



# Canonical Graph Decompositions

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*References* : BC+JE: **The Great Book of MSO**, Cambridge University Press, May 2012

BC: MSOL of graphs: X, XI, XII, XVI, *Theoret. Comp. Sci. and Logical Meth. in Comp. Sci.*  
(1996-2006)

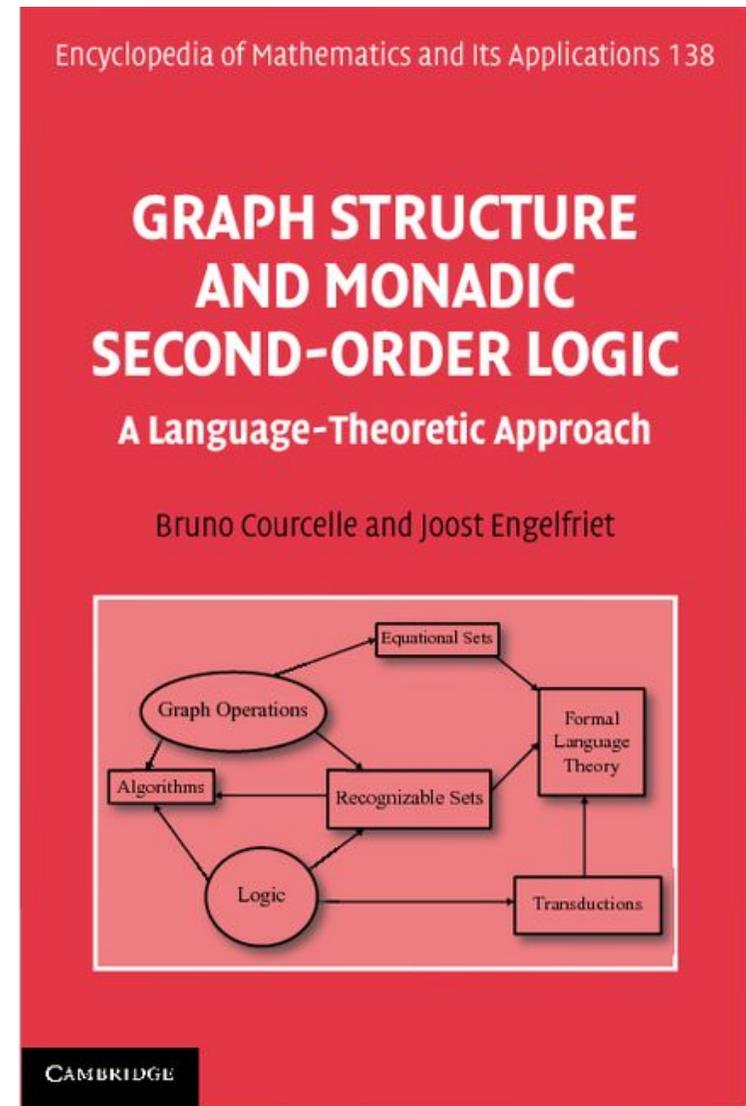
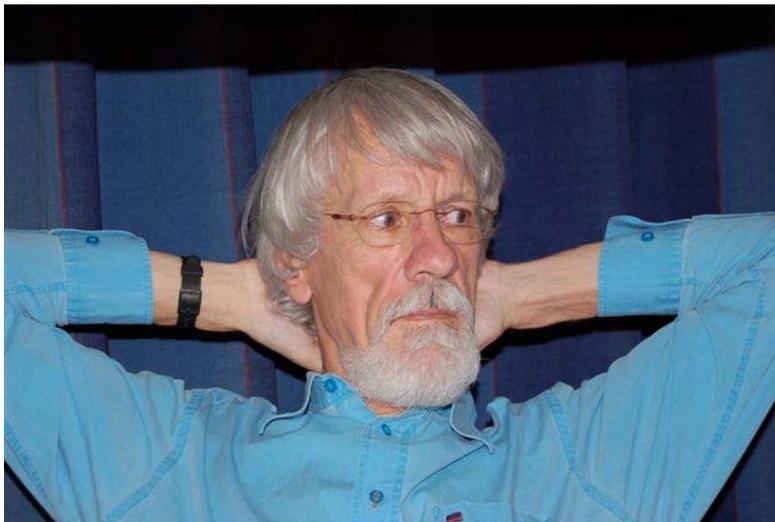
BC : Circle graphs and MSOL, *J. of Applied Logic*, 2008

BC +AB : MSO definable graph orderings, 2011, submitted.

BC: The atomic decomposition of strongly connected graphs and application to Gauss words,  
2012, submitted.

Other title : **The missing chapter**

Perhaps Chapter 10 of the second edition.



Dear Participants of Bruno's Workshop,

I was surprised, impressed and inspired when I heard, many years ago, that Bruno had generalized Büchi's Theorem from strings to graphs.

*Surprised* because I had not expected such a result to be possible.

*Impressed* because the proof was ingenious and elegant.

*Inspired* because it initiated a fascinating development.

So, in the past few years, it has been a great pleasure for me to work together with Bruno on a book (**The Book**) with this and related results as subject matter.

During this work we never met, but we exchanged 576 emails, by a rough count of the ones I kept. Since the work is now finished, these mails can be thrown in the dustbin, together with all previous versions of **The Book**.

But in case Bruno is planning "**The Book, Part Two**", everything will start again...

Unfortunately I cannot be present at the workshop. From here I greet all my old friends, and I wish all participants a wonderful workshop.

Joost Engelfriet

Graph structure is a broad concept, covering

*Hierarchical graph decompositions*

And also (Robertson and Seymour) :

*Embeddings in surfaces*

*Existence of a substructure of a certain type*

(subgraph, minor, vertex-minor)

*Orientations or orderings* with particular properties.

*Colorings* (many notions)

*Many links between these different types.*

**MSO** (Monadic Second-Order) logic is useful:

to express **properties** of graphs or relational structures,

to express **queries** (sets of tuples satisfying a formula with free variables) in relational structures,

to express **transformations** of relational structures by

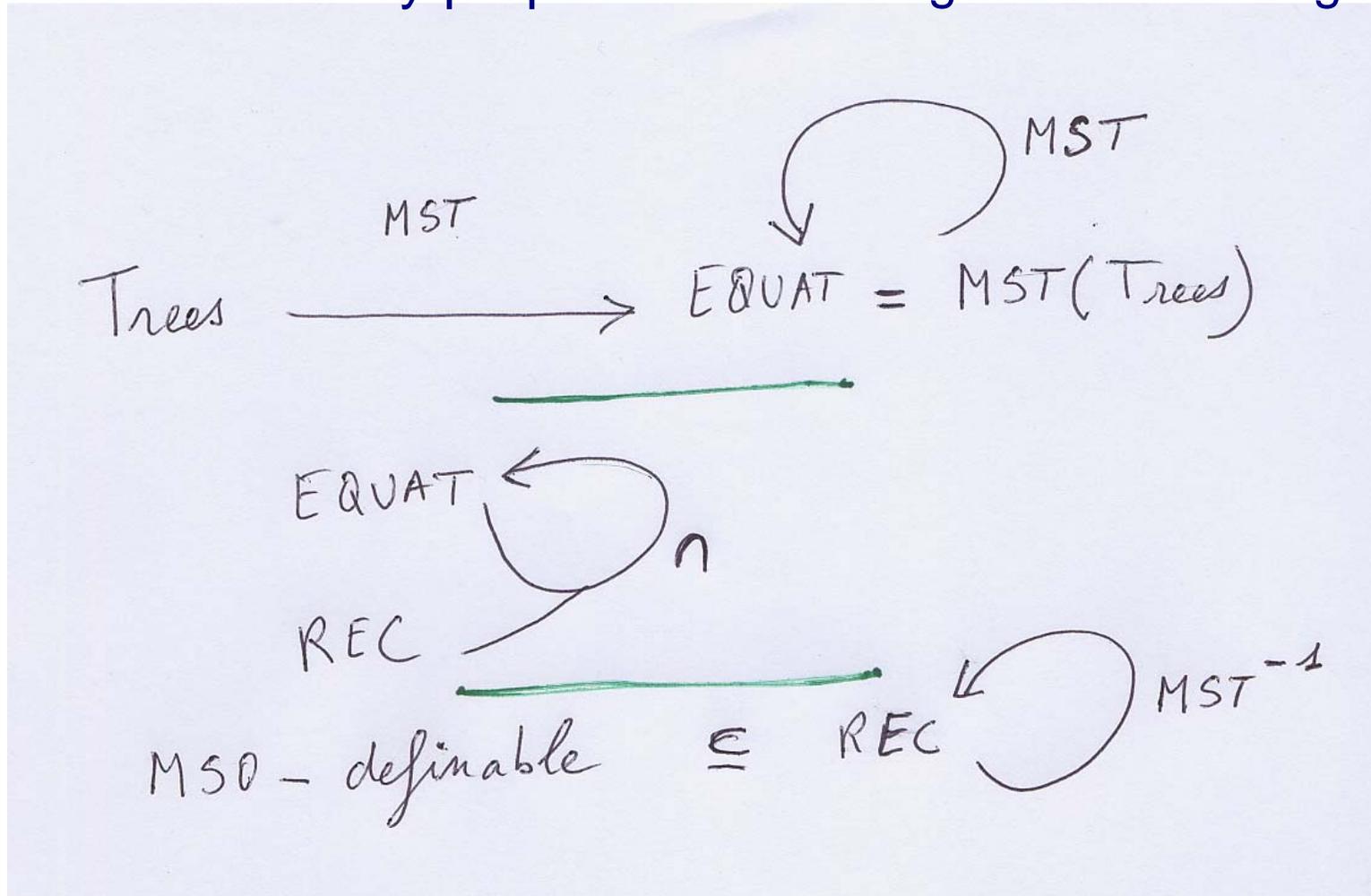
**MSTs** : monadic second-order transductions.

