A multivariate interlace polynomial

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General objectives :

Logical descriptions of graph polynomials

Application to their computations

Systematic construction of recursive definitions

Here : the multivariate *Interlace polynomial*

Full article on ArXiv and HAL.

Multivariate polynomials

Example: Sokal's multivariate version of Tutte's two-variable polynomial

For a graph *G=(V,E)* :

$$Z(G) = \sum_{A \subseteq E} u^{k(G[A])} \prod_{e \in A} x_e$$

where *G*[*A*] is the subgraph of *G* with set of vertices *V* and set of edges *A*,

k(G[A]) is the number of its connected components,

x_e is an indeterminate associated with each *e* in *E*,

u is an "ordinary indeterminate".

Multivariate polynomials not only count *configurations* (spanning trees, colorings,...) but they enumerate them (with associated values).

Logical view : typical form :

$$P(G) = \sum_{\varphi(A,B)} x_A y_B u^{f(A,B)}$$

A,B sets of vertices or edges
satisfying condition $\varphi(A,B)$
f(*A,B*) nonnegative integer function of *A,B*,
x_A and y_B denote

$$\prod_{a\in A} x_a, \quad \prod_{a\in B} y_a$$

Logical expression of Condition φ

Second-order logic (SO) : very (too) powerful

= First-order logic with set and relation quantifications.

Monadic second-order logic (MS) : quite powerful, very good algorithmic properties,

= First-order logic with set quantifications (only).

Typical properties :

MS : X is a set of edges defining a spanning forest, or X is a set of vertices inducing a connected component.

SO not MS : two sets X and Y are in bijection.

Example of MS formula :

The induced subgraph G[X] is not connected

$$\exists Y \bullet [\exists x \bullet (x \in X \wedge x \in Y) \wedge \exists y (y \in X \wedge y \notin Y) \wedge$$
$$\forall x, y \bullet (x \in X \wedge y \in X \wedge A(x,y)$$
$$\Rightarrow \{(x \in Y \wedge y \in Y) \vee (x \notin Y \wedge y \notin Y)\})].$$

where A(x,y) is the adjacency relation.

Logical expression of functions with values in ***N*** in

$$P(G) = \sum_{\varphi(A,B)} x_A y_B u^{f(A,B)}$$

If f(A,B) = | C | for a unique set C satisfying $\psi(A,B,C)$

then $P = P'[\, u_c \leftarrow u \,]$ (substitution) where

$$P'(G) = \sum_{\psi(A,B,C)} x_A y_B u_C$$

Example (for Sokal's polynomial) :

$k(G[A])$ = number of connected components is defined as $|C|$ where :

$\psi(A,C) \Leftrightarrow C$ is the set of vertices which are minimal in each connected component of the graph (V,A) ; "minimal" is with respect to an arbitrary linear ordering of V. (sets C depend on this order but not $|C|$).

The polynomial $Z(G)$ can thus be rewritten as :

$$Z(G) = \sum_{A \subseteq E} x_A u^{k(G[A])}$$

$$= (\sum_{\psi(A,C)} x_A u_C)[u_c := u \quad \text{all} \quad c \in V]$$

Why Monadic Second-order logic ?

Polynomials and *truncations of multivariate* MS-definable polynomials can be computed in polynomial time for input graphs of bounded tree-width or clique-width.

This result covers such results for (nonmultivariate) :

1) Tutte polynomial for graphs of bounded tree-width (Noble, Andrzejak, 1998)

2) Interlace polynomial for distance-hereditary graphs (of clique-width 3) (Ellis-Monagham,Sarmiento, 2006)

Interlace polynomials

Several (incompatible) *interlace* polynomials by : Arratia, Bollobas and Sorkin (2 articles in 2004), and Aigner and van der Holst (2004).

Motivations :

- Counting partitions into k circuits (Las Vergnas, Martin)
- Counting the number of Eulerian trails in 4-regular graphs
- Related to Kauffman's polynomial of link diagrams.

This communication :

- a common multivariate interlace polynomial,
- generalizing all the existing ones,
- MS-definable (hence truncations are poly-time computable for graphs of bounded clique-width),
- with a recursive definition from which the existing ones can be established.

