1 About tree-width

1-1) Let $T$ be a binary tree where each node either is a leaf or has a left and a right son. Let $\hat{T}$ be the graph made of $T$ with an edge between each leaf and the next one in the left-right order. What is the tree-width of $\hat{T}$? Build an HR-equational system defining the set of all graphs $\hat{T}$.

1-2) Let us consider $T$ as above with $m$ leaves numbered from 1 to $m$ in the left-right order. The $i$-th leaf is made into the $i$-source. What is the tree-width of $fg_{All}(T//T)$? Build an HR-equational system defining the set of graphs $fg_{All}(T//T)$ for all trees $T$. ($fg_{All}$ forgets all sources).

1-3) Prove that the set of graphs $fg_{All}(S//T)$ for all trees $S$ and $T$ with same number of leaves has unbounded tree-width.

1-4) The graph $G(f)$ of a partial function $f : V \rightarrow V$ where $V$ is finite has vertex set $V$ and arcs $x \rightarrow f(x)$ for all $x \in V$. What is the maximum tree-width of such graphs? Prove that the set of all graphs $G(f)$ is HR- and also VR-equational.

1-5) For $f$ a bijection $V \rightarrow V$ where $V$ is of the form $\{1, ..., k\}$, we let $H(f)$ consist of $G(f)$ augmented with edges between $i$ and $i+1$ for every $i = 1, ..., k-1$. Prove that the set of all graphs $H(f)$ has unbounded tree-width.
2 Monadic second-order logic

2-1) For $H$ a finite simple loop-free undirected graph, show that the property "$G$
has a minor isomorphic to $H$" is MS expressible without edge set quantifications.
Conclude that planarity is MS expressible without edge set quantifications.

2-2) Let $C$ be an MS-definable class of graphs. Let $\text{Apex}(C)$ be the class of
graphs such that, by deleting one vertex, one obtains a graph in $C$. Show that
this class is MS-definable.

2-3) Prove that the class $\text{Apex}(\text{PLANAR})$ is minor-closed and that the
 corresponding set $K$ of minimal excluded minors is MS-definable. Do we obtain
from this fact an algorithm that enumerates the set $K$ which is finite up to
isomorphism by the Graph Minor Theorem (that is can one get one concrete
graph for each element of $K$)?

2-4) The rank of an undirected graph is defined as the rank over GF(2) of
its adjacency matrix. Show that this number is the common cardinality of the
sets $X$ of vertices that satisfy an MS-formula with free variable $X$ and written
with the even cardinality set predicate $\text{Even}(U)$.

2-5) Assume we augment MS logic with a set predicate $\text{Eq}(X,Y)$ expressing
that two sets $X$ and $Y$ have the same cardinality. Try to establish the
Fefermann-Vaught theorem and explain what does not work.

2-6) How to obtain disjoint union in the Fefermann-Vaught theorem?

2-7) Find classes of graphs on which a linear order of the set of vertices can
be defined by monadic second-order formulas. (That is, for each such class, there
exists a formula $\varphi(x, y, X, ..., Z)$ that defines a linear order $x < y$ on vertices,
with the help of auxiliary sets $X, ..., Z$ that have to satisfy another MS formula
$\theta(X, ..., Z)$. For each graph in the class, there must exist $X, ..., Z$ satisfying
$\theta(X, ..., Z)$.)

There are two cases, depending on whether one uses edge set quantifications.