*Example:* **MSO** expression of planarity via Kuratowsky;

*Better:* construction by an **MST** of the unique plane embedding of a 3-connected planar graph.

# Language Theoretic Aspects

**Equational** (“context-free”), **recognizable** sets of graphs and **MST** are related by properties extending the case of languages:



# Graph structuring by Monadic Second-order Transductions

Can one define by an MST:

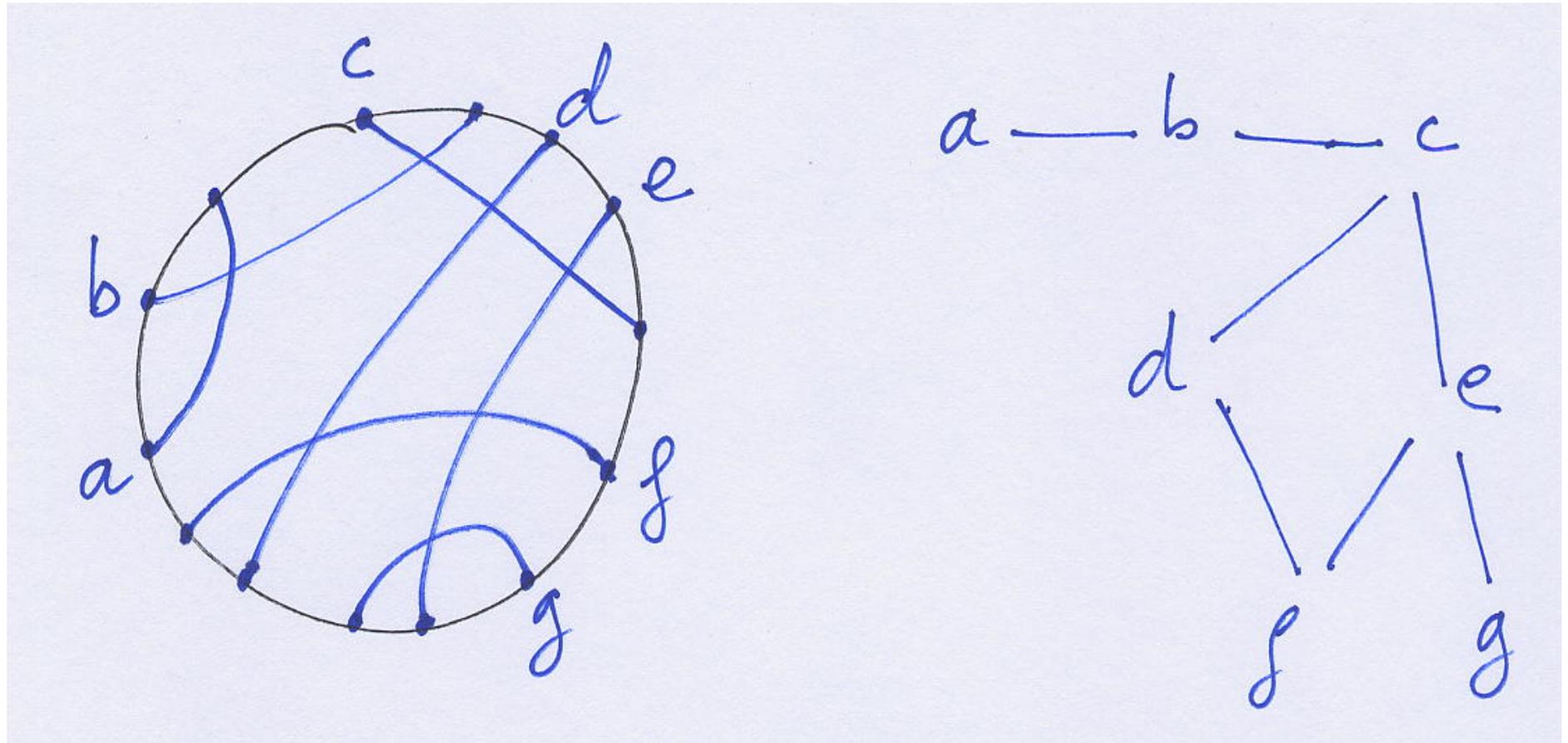
All (or only some) orientations of a graph ? (depends)

Some linear order ? (depends, A.Blumensath+B.C.)

Some planar embedding of a planar graph ? (yes, if ordered)

Some chord diagram representing a circle graph?

(yes, if the graph is linearly ordered)



Can one define *hierarchical decompositions* by MSTs:

Some tree-decomposition of width  $k$  ? (yes, if  $k \leq 3$  ; other  $k$ ?)

Some clique-width expression or rank-decomposition of width at most  $k$  ? (yes, for ordered cographs ; other cases??)

*Canonical decompositions :*

The decomposition in 3-connected components (Tutte 1984) ? (yes)

The modular decomposition (Gallai 1967) ? (yes, if lin. ordered)

The split decomposition (Cunningham 1980) ? (yes, if lin. ordered)

## *Observations:*

1. Canonical decompositions are **easier** to define by MSTs than tree-decompositions, for which some choices must be made, that are not (easily) MSO expressible.
2. Several constructions use (**need ?**) a linear order on the given graph. (**Why ?**)
3. Some of these constructions extend to countable graphs.

*Application* : Construction by an MST of planar embeddings ?

- $G \rightarrow$  *the* decomposition in 3-connected (planar) components, “bonds” (several parallel edges) and cycles;
- 3-connected planar graphs and cycles have **unique** planar embeddings, that are constructible by MSTs;
- the *linear order* yields planar embedding of bonds;
- the embeddings of the components can be combined into one of the given graph.

The *very same* schema can be used for constructing :

- All **transitive orientations** of a **comparability** graph (using the *modular* decomposition)
- Some **chord diagram** for a **circle** graph (using the *split* decomposition)

Unicity properties are used twice : for building the decompositions and for constructing the objects attached to the “prime” components.

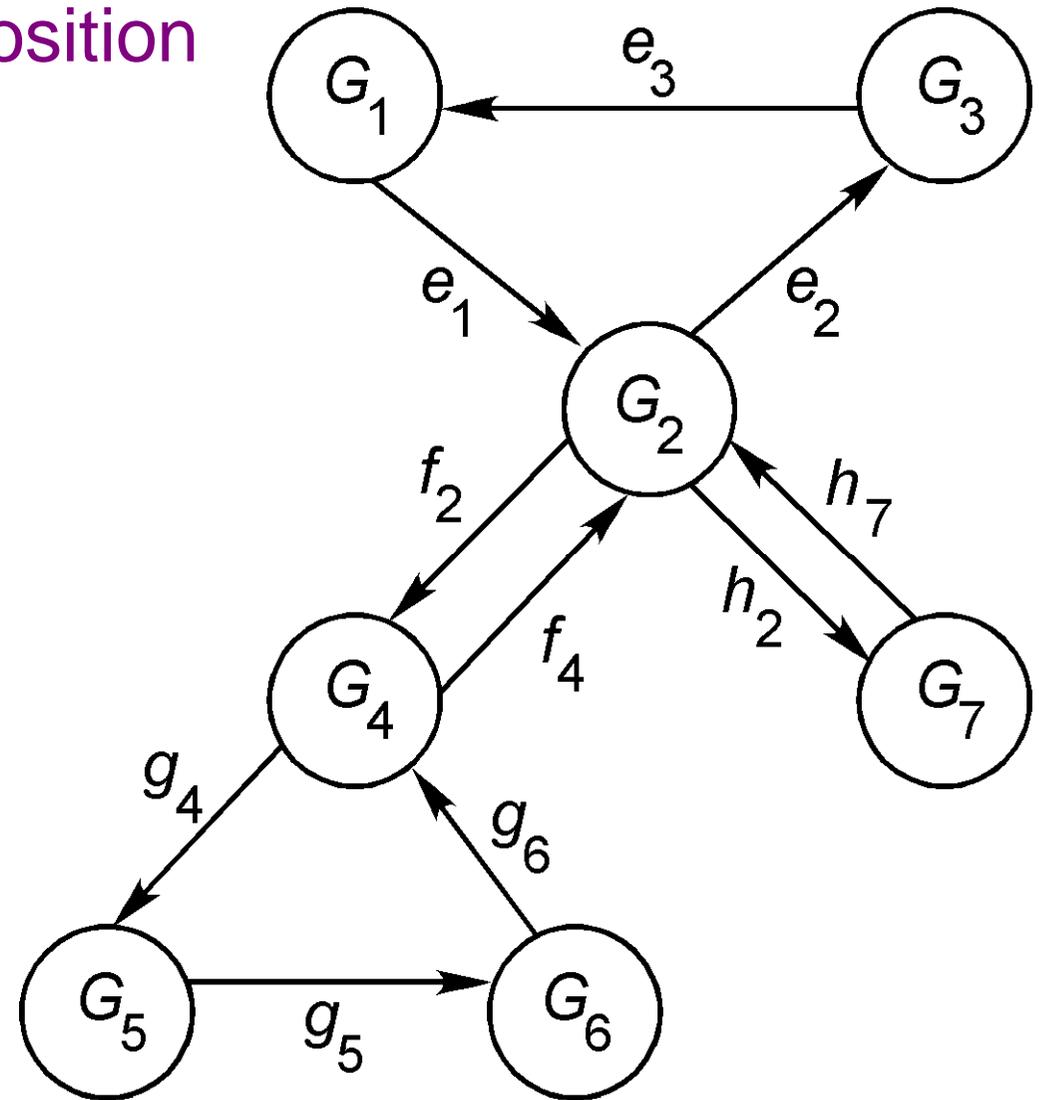
A new (I think!) canonical graph decomposition:

The atomic decomposition  
of a strongly  
connected graph

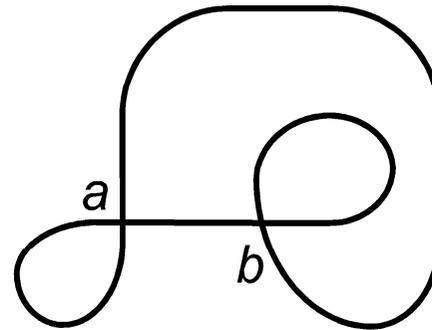
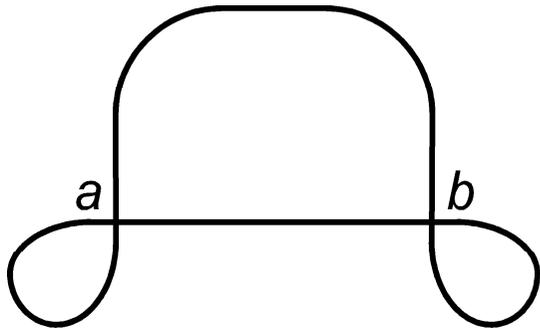
Quick look :

The subgraphs

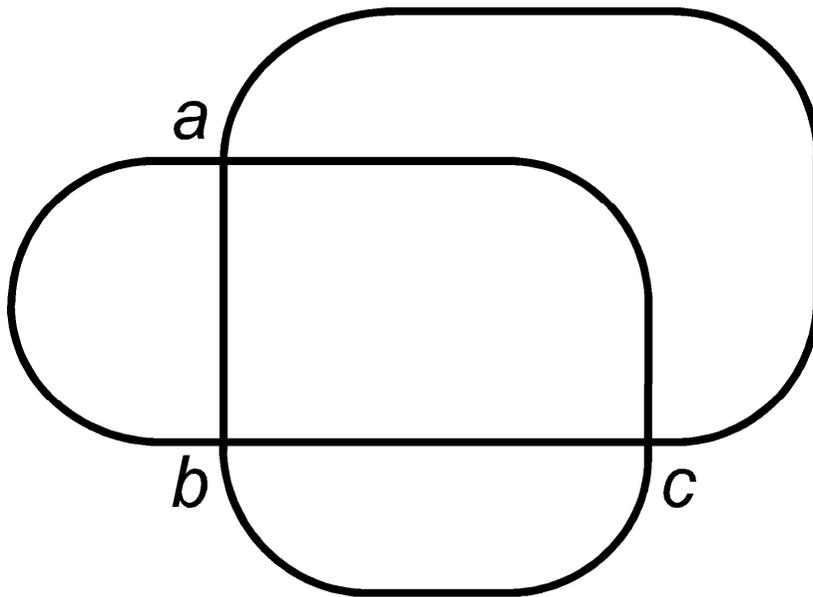
$G_i$  are its *atoms*.



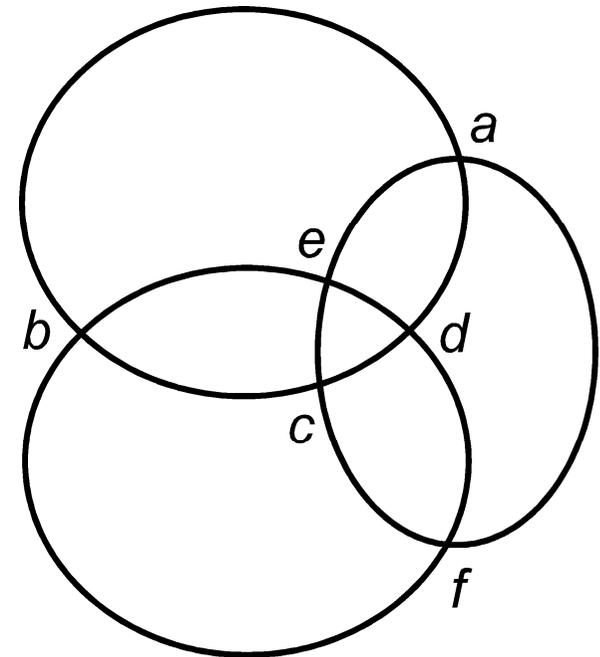
Motivation : Gauss words describing intersections of curves



*word* : aabb

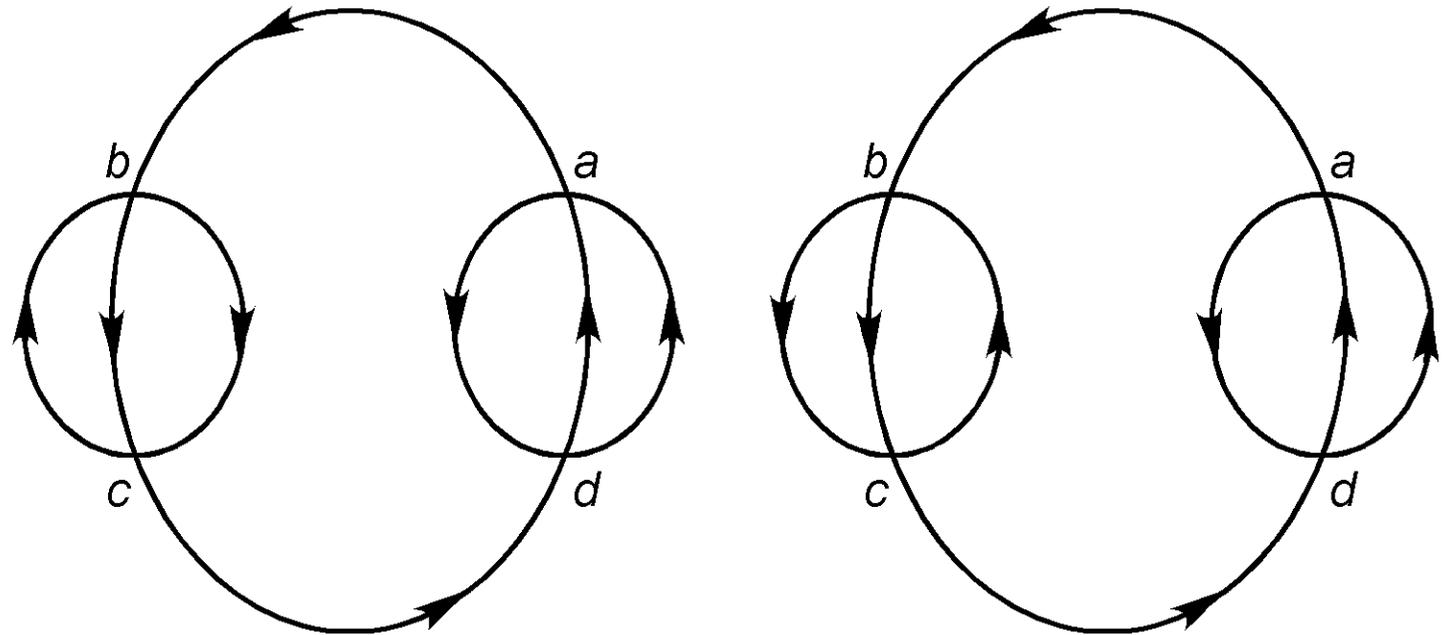


*word* : abcabc



*multiword* : (abcd, bfde, aecf )

**Question :** Can one describe by an MST all (tuples of) curves (up to homeomorphism) corresponding to a given Gauss (multi)word ?

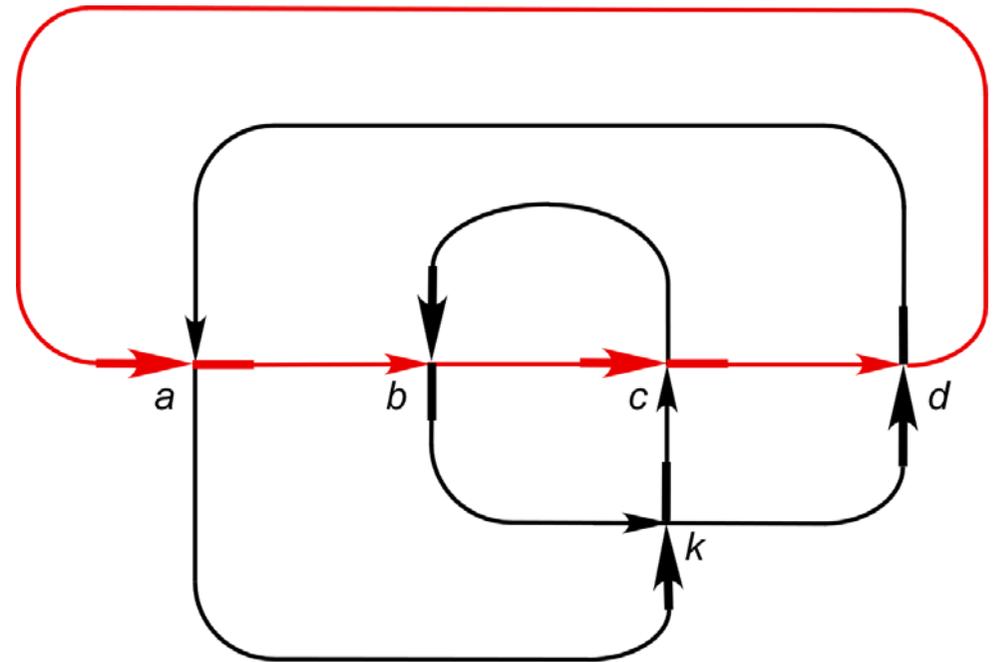


An **ambiguous** multiword  $(abcd, bc, ad)$  (the circle “bc” is “flipped”).  
It is **not** ambiguous for *nonoriented* curves (invariance under flipping).

*Method*: Multiword  $W$  yields (2,2)-regular graph  $\text{Gra}(W)$  with transitions.

*Ex.:*  $\text{Gra}(Z)$  for  
 $Z = (abcd, akcbkd)$

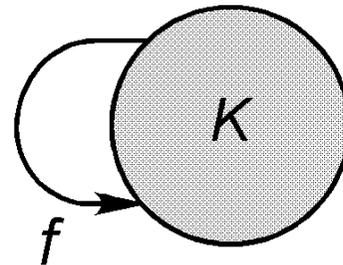
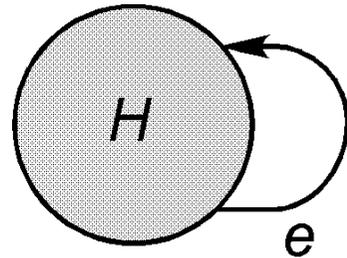
What are the planar embeddings of  $\text{Gra}(W)$  that respect its transitions ?



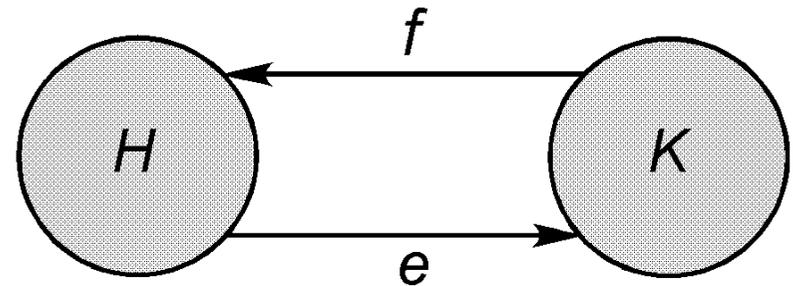
Planarity of (2,2)-regular  $t$ -graphs ( $t$ - means “with transitions”) can be checked in **linear time**.

(Easy reduction to the usual planarity test).

## Circular composition of graphs, t-graphs and maps



$$G = H \boxplus_{e,f} K$$



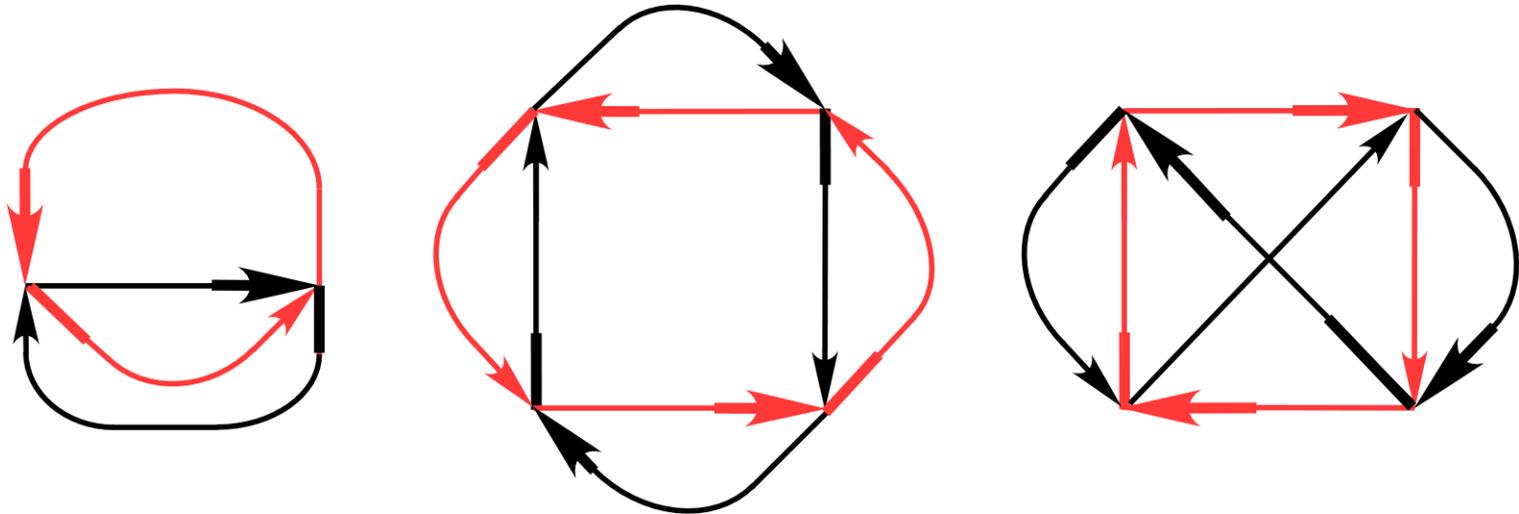
**Atoms** : those that cannot be decomposed.

A planar **t**-atom has a **unique** planar embedding.

By “flipping the embedding of  $K$ ”, one gets a different embedding of the **t**-graph  $G$ .

(Flipping is like turning over a pancake.)

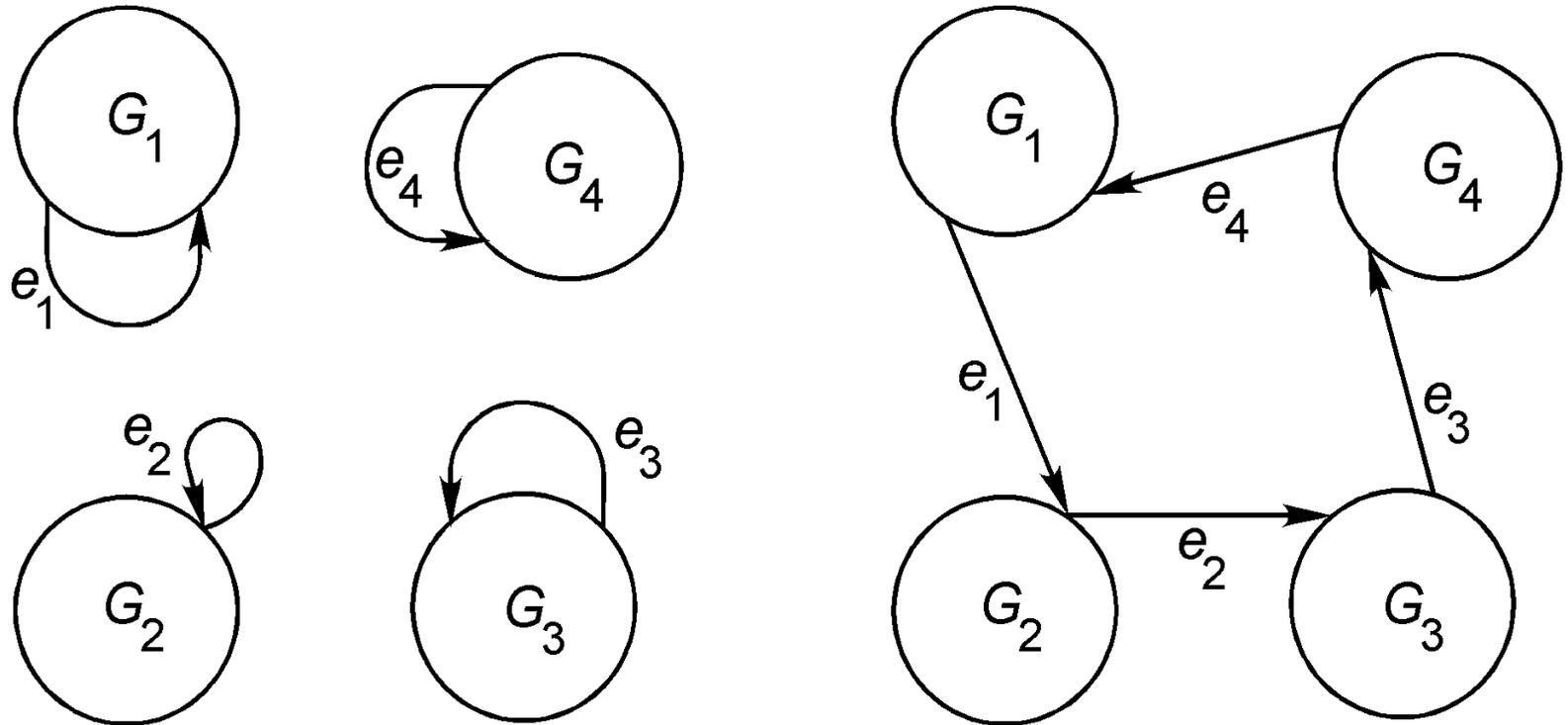
Examples of *t*-atoms (with transitions):



not planar

*Circular composition* satisfies some laws (*circular associativity*).

Graphs and *not trees*, represent “normal forms” of terms.



Circular composition  $\boxplus_{e_1, e_2, e_3, e_4} (G_1, G_2, G_3, G_4)$  defined as

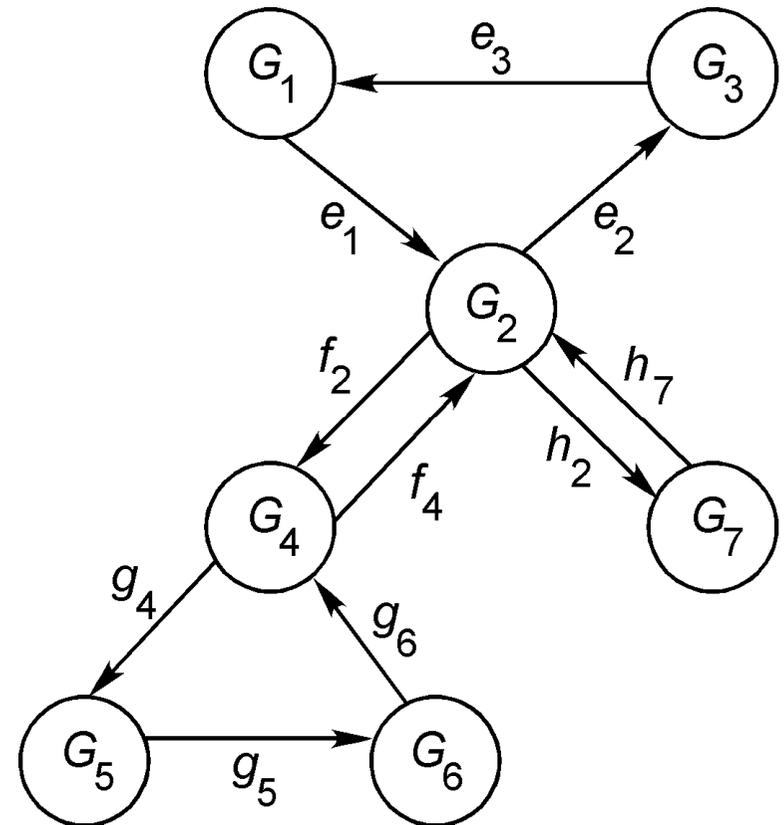
$$((G_1 \boxplus_{e_1, e_2} G_2) \boxplus_{e_2, e_3} G_3) \boxplus_{e_3, e_4} G_4$$

Two vertices belong to an atom  $\leftrightarrow$  they are not **separated** by an “edge-cut” of 2 edges.

This equivalence is **MSO expressible**.

The quotient graph is a **cactus** :

It defines the **atomic decomposition**.

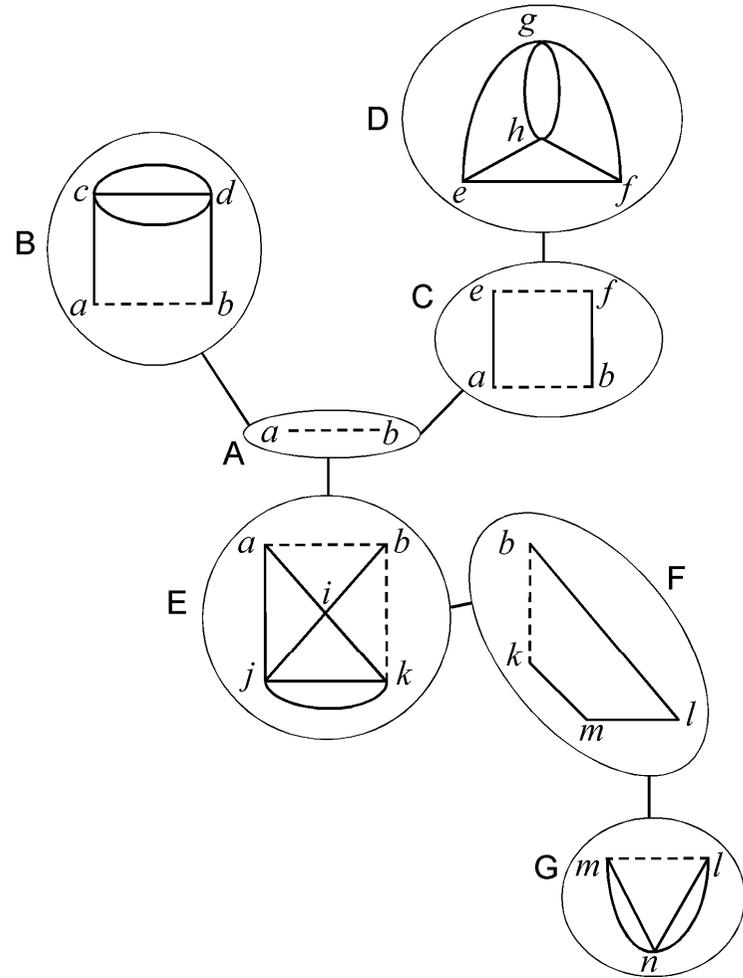
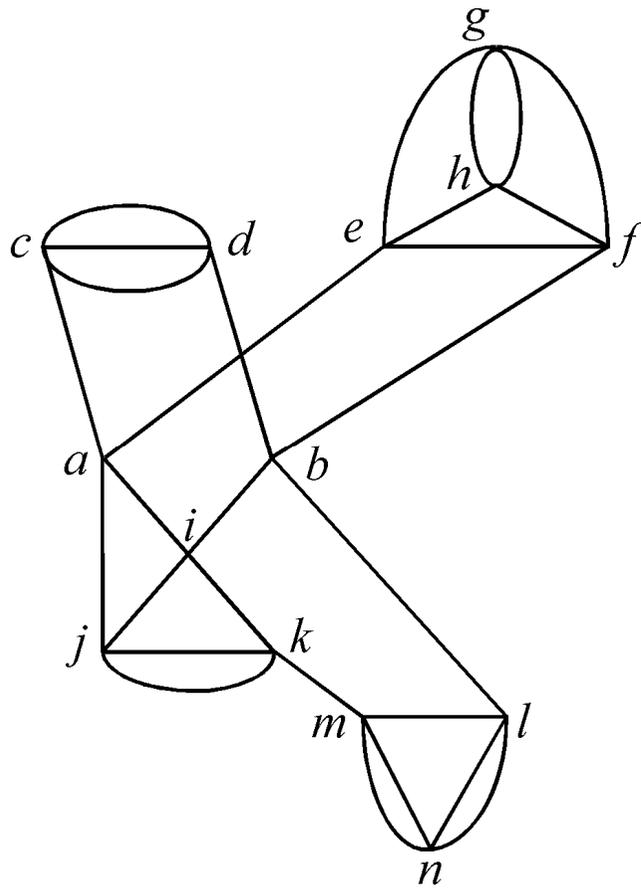


**Cactus** : the 2-connected components are circuits

$\leftrightarrow$  for all  $x,y$ , there is a **unique path** from  $x$  to  $y$

(the directed version of one characterization of undirected trees).

The Tutte decomposition of a 2-connected graph:



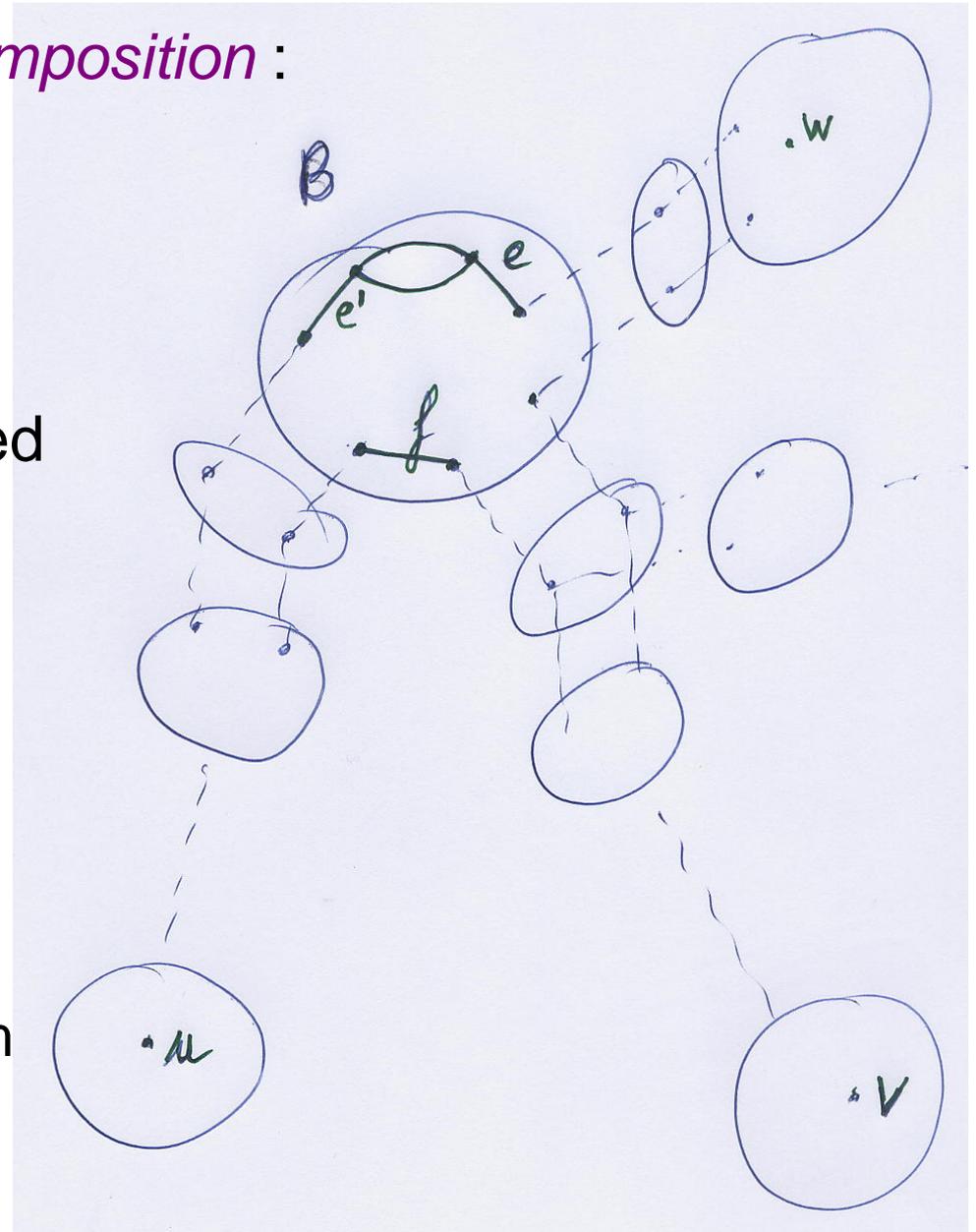
The *blocks* have vertex sets  $\{c, d\}$ ,  $\{a, b, i, j, k\}$ ,  $\{e, f, g, h\}$ ,  $\{l, m, n\}$ .

*Atoms from the Tutte decomposition :*

- u and v are **separated** at block **B** by  $\{ e, f \}$
- v and w are **not** separated at block **B** by  $\{ e, f \}$

(An atom is **not** a subgraph of a block in general.)

The atomic decomposition can be computed in linear time from the Tutte decomposition.

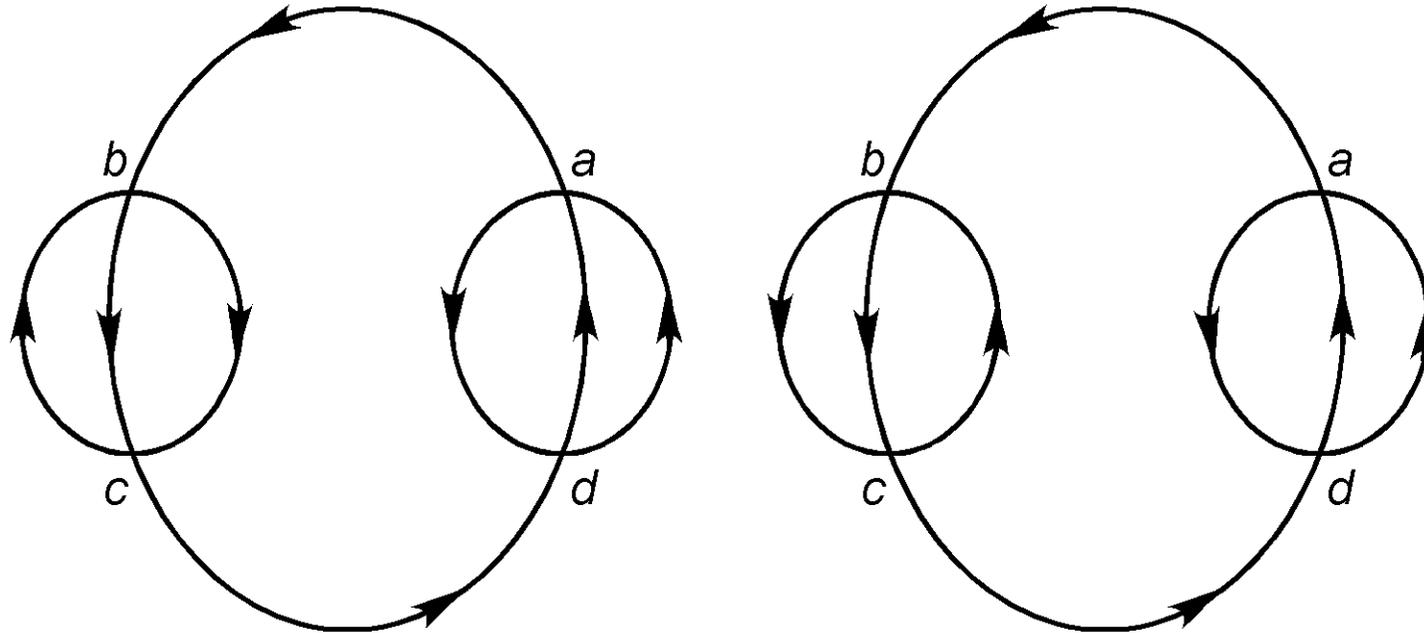


## Application to Gauss multiwords

- Given a multiword  $W$ , construct  $\text{Gra}(W)$ ;
- Check if planar; check if a  $t$ -atom;
- If planar and not atom, build the atomic decomposition  
(using the Tutte decomposition);
- Build the unique planar embedding of each  $t$ -atom;
- By flipping each of the  $p$   $t$ -atoms, one gets the  $2^{p-1}$   
different planar embeddings of  $\text{Gra}(W)$ .

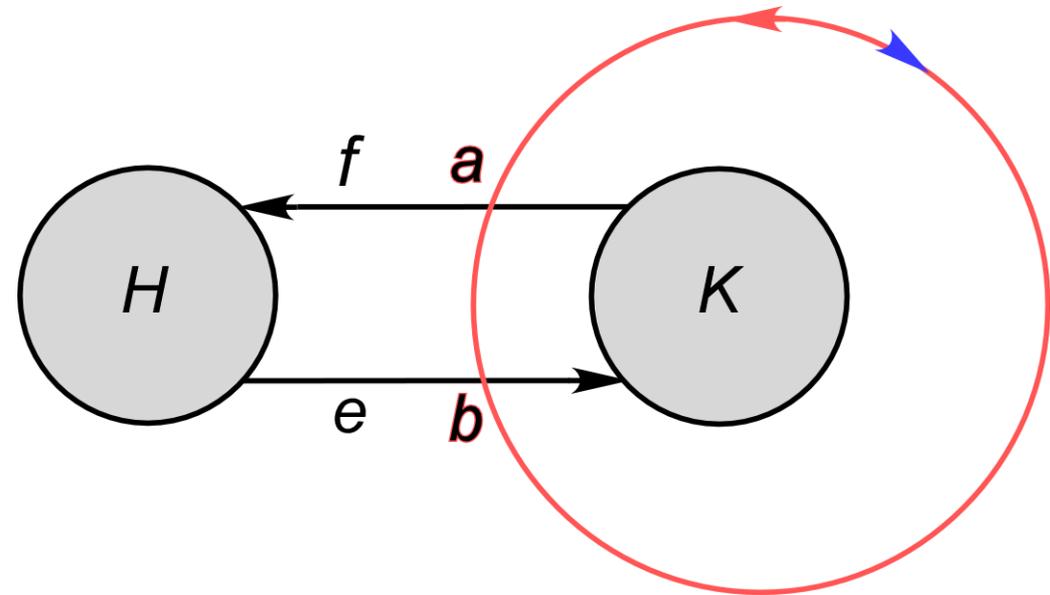
All these tasks can be done by **MSO** formulas and **MSTs** and by **linear time** algorithms.

Gauss multiwords representing tuples of **nonoriented** curves



The multiword  $(abcd, bc, ad)$  is ambiguous for oriented curves but not for nonoriented ones. The problem comes (*only*) from **small curves**: those with 2 intersections.

Both orientations of the **red circle** give the same multiword.



$$W = (\mathbf{ab}, w_1, \dots, w_n) \rightarrow W' = (w'_1, \dots, w'_n) \quad (\text{we remove } \mathbf{a,b})$$



$\mathbb{N}$  tuples of nonoriented  $\leftarrow$   $\mathbb{N}$  planar embeddings of  $\text{Und}(\text{Gra}(W'))$   
 curves for  $W$ .

## General case for nonoriented curves

**Theorem:** Let  $W$  be a Gauss multiword of the form

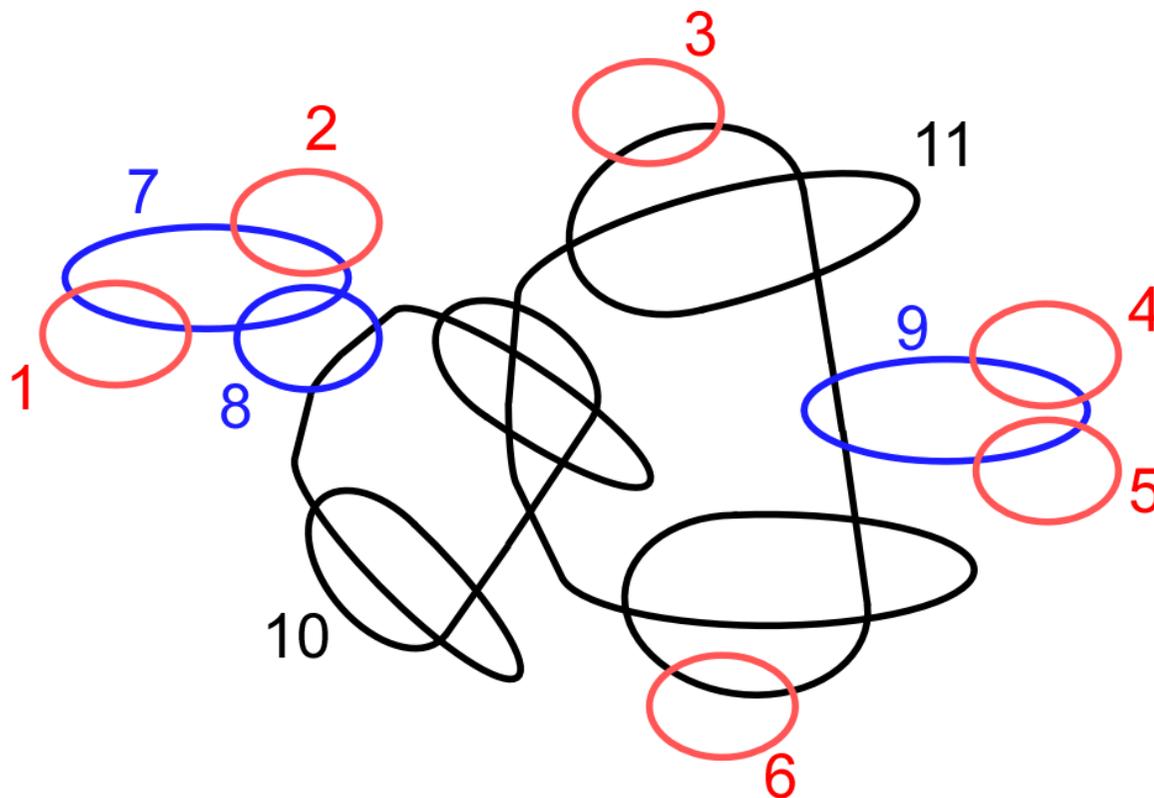
$W = (ab, \dots, cd, w_1, \dots, w_n) \rightarrow W' = (w'_1, \dots, w'_n)$  (removing  $a, b, \dots, c, d$ ),

$w_1, \dots, w_n$  have length  $\geq 3$  (actually  $\geq 4$  by planarity),

$W$  has  $q$  *small components* :  $ab, \dots, cd$ ,

$\text{Gra}(W')$  has  $p$  atoms,

$\rightarrow$  there are  $2^{p-1}$   $(q+n)$ -tuples of nonoriented curves with Gauss multiword  $W$ .



There are 6 small components (in red).

The remaining curves (in black and blue) represent

$\text{Gra}(W')$  that has 7 atoms  $\rightarrow$  64 different 11-tuples of curves

## Extensions and open questions

Curves with **multiple intersections**,

**Knot** diagrams,

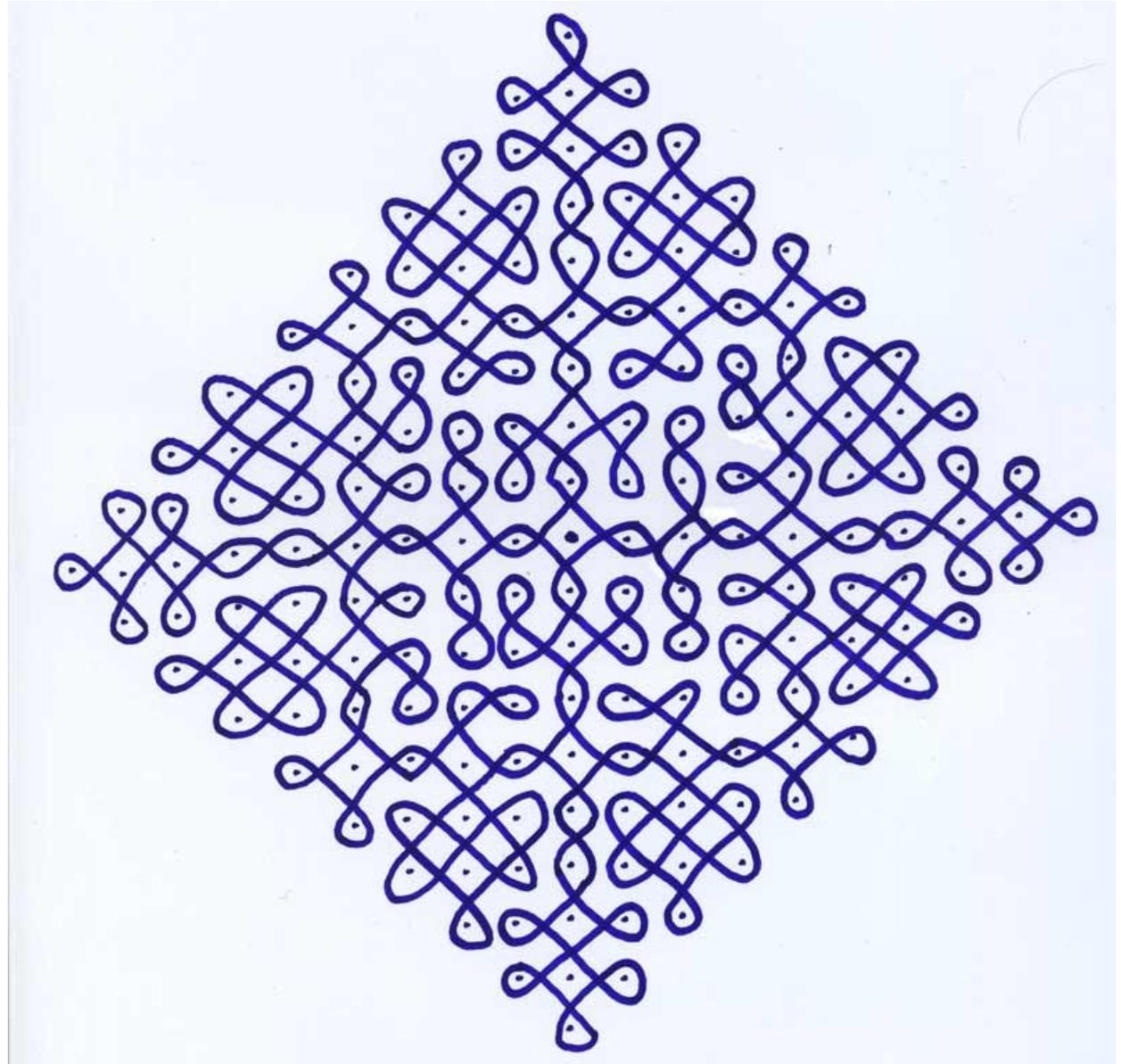
Decompositions based on circular composition  
of **directed graphs**,

Gauss multiwords on other **surfaces**,

**Forbidden** configurations (minors ?) for planar  
4-regular graphs with transitions,

**Applications to be developed ? Yes !**

Grammar for  
Gauss words  
of **kolams** ?



## A wooden knot diagram