Only bounded portions can be evaluated in polynomial time. The full computation is necessarily exponential because of the size of the result.

Description of the new animal (for Janos's Zoo).

Configurations : sets A of vertices ;

Their values based on the *rank* of the induced subgraph.

To generalize the polynomial by Aigner, we put also in a configuration a set B of vertices for "toggling loops": if a vertex in B has no loop, we add one ; if it has a loop, we delete it, giving graph $G\nabla B$

Ranks of adjacency matrices are over GF(2).

The definition

$$B(G) = \sum_{A \cap B = \varnothing} x_A y_B u^{rk(G \nabla B[A \cup B])} v^{n(G \nabla B[A \cup B])}$$

where *A,B* range over subsets of *V* (vertices)

rk(H) denotes the rank of a graph H,

n(H)= $|V|$ - *rk(H)* denotes the *nullity*.

The existing interlace polynomials are specializations:

$$q(G,x,y) = \sum_{A \subseteq E} (x-1)^{rk(G[A])} (y-1)^{n(G[A])}$$

= $\sigma(B(G))$ where σ is the substitution

$$[u := x-1; v := y-1; x_a := 1, y_a := 0 \quad \text{all} \quad a \in V],$$

$$Q(G,x) = \sum_{A \cap B = \varnothing} (x-2)^{n(G \nabla B[A \cup B])}$$

idem with $\sigma = [u := 1; v := x-2; x_a := y_a := 1$ all $a \in V]$.

B(G) is MS-definable

Because $rk(G[A]) = |C|$ if C is a maximal subset of A with associated row vectors in the adjacency matrix of $G[A]$ that are independent.

This is expressible in MS logic augmented with the Even cardinality set predicate : *Even(X)*

This extension satisfies all good algorithmic properties of MS logic.

The recursive definition : For every vertex *a* :

$$B(\varnothing) = 1$$

$$B(G) = (1 + z_a v + w_a u)B(G - a) \quad \text{if} \quad N(G,a) = \varnothing \ ,$$

$$B(G) = z_b u^2 \{z_a B(G^{ab} - a - b) + w_a B((G^a)^b - a - b)\}$$
$$+ w_b u\{B(G^b - b) - B(G^b - a - b)\}$$
$$+ B(G - a) + B(G - b) - B(G - a - b).$$

if $b \in N(G,a)$ (set of neighbours of *a*).

We use "metavariables", for each *c* in *V* :

$z_c = x_c$ and $w_c = y_c$ if *c* is not a loop,

$z_c = y_c$ and $w_c = x_c$ if *c* is a loop.

This definition involves the following graph transformations :

G^a : local complementation at a : toggles edges and loops in the neighbourhood of a ;

G^{ab} : pivoting edge ab: toggles edges between $N(G,a)$-$N(G,b)$, $N(G,b)$-$N(G,a)$ and $N(G,a) \cap N(G,b)$;

G-a : deletion of vertex a

Its determination is based on the Lemma (ABS+C):

(1) $rk(G) = 1 + rk(G^a - a)$ if $a \in Loops(G)$;

(2) $rk(G) = 2 + rk(G^{ab} - a - b)$ if $a - b$;

(3) $rk(G - a) = rk(G^{ab} - a)$ if $a - b$;

(4) $rk(G) = 2 + rk((G^a)^b - a - b) = 1 + rk(G^{ab} - b)$ if $a^{\ell} - b$.

We can derive the known recursive definitions, sometimes with the necessity of proving nontrivial properties.

Property: For every graph *G* and vertex *a* the polynomial $B(G) - B(G - a)$ has positive coefficients.

