds have planar embeddings described by permutations.
4. The embeddings of the components can be combined into one of the given graph.

Similar situations :

1. All transitive orientations of a comparability graph:

using the **modular decomposition** :

a *prime module* has a **unique** transitive orientation (up to reversal); the other components are edgeless and cliques (all linear orders fit).

2. All chord diagrams of a circle graph :

using the **split decomposition** :

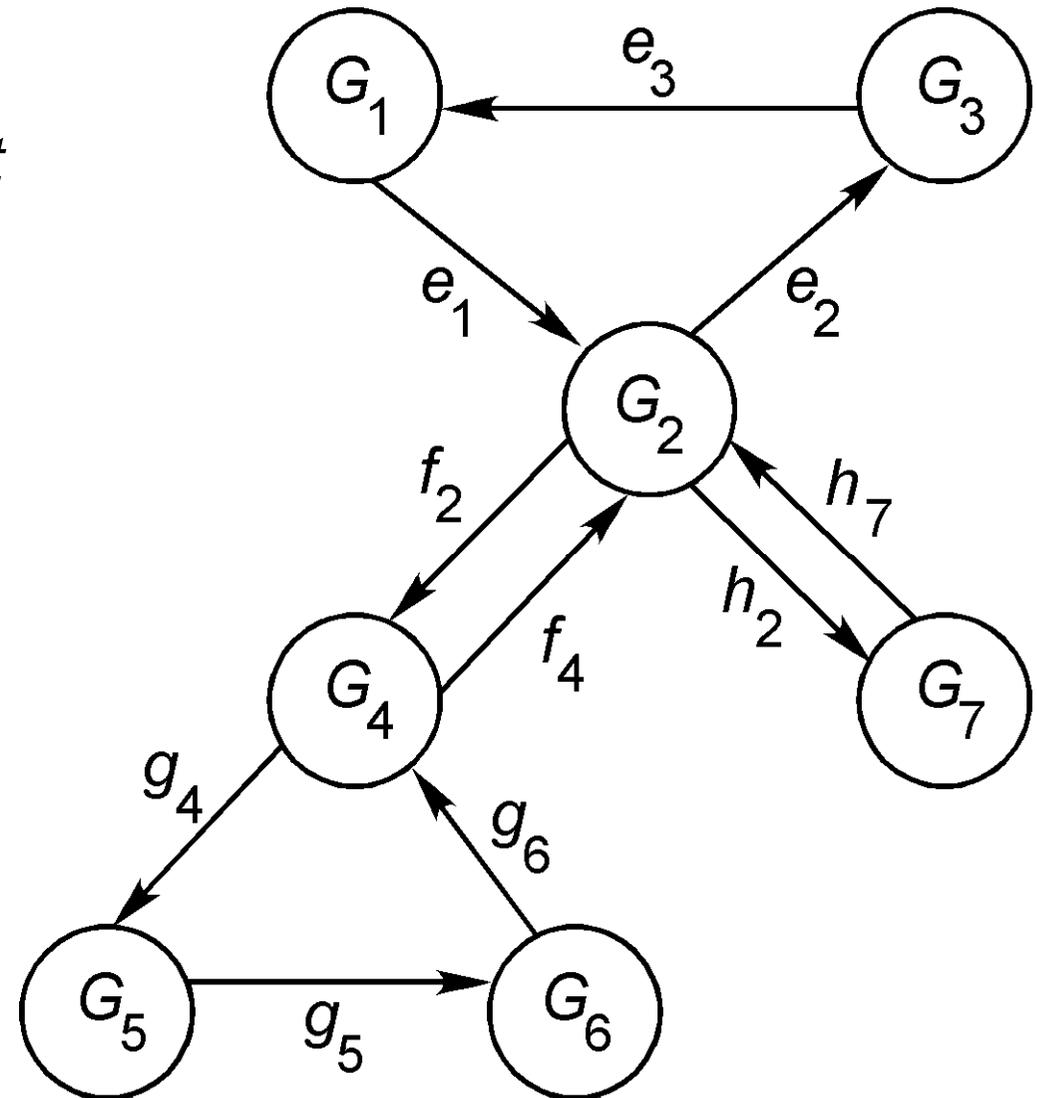
a *prime component* has a **unique** chord diagram (up to flipping, rotation); the others are as above (their chord diagrams are defined from linear orders).

The atomic decomposition of a strongly connected graph

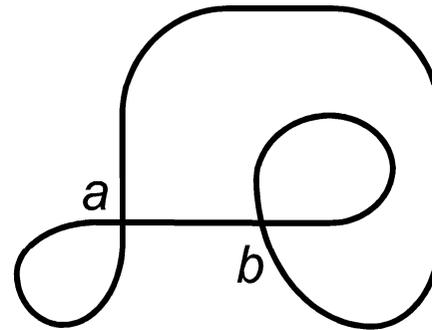
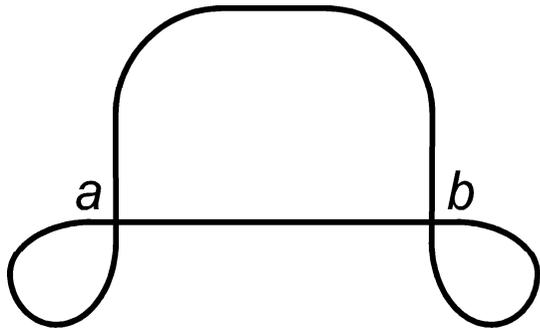
Different from the *split decomposition* of the same graphs by Cunningham (1982).

Close to the *Tutte decomposition*.

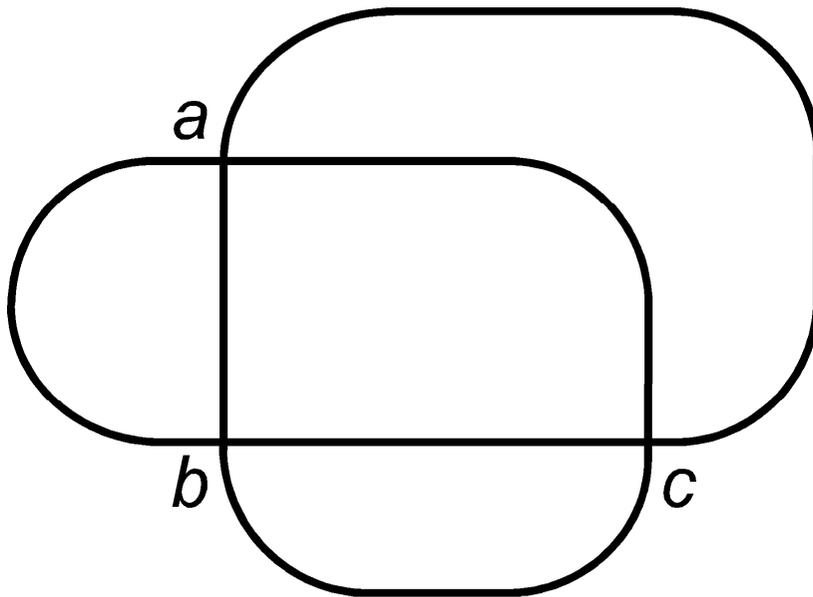
The subgraphs G_i are its *atoms*.



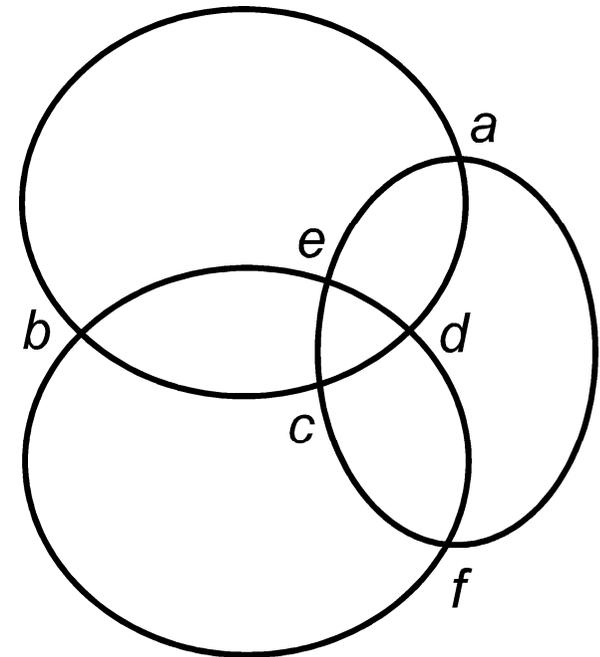
Motivation : Gauss words describing intersections of curves



word : aabb

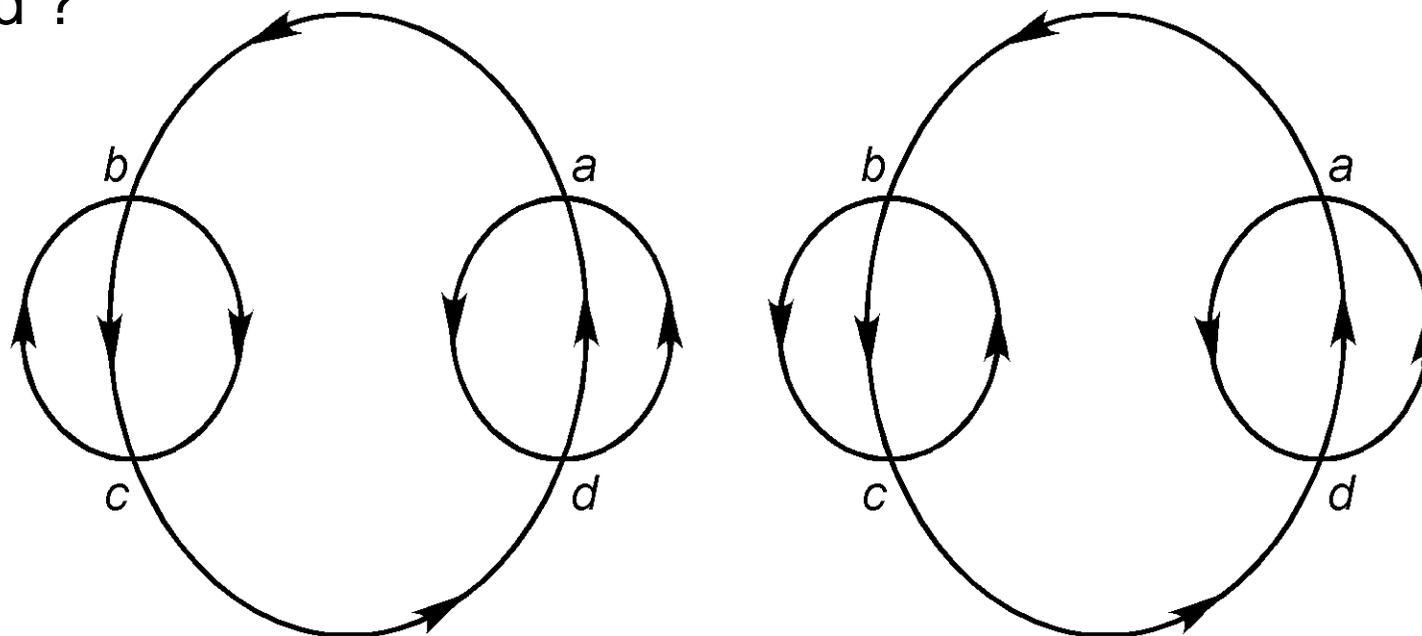


word : abcabc



multiword : (abcd, bfde, aecf)

Question : Can one describe all (tuples of) curves
 (up to homeomorphism) corresponding to a given Gauss
 (multi)word ?

