



Undecidable Extensions of Monadic Second-Order Logic
over Words Using Cardinality Set Predicates
relative to Recursive Sets.

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References : B. C. and J. Engelfriet : *Graph structure and monadic second-order logic, A language theoretic approach*, book to be published by Cambridge University Press,

Readable on: <http://www.labri.fr/perso/courcell/ActSci.html>

Monadic Second-order Logic is a favourite language to many researchers because :

1) It is **decidable** on many infinite structures and classes of finite and infinite structures;

2) On finite structures, there are **fixed-parameter tractable** model-checking algorithms on words, trees and graphs for structural parameters : tree-width, clique-width, rank-width;

Also:

3) MSOL **replaces finite automata and transducers** in the extension to graphs of formal language theory.

For a class of finite graphs and some logic, 2 problems :
 Decidability ? Time complexity of model-checking ?

	<i>Decidability</i>	<i>Model-checking</i>
FO, all graphs	Undecidable	Polynomial-time
MSOL2, tree-width < k	Decidable	Linear-time
MSOL2, unbdd tree-wd.	Undecidable	Not FPT (*)

MSOL2 = MSOL over incidence graphs, allows edge set quantification.

(*) by S. Kreutzer, LICS 2010: exact statement is very technical.

Remarks: The positive results (decidability, linear-time) are based on **finite automata**. The negative results for MSOL2 are based on **large grid minors**. However, there is no implication between them.

How to improve the positive results ?

No possible improvement based on waiving the conditions on tree-width for MSOL2. (But possible with *bounded clique-width* for MSOL without edge set quantifications. Nothing new for words and trees.)

Keeping bounded tree-width, can one enrich the language?

Cardinality set predicates

If R is a recursive set of integers, we let $\text{Card}_R(X)$ be the set predicate that means :

the cardinality of X belongs to R .

Notation : language $\text{MSOL}+R$.

Validity is decidable on a single finite structure.

The constructions of finite automata for decidability and model-checking *only work* if R is a **semi-linear set** (e.g., even numbers; more generally “equivalent to p modulo q if $> r$ ”).

Proposition: The satisfiability problem for MSOL+R is undecidable on words if $R = \{2^n, 3^n / n > 0\}$.

Open questions :

Cases of $R = \{2^n / n > 0\}$ or $R =$ prime numbers.

Method : Using grids, as for the result for MSOL2.

Fact: If a set of graphs L contains square grids $G_{n \times n}$ for infinitely many integers n , then the satisfiability problem for MSOL is **undecidable** on L .

Proof : A finite computation sequence of a Turing machine T is a sequence of words that one can put on a large enough square grid. The existence of a finite terminating computation can be expressed by:

there is a large enough square grid and a choice of letters (from a fixed finite alphabet) for its vertices that encode a terminating computation. This is MSO expressible.

Reduction to this case.

If L is a set of finite structures, if there exist MSO+R formulas that define arbitrarily large square grids in (some of) the structures of L , then the satisfiability problem for MSOL+R is **undecidable** on L .

We will do that for words over 1 letter: $D_n = ([n], \leq)$.

We will specify “large” sets A, B, C and bijections $f : A \times B \rightarrow C$ (*pairing functions*).

From that one gets a rectangular grid with vertex set C .

We let for p, q :

$$A = \{ 2^k / 2 \leq k \leq p \}.$$

$$B = \{ 3^m / 1 \leq m \leq q \}.$$

$$C = \{ 2^k 3^m / 2 \leq k \leq p, 1 \leq m \leq q \}.$$

Then a number $x > 3$ is a power of 2

\leftrightarrow : $\text{Card}_{\mathbb{R}}([x])$ & $\text{Even}([x])$ (“even cardinality”).