Recursive definitions of q(G,x,y) and Q(G,x)

(q1) $q(G) = y^n$ if G = n isolated non-looped vertices,

(q2) $q(G) = (x-1)q(G^a - a) + q(G - a)$ if *a* loop, isolated or not,

(q3) $$q(G) = (x-1)^2 q(G^{ab} - a - b) + \\ + q(G - a) + q(G - b) - q(G - a - b)$$ if *a-b*

It is not the one of [ABS].; they replace (q3) by :

(q'3) $$q(G) = x(x-2)q(G^{ab} - a - b) + \\ + q(G - a) + q(G^{ab} - b)$$

(Q1) $Q(\varnothing) = 1$

(Q2) $Q(G) = x \cdot Q(G - a)$ if $N(G,a) = \varnothing$

(Q3) $$Q(G) = Q(G^{ab} - a - b) + Q((G^a)^b - a - b) \\ + Q(G^b - b) - Q(G^b - a - b) \\ + Q(G - a) + Q(G - b) - Q(G - a - b)$$ if $b \in N(G,a)$

(Q'3) $Q(G) = Q(G - b) + Q(G * b - b) + Q(G^{ab} - a)$ if $b \in N(G,a)$

where $G * b = G^b \nabla N(G,b)$

Computing B(G) and multivariate polynomials

Such polynomials have exponential size.

We may ask for a polynomial time computation (not an evaluation for particular values) of the

d-*truncation* of *B(G),* i.e., of its part limited to the monomials of *quasi-degree* at most d (number of distinct *indexed indeterminates*; indexed by vertices or edges).

Because *B(G)* is MS-definable, this is possible for graphs of bounded clique-width. This includes cographs and distance-hereditary graphs.

Theorem : For each k, the following polynomials are computable for graphs G of clique-width < k :

d-truncation of B(G) in time $O(n^{3d+O(1)})$,

q(G) in time $O(n^7)$,

Q(G) in time $O(n^3)$,

n = number of vertices.

The basic lemma (simplified formulation) :

For every MS formula ξ there exists a finite subset Φ of MS formulas containing ξ such that :

(1) For every φ in Φ, for every unary operation *op* there exists a formula φ^{op} such that, for every graph *G*, for every substitution σ :

$$(\sigma \circ P_{\varphi})(op(G)) = (\sigma \circ P_{\varphi^{op}})(G).$$

(2) For every φ in Φ, there exists

$$(\theta_1, \ldots, \theta_p, \psi_1, \ldots, \psi_p) \in \Phi^{2p}$$

such that for every disjoint graphs *G* and *H*, for every substitution σ :

$$(\sigma \circ P_{\varphi})(G \overrightarrow{\oplus} H) = \sum_{1 \leq i \leq p} (\sigma \circ P_{\theta_i})(G) \bullet (\sigma \circ P_{\psi_i})(H).$$

Remark :

In case of truncations we also have :

$$\sigma(P_{\varphi}(G\overrightarrow{\oplus}H)) \restriction d = \sum_{1\leq i\leq p}((\sigma(P_{\theta_i}(G)) \restriction d) \bullet (\sigma(P_{\psi_i}(H)) \restriction d)) \restriction d.$$

Consequence : One can compute simultaneously the finitely many polynomials $\sigma(P_{\varphi}(G)) \restriction d$ by induction on the structure of a term that describes G by using the following operations :

ordered disjoint union : $\overrightarrow{\oplus}$

finitely many unary graph operations expressible without quantifiers.

This applies to graphs of clique-width at most *k*.

Clique-width

A graph complexity measure like tree-width that yields many Fixed Parameter Tractable algorithms, especially for problems expressed in Monadic Second-order logic.

Graphs have vertices labelled in {1,2,…,k}.

(1) For each *i* in {1,2,…,k}, constants $\mathbf{i}$ and $\mathbf{i}^{loop}$ denote isolated vertices labelled by *i*, the second one with a loop.
(2) For *i,j* in {1,2,…,k}, the unary function $add_{i,j}$ adds undirected edges between any two vertices, one labelled by *i*, the other by *j*.
(3) The unary function $ren_{i,j}$ relabels *i* into *j*.

(4) The binary operation $\oplus$ makes the union of disjoint copies of its arguments.

The clique-width of a graph is the minimal k such that this graph can be constructed with these operations.

Cliques have clique-width 2 and unbounded tree-width.

The unary operations are quantifier-free definable.

A variant is used where graphs are linearly ordered and the disjoint union $\overrightarrow{\oplus}$ concatenates the linear orders.

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