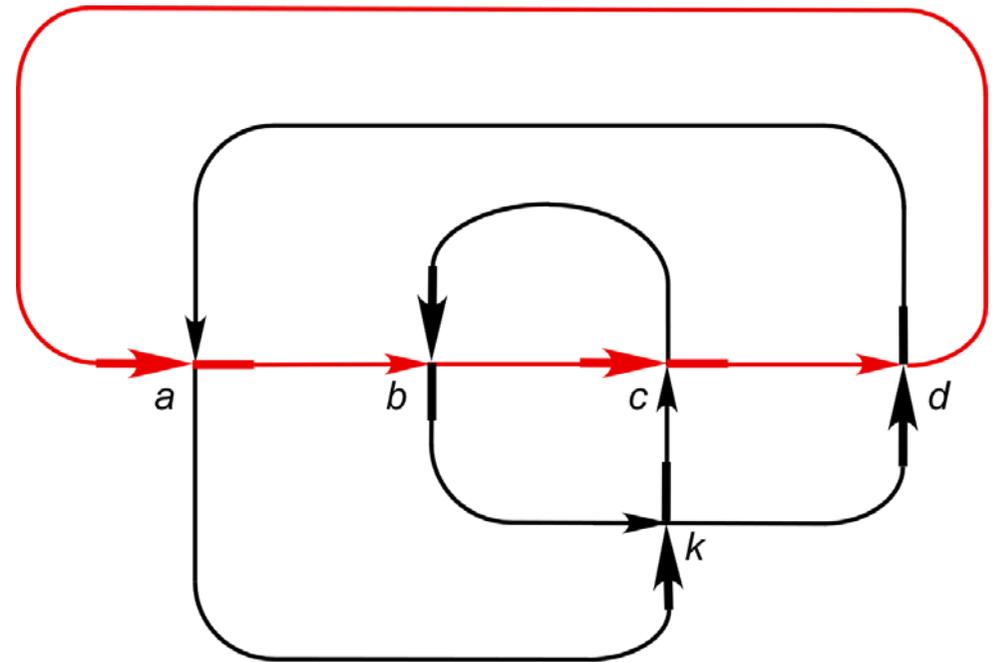


An **ambiguous** multiword $(abcd, bc, ad)$ (the circle “bc” is “flipped”).
 It is **not** ambiguous for nonoriented curves (invariance under flipping).

Method : A multiword W yields a (2,2)-regular graph $\text{Gra}(W)$ with *transitions*.

Ex.: $\text{Gra}(Z)$ for
 $Z = (abcd, akcbkd)$

What are the planar embeddings of $\text{Gra}(W)$ that respect its transitions ?

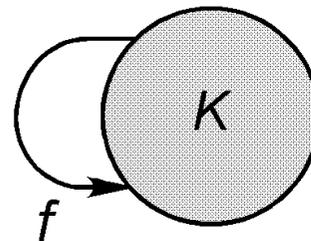
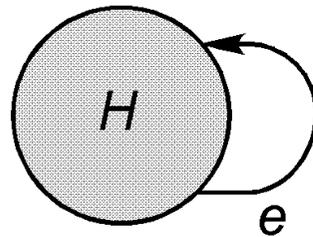


Planarity of (2,2)-regular t -graphs (t means “with transitions”)

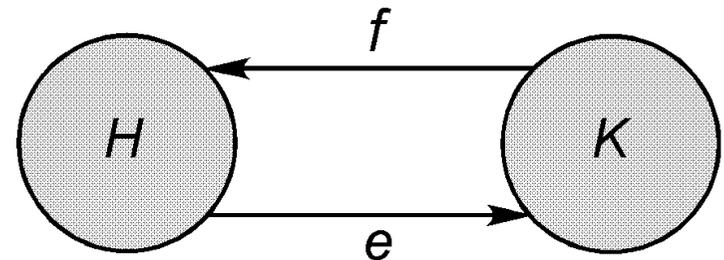
Can be checked in **linear time**.

(Easy reduction to the usual planarity test).

Circular composition of graphs, t-graphs and maps



$$G = H \boxplus_{e,f} K$$

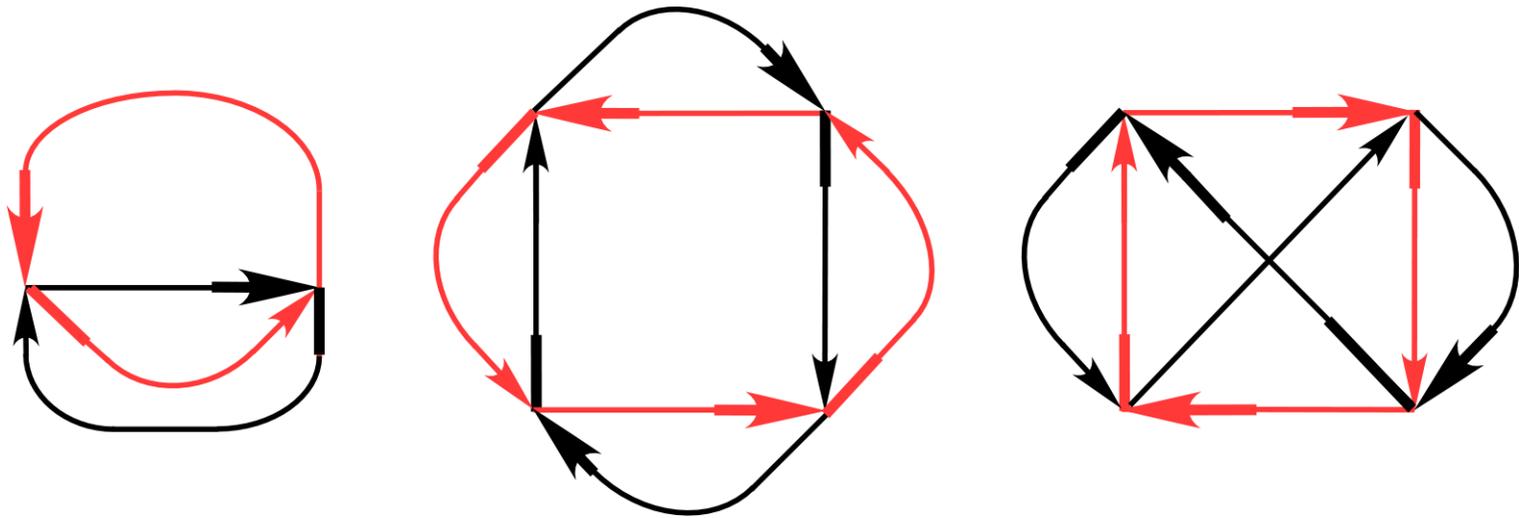


Atoms : those that cannot be decomposed (as G)

Theorem : A planar t-atom has a **unique** planar embedding.

By “flipping” the embedding of K , one gets a different embedding of the t-graph G . (Flipping is like turning over a pancake.)

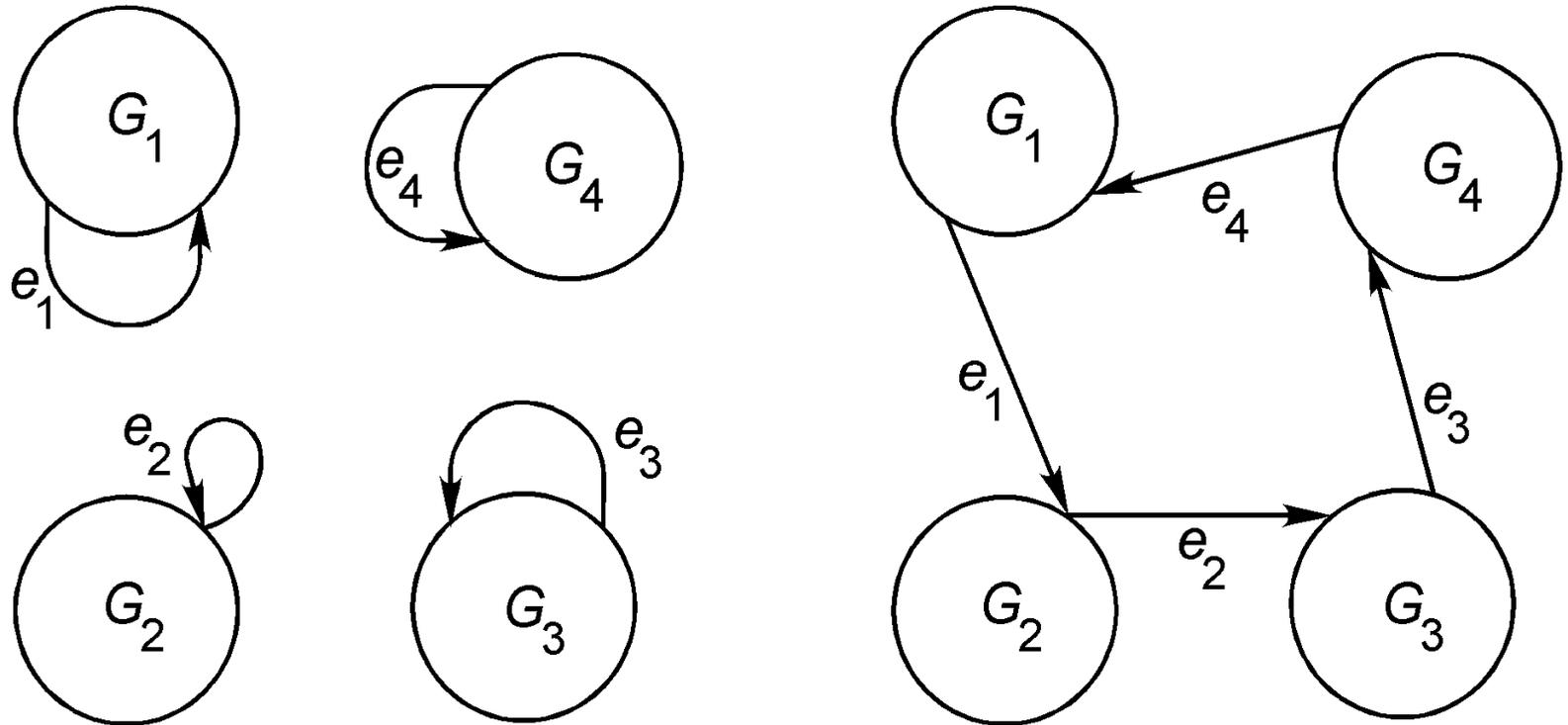
Examples of *t*-atoms :



not planar
(as *t*-atom;
the graph is planar).

Circular composition satisfies some laws (*circular associativity*).

Graphs and *not trees*, represent “normal forms” of terms.



Circular composition $\boxplus_{e_1, e_2, e_3, e_4} (G_1, G_2, G_3, G_4)$ defined as

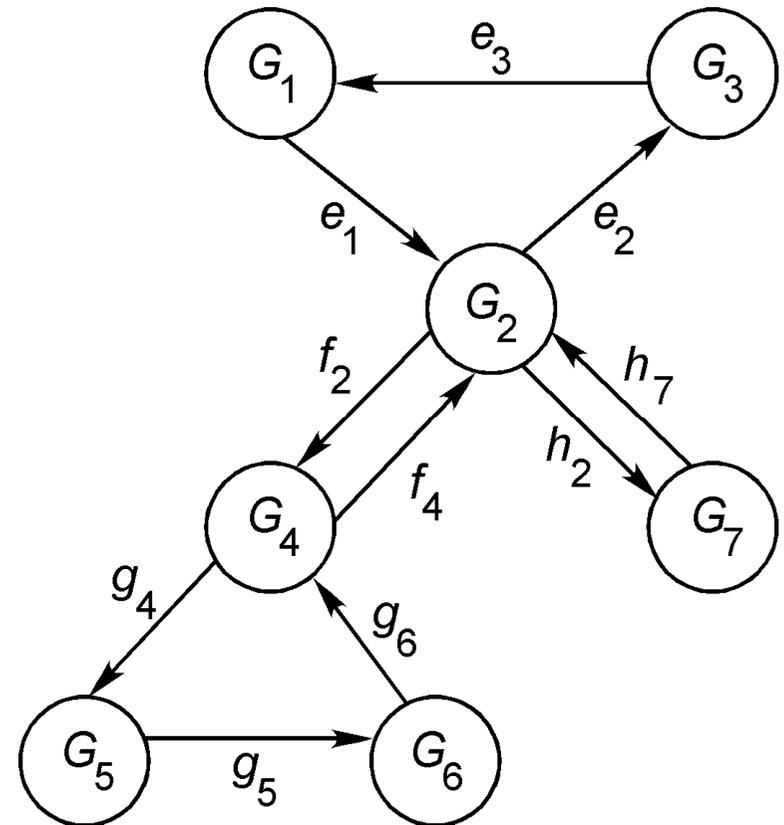
$$((G_1 \boxplus_{e_1, e_2} G_2) \boxplus_{e_2, e_3} G_3) \boxplus_{e_3, e_4} G_4$$

Two vertices belong to an atom \leftrightarrow they are not **separated** by an “edge-cut” of 2 edges.

The quotient graph is a *cactus* :

It defines the

atomic decomposition.

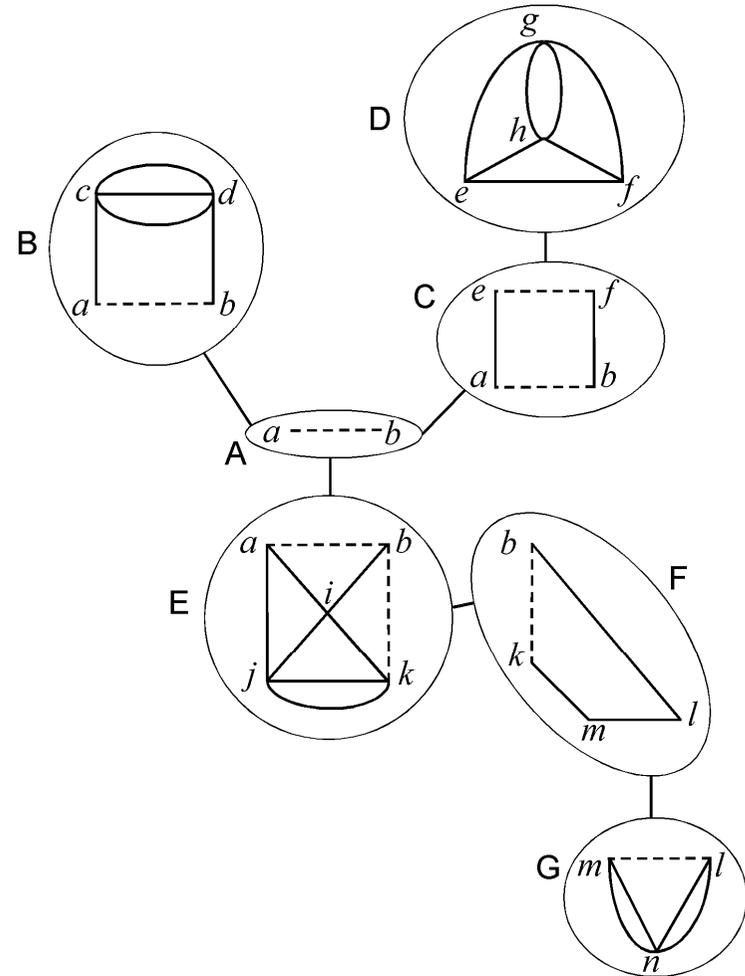
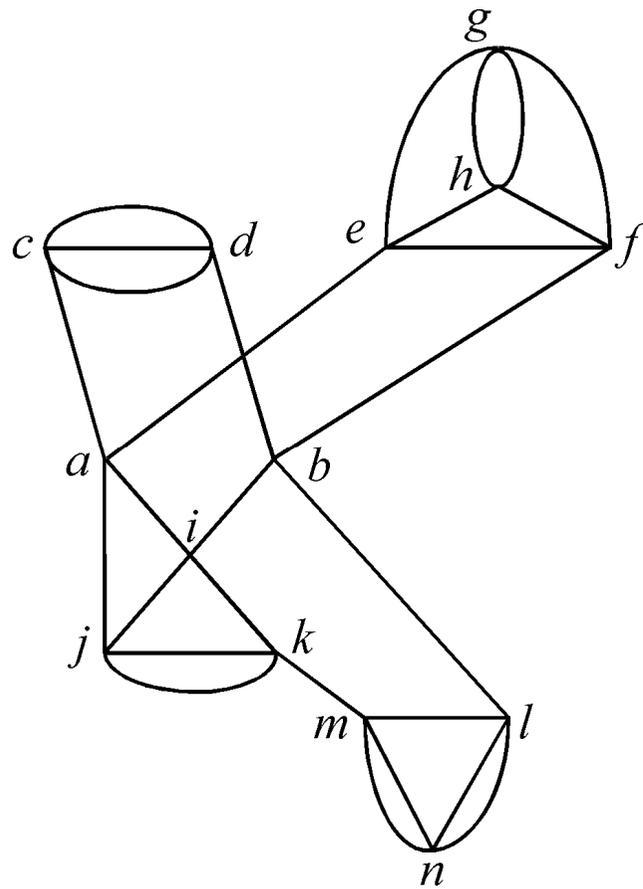


Def. of cactus : The 2-connected components are circuits.

Equiv. : for all x,y , there is a **unique path** from x to y

(the directed version of one characterization of undirected trees).

The Tutte decomposition of a 2-connected graph:



The *blocks* have vertex sets $\{c, d\}$, $\{a, b, i, j, k\}$, $\{e, f, g, h\}$, $\{l, m, n\}$.

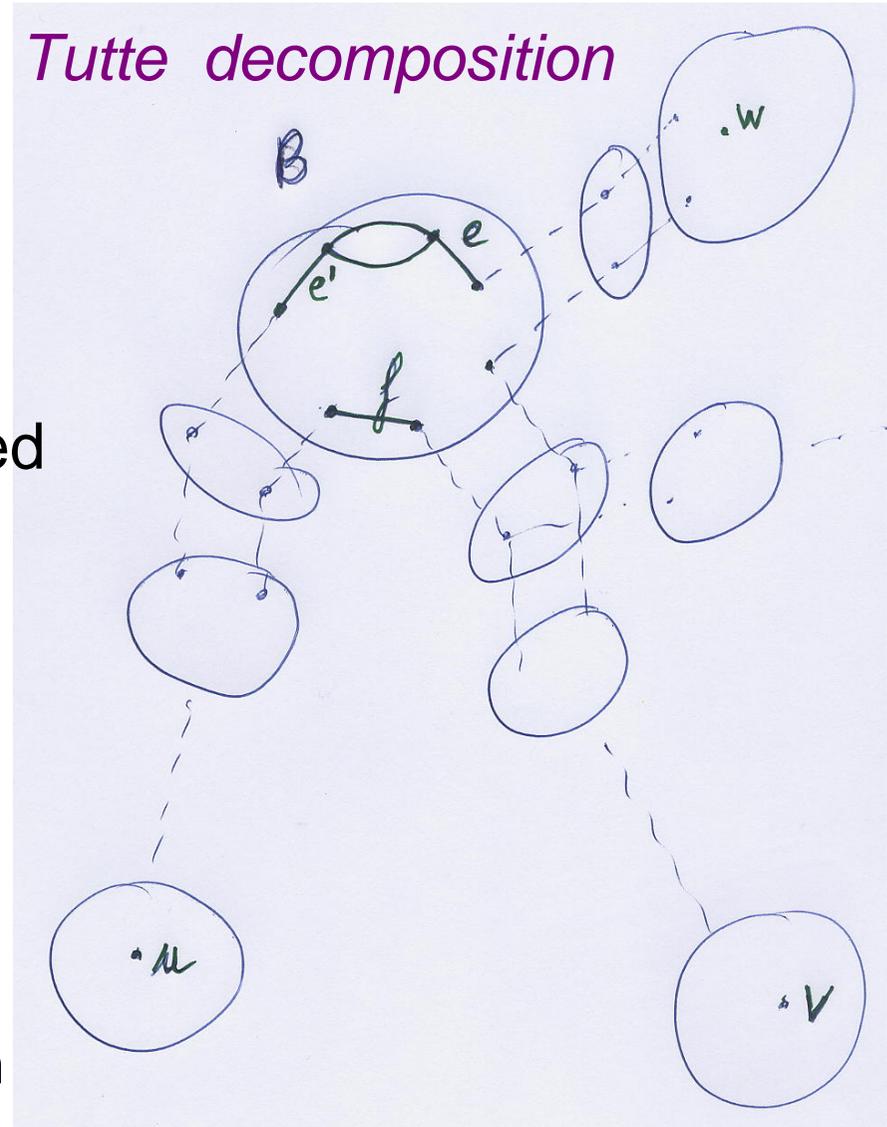
Atoms determined from the Tutte decomposition

- u and v are **separated** at block B by $\{e, f\}$
- v and w are **not** separated at block B by $\{e, f\}$

An atom is not in general a (subgraph of a) **block** of the Tutte decomposition.

The atomic decomposition can

be computed in linear time from the Tutte decomposition.

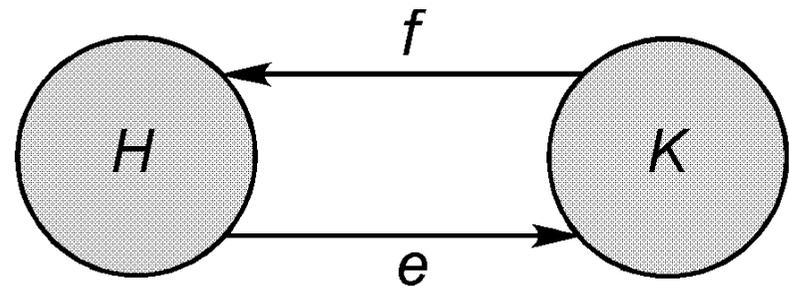


What about undirected graphs ?

The 2-edge connected graphs (*equiv.*: connected without bridges) are those having a strongly connected orientation.

The atomic decomposition is (essentially) *invariant* under a change of strongly connected orientation. Because so are the notions of *separation of 2 vertices* and of atom .

$$G = H \boxplus_{e,f} K$$



Idea : If one reverses e , one must also reverse f to keep a strongly connected orientation.

Application: Number of strongly connected (or of Eulerian orientations), denoted by $\#(G)$.

Fact:

$$G = H \boxplus_{e,f} K \quad \rightarrow \quad \#(G) = \#(H) \cdot \#(K) / 2$$

Hence if G has atoms H_1, \dots, H_p , then :

$$\#(G) = \#(H_1) \cdot \dots \cdot \#(H_p) / 2^{p-1}$$

oOo

Relation to tree-width :

$\text{twd}(G) =$ Maximum tree-width of the atoms of G

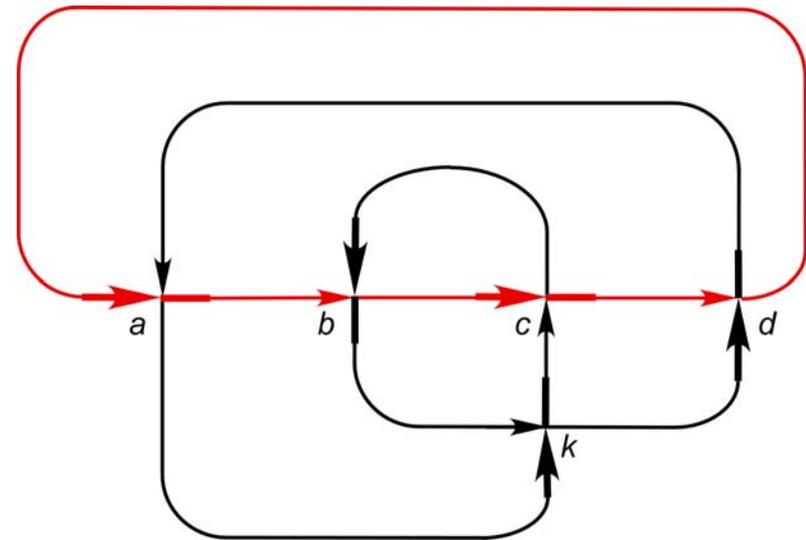
(if $\text{twd}(G) \geq 2$)

Application to Gauss multiwords

1. For multiword W , construct the t -graph $\text{Gra}(W)$.

Ex.: $\text{Gra}(Z)$ for

$Z = (abcd, akcbkd)$

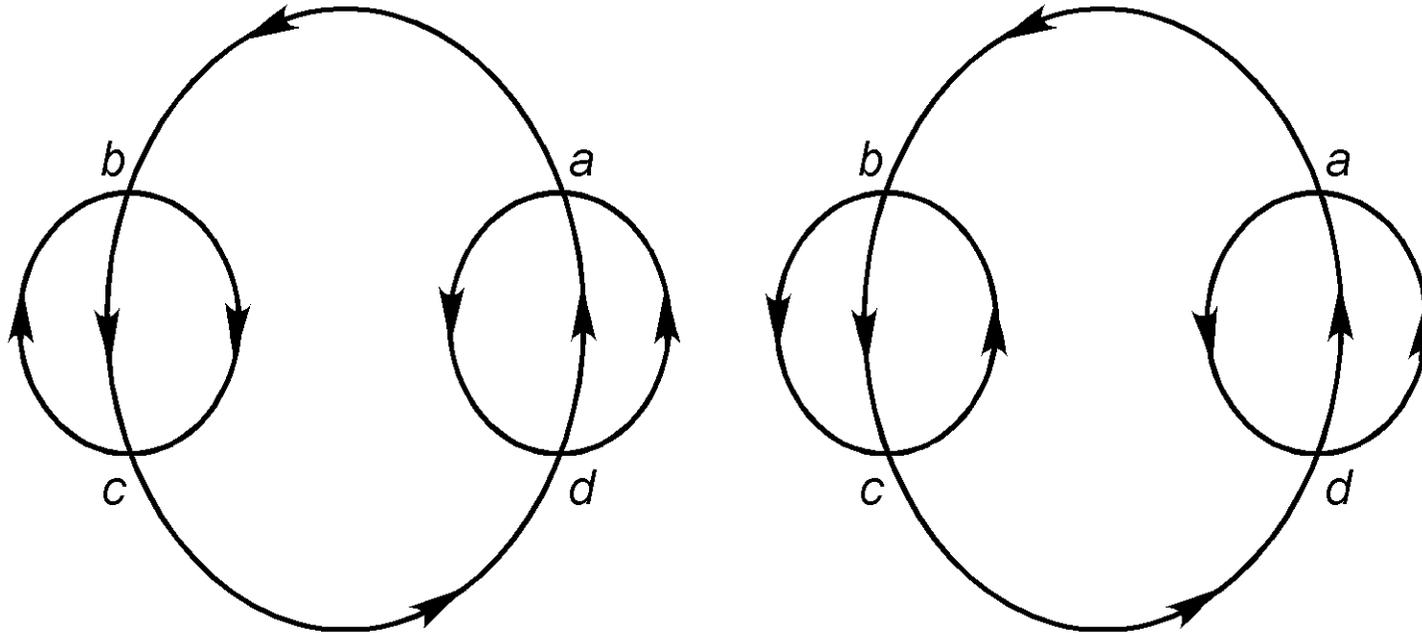


2. Check if $\text{Gra}(W)$ is planar ; check if it is a t -atom.

3. If planar, build the atomic decomposition (trivial if atom).
4. Build the unique planar embedding of each **t**-atom.
5. By flipping each of the p **t**-atoms, we get the 2^{p-1} different planar embeddings of **Gra(W)**.

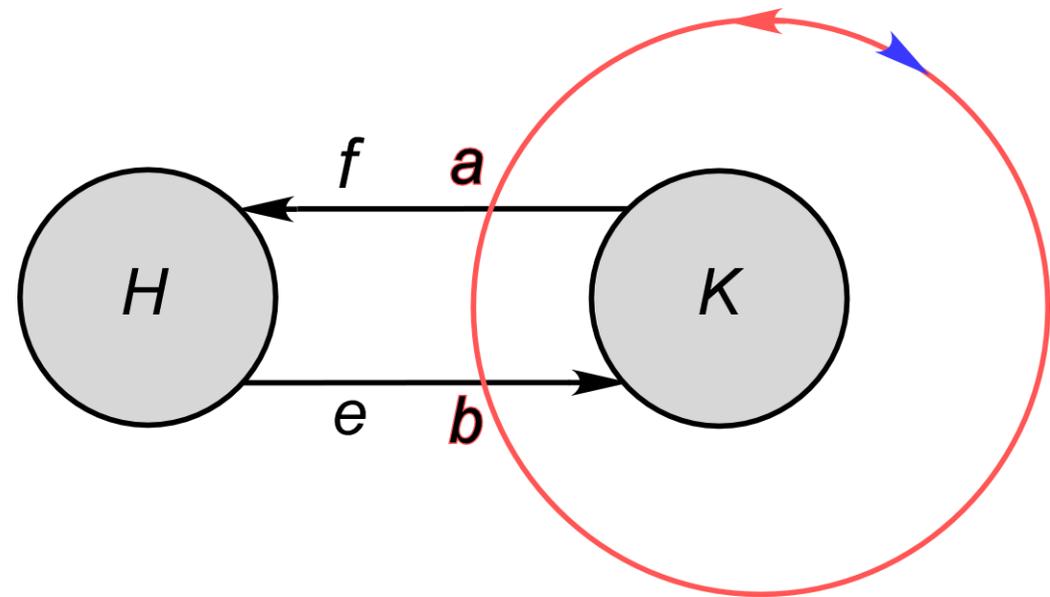
All these tasks can be done by **linear time** algorithms.

Gauss multiwords representing tuples of **nonoriented** curves



The multiword $(abcd, bc, ad)$ is ambiguous for oriented curves but not for nonoriented ones. The problem comes (**only**) from **small curves**: those with 2 intersections.

Both orientations of the small red curve give the same multiword.



$$W = (\mathbf{ab}, w_1, \dots, w_n) \rightarrow W' = (w'_1, \dots, w'_n) \quad (\text{we remove } \mathbf{a,b})$$



\mathbb{N} tuples of nonoriented \leftarrow \mathbb{N} planar embeddings of $\text{Und}(\text{Gra}(W'))$
 curves for W .

General case for nonoriented curves

Theorem: If W is a Gauss multiword of the form

$W = (ab, \dots, cd, w_1, \dots, w_n) \rightarrow W' = (w'_1, \dots, w'_n)$ (removing a, b, \dots, c, d),

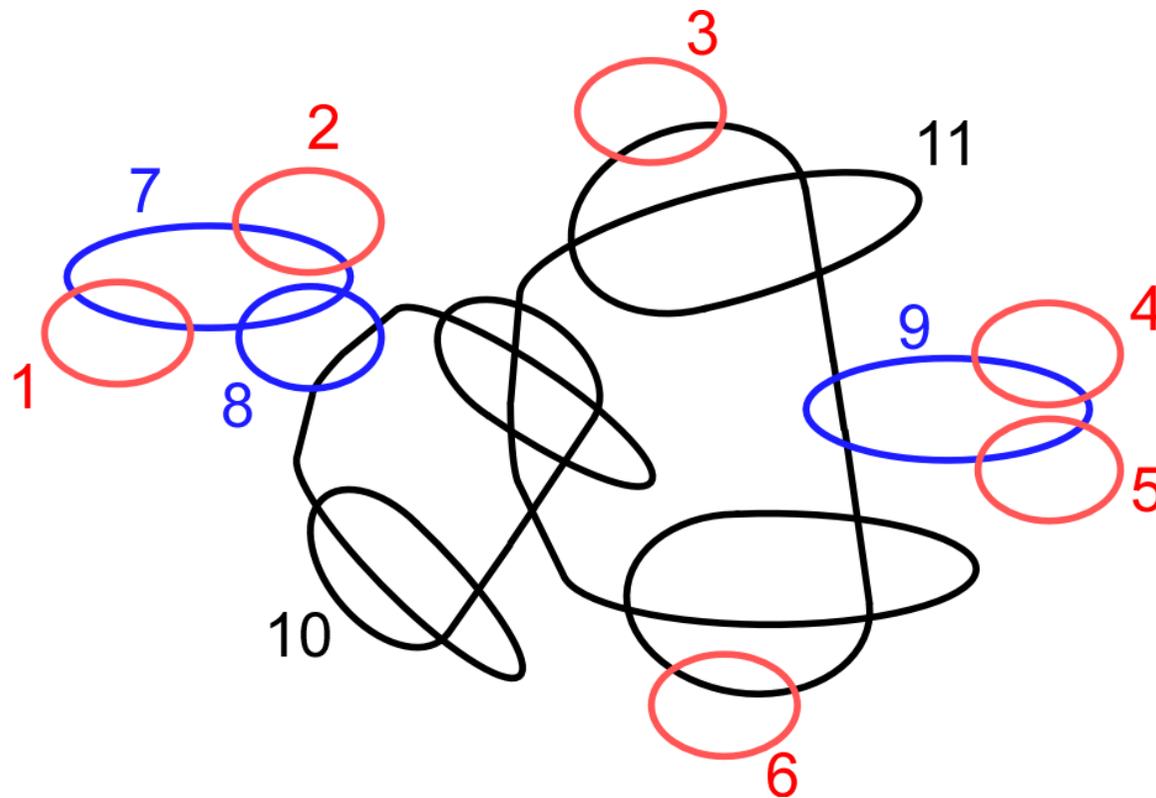
w_1, \dots, w_n have length ≥ 3 (actually ≥ 4 by planarity),

if W has q small components : ab, \dots, cd ,

then $\text{Gra}(W')$ has p t -atoms, and

there are 2^{p-q-1} $(q+n)$ -tuples of nonoriented curves with Gauss multiword W .

Remark: Each small component yields a t -atom, whence the number $p-q$.



There are 6 small components (in red).

The remaining curves (in black and blue) represent

$\text{Gra}(W')$ that has 7 atoms \rightarrow 64 different 11-tuples of curves

Extensions and open questions

Curves with **multiple intersections**,

Knot diagrams,

Decompositions based on circular composition
of **directed graphs**,

Gauss multiwords on other **surfaces**,

Forbidden configurations (minors ?) for planar
4-regular graphs with transitions,

Algorithmic applications ? Thanks for suggestions !

Artistic applications: Possible.

Find grammar for the
Gauss words
of **kolams** ?