A number $x > 3$ is a power of 3

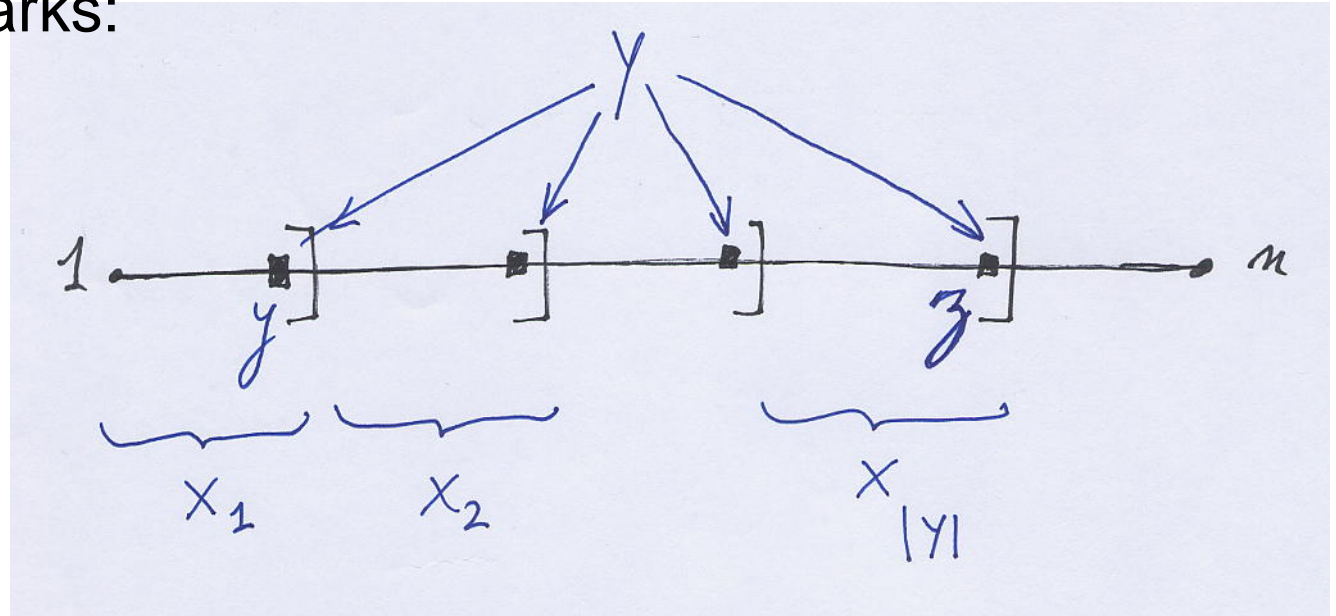
\leftrightarrow : $\text{Card}_{\mathbb{R}}([x])$ & $\text{Odd}([x])$.

Now :

$2^a + 2^b$ is a power of 2 $\leftrightarrow a = b$.

$3^a + 3^b + 3^c$ is a power of 3 $\leftrightarrow a = b = c$.

With these remarks:



Card(Y) is power of 2 & y is power of 3 & the union of any three intervals X_a has cardinality = power of 3

$$\rightarrow z = y \cdot 2^k \text{ for some } k.$$

Similarly, we express that $z = x \cdot 3^m$, x power of 2.

$$\rightarrow z = x \cdot y \text{ with } x = 2^k, y = 3^m$$

\rightarrow Definition of arbitrary large rectangular and square grids

Similar constructions :

Undecidable extensions of MSOL (on words or trees) :

$\text{Eq}(X, Y)$ to mean $\text{Card}(X) = \text{Card}(Y)$.

$\text{Auto}(X)$ to that the graph induced on X has a nontrivial automorphism.

Related results :

- 1) Assuming “the linear case of Schinzel’s Hypothesis” about sequences of consecutive prime numbers :

The MSO-theory of $(\mathbf{N}, \leq, \textit{Prime}(\cdot))$ is **decidable**, (Bateman *et al.* 1993), hence the MSO-satisfiability problem for the structures $([n], \leq, \textit{Prime}(\cdot))$ is also, which *does not imply* the decidability of MSOL+***Prime*** on the structures $([n], \leq)$ (because bijections are not MSO-definable)

2) The MSO-theories of $(\mathbf{N}, \leq, 2^{\text{some}}(.))$ and of $(\mathbf{N}, \leq, 3^{\text{some}}(.))$ are **decidable** (Elgot & Rabin 1966), but this *does not imply* that that of $(\mathbf{N}, \leq, 2^{\text{some}}(.), 3^{\text{some}}(.))$ is. (Semenov)

Question : Is the MSO-satisfiability problem of the structures $([n], \leq, 2^{\text{some}} \vee 3^{\text{some}}(.))$ is decidable ?

Open question : For which recursive sets R is the satisfiability problem for **MSOL+R** *decidable on words* ?

What about $R = \{ 2^n / n > 0 \}$ and $R = \text{prime numbers